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Existence of periodic solutions for a second order nonlinear neutral differential equation with variable delay

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Abstract In this article we study the existence of periodic solutions of the second order nonlinear neutral differential equation with variable delay

$$x''(t)+p(t)x'(t)+q(t)x^3(t)=c(t)x'(t-\tau(t))+f\left(t,x^3(t-\tau(t))\right).$$

The main tool employed here is the Burton-Krasnoselskii's hybrid fixed point theorem dealing with a sum of two mappings, one is a large contraction and the other is compact.

1 Introduction

Due to their importance in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential equations, see $[1-18]$ and references therein.

In this paper, we are interested in the analysis of qualitative theory of periodic solutions of delay differential equations. Motivated by the papers $[1-4, 7-8, 14-18]$ and the references therein, we concentrate on the existence of periodic solutions for the second order nonlinear neutral differential equation

$$x''(t)+p(t)x'(t)+q(t)x^3(t)=c(t)x'(t-\tau(t))+f\left(t,x^3(t-\tau(t))\right), \tag{1.1}$$

where p, q are positive continuous real-valued functions, c is continuously differentiable and τ is positive twice continuously differentiable. The function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous with respect to its arguments. To reach our desired end we have to transform (1.1) into an integral equation and then use Burton-Krasnoselskii's fixed point theorem to show the existence of periodic solutions. The obtained integral equation splits in the sum of two mappings, one is a large contraction and the other is compact.

The organization of this paper is as follows. In Section 2, we introduce some notations and lemmas, and state some preliminary results needed in later sections, then we give the Green's function of (1.1), which plays an important role in this paper. Also, we present the inversion of (1.1) and Burton-Krasnoselskii's fixed point theorem. In Section 3, we present our main results on existence.

2 Preliminaries

For $T>0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T,\|.\|)$ is a Banach space with the supremum norm

$$\|x\|=\sup_{t\in\mathbb{R}}|x(t)|=\sup_{t\in[0,T]}|x(t)|.$$

Since we are searching for the existence of periodic solutions for equation (1.1), it is natural to assume that

$$p(t+T)=p(t),\ q(t+T)=q(t),\ c(t+T)=c(t),\ \tau(t+T)=\tau(t). \tag{2.1}$$

Further, we ask that

$$\int_0^T p(s)\,ds>0,\ \int_0^T q(s)\,ds>0. \tag{2.2}$$

Function $f(t,x)$ is periodic in t of period T and globally Lipschitz continuous in x. That is

$$f(t+T,x)=f(t,x), \tag{2.3}$$

and there is a positive constants k such that

$$|f(t,x)-f(t,y)|\leq k\,\|x-y\|. \tag{2.4}$$

Also, we assume that for all t, $0\leq t\leq T$,

$$\tau'(t)\neq 1. \tag{2.5}$$

Lemma 2.1 ([15]). *Suppose that* (2.1) *and* (2.2) *hold and*

$$\frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right)-1\right]}{Q_1 T}\geq 1, \tag{2.6}$$

where

$$R_1=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right)-1}q(s)\,ds\right|,\ Q_1=\left(1+\exp\left(\int_0^T p(u)\,du\right)\right)^2R_1^2.$$

Then there are continuous T-periodic functions a and b such that $b(t)>0$, $\int_0^T a(u)\,du>0$ and

$$a(t)+b(t)=p(t),\ b'(t)+a(t)b(t)=q(t),\ \textit{for } t\in\mathbb{R}.$$

Lemma 2.2 ([17]). *Suppose the conditions of Lemma* 2.1 *hold and $\phi\in P_T$. Then the equation*

$$x''(t)+p(t)x'(t)+q(t)x(t)=\phi(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t)=\int_t^{t+T}G(t,s)\phi(s)\,ds,$$

where

$$G(t,s)=\frac{\int_t^s\exp\left[\int_t^u b(v)\,dv+\int_u^s a(v)\,dv\right]du+\int_s^{t+T}\exp\left[\int_t^u b(v)\,dv+\int_u^{s+T}a(v)\,dv\right]du}{\left[\exp\left(\int_0^T a(u)\,du\right)-1\right]\left[\exp\left(\int_0^T b(u)\,du\right)-1\right]}.$$

Corollary 2.3 ([17]). *Green's function G satisfies the following properties*

$$G(t,t+T)=G(t,t),\ G(t+T,s+T)=G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s)=a(s)G(t,s)-\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1},$$

$$\frac{\partial}{\partial t}G(t,s)=-b(t)G(t,s)+\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right)-1}.$$

The following lemma is fundamental to our results.

Lemma 2.4. *Suppose* $(2.1)-(2.3)$, (2.5) *and* (2.6) *hold. If $x\in P_T$, then x is a solution of equation* (1.1) *if and only if*

$$\begin{aligned}x(t)&=\int_t^{t+T}G(t,s)q(s)\left[x(s)-x^3(s)\right]ds\\&\quad+\int_t^{t+T}x(s-\tau(s))\left[E(t,s)-R(s)G(t,s)\right]+G(t,s)f\left(s,x^3(s-\tau(s))\right)ds,\end{aligned} \tag{2.7}$$

where

$$E(t,s)=\frac{c(s)}{1-\tau'(s)}\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1}, \tag{2.8}$$

$$R(s)=\frac{\left(c'(s)+a(s)c(s)\right)\left(1-\tau'(s)\right)+c(s)\tau''(s)}{\left(1-\tau'(s)\right)^2}. \tag{2.9}$$

Proof. Let $x \in P_T$ be a solution of (1.1). Rewrite (1.1) as

$$x''(t) + p(t)x'(t) + q(t)x(t) = q(t)\left[x(t) - x^3(t)\right] \\ + c(t)x'(t-\tau(t)) + f\left(t, x^3(t-\tau(t))\right).$$

From Lemma 2.4, we have

$$x(t) = \int_t^{t+T} G(t,s)q(s)\left[x(s) - x^3(s)\right]ds \\ + \int_t^{t+T} G(t,s)\left[c(s)x'(s-\tau(s)) + f\left(s, x^3(s-\tau(s))\right)\right]ds. \tag{2.10}$$

Performing an integration by parts, we have

$$\begin{aligned}
&\int_t^{t+T} G(t,s)c(s)x'(s-\tau(s))\,ds \\
&= \int_t^{t+T} \frac{c(s)x'(s-\tau(s))(1-\tau'(s))}{1-\tau'(s)}G(t,s)\,ds \\
&= \int_t^{t+T} \frac{c(s)}{1-\tau'(s)}G(t,s)\,dx(s-\tau(s)) \\
&= -\int_t^{t+T} \frac{\partial}{\partial s}\left[\frac{c(s)}{1-\tau'(s)}G(t,s)\right]x(s-\tau(s))\,ds \\
&= \int_t^{t+T} x(s-\tau(s))\left[E(t,s) - R(s)G(t,s)\right]ds,
\end{aligned} \tag{2.11}$$

where E and R are given by (2.8) and (2.9), respectively. Then substituting (2.11) in (2.10) completes the proof. □

Lemma 2.5 ([17]). *Let* $A = \int_0^T p(u)\,du$, $B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln(q(u))\,du\right)$. *If*

$$A^2 \geq 4B, \tag{2.12}$$

then we have

$$\min\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \geq \frac{1}{2}\left(A - \sqrt{A^2-4B}\right) := l,$$

$$\max\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \leq \frac{1}{2}\left(A + \sqrt{A^2-4B}\right) := m.$$

Corollary 2.6 ([17]). *Functions G and E satisfy*

$$\frac{T}{(e^m-1)^2} \leq G(t,s) \leq \frac{T\exp\left(\int_0^T p(u)\,du\right)}{(e^l-1)^2}, \quad |E(t,s)| \leq \left|\frac{c(s)}{1-\tau'(s)}\right|\frac{e^m}{e^l-1}.$$

In the analysis, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. The following definition, due to T. A. Burton, can be found in [3, 4].

Definition 2.7 (Large Contraction). Let $(\mathbb{M}, d)$ be a metric space and consider $\mathcal{B} : \mathbb{M} \to \mathbb{M}$. Then $\mathcal{B}$ is said to be a large contraction if given $\phi, \varphi \in \mathbb{M}$ with $\phi \neq \varphi$ then $d(\mathcal{B}\phi, \mathcal{B}\varphi) \leq d(\phi, \varphi)$ and if for all $\varepsilon > 0$, there exists a $\delta < 1$ such that

$$[\phi, \varphi \in \mathbb{M},\ d(\phi,\varphi) \geq \varepsilon] \Rightarrow d(\mathcal{B}\phi, \mathcal{B}\varphi) \leq \delta d(\phi,\varphi).$$

The next theorem is also a result of T. A. Burton. This captivating theorem, which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem and have been used successfully in existence and stability in differential equations (see [3, Theorem 3] and [4]).

Theorem 2.8 (Burton-Krasnoselskii). *Let $\mathbb{M}$ be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{M}$ such that*
(i) $x, y \in \mathbb{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,
(ii) $\mathcal{A}$ is compact and continuous,
(iii) $\mathcal{B}$ is a large contraction mapping.
Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

We will use this theorem to prove the existence of periodic solutions for equation (1.1). We begin with the following proposition (see [3], [4]) and for convenience we present its proof.

Proposition 2.9. *If $\|.\|$ is the maximum norm,*

$$\mathbb{M} = \left\{\varphi \in P_T : \|\varphi\| \leq \sqrt{3}/3\right\},$$

and $(\mathfrak{B}\varphi)(t) = \varphi(t) - \varphi^3(t)$, then $\mathfrak{B}$ is a large contraction of the set $\mathbb{M}$.

Proof. For each $t \in \mathbb{R}$ we have for the real functions φ, ψ

$$\begin{aligned}
&|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \\
&= |\varphi(t) - \psi(t)| \left|1 - \left(\varphi^2(t) + \varphi(t)\psi(t) + \psi^2(t)\right)\right|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\varphi(t) - \psi(t)|^2 &= \varphi^2(t) - 2\varphi(t)\psi(t) + \psi^2(t) \\
&\leq 2\left(\varphi^2(t) + \psi^2(t)\right).
\end{aligned}$$

Using $\varphi^2(t) + \psi^2(t) < 1$ we have

$$\begin{aligned}
&|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \\
&\leq |\varphi(t) - \psi(t)| \left[1 - \left(\varphi^2(t) + \psi^2(t)\right) + |\varphi(t)\psi(t)|\right] \\
&\leq |\varphi(t) - \psi(t)| \left[1 - \left(\varphi^2(t) + \psi^2(t)\right) + \frac{\varphi^2(t) + \psi^2(t)}{2}\right] \\
&\leq |\varphi(t) - \psi(t)| \left[1 - \frac{\varphi^2(t) + \psi^2(t)}{2}\right] \\
&\leq \|\varphi - \psi\|.
\end{aligned}$$

Then

$$\|\mathfrak{B}\varphi - \mathfrak{B}\psi\| \leq \|\varphi - \psi\|.$$

Now, let $\epsilon \in (0, 1)$ be given and let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi - \psi\| \geq \epsilon$.
a) Suppose that for some t we have

$$\epsilon/2 \leq |\varphi(t) - \psi(t)|.$$

Then

$$(\epsilon/2)^2 \leq |\varphi(t) - \psi(t)|^2 \leq 2\left(\varphi^2(t) + \psi^2(t)\right),$$

that is

$$\varphi^2(t) + \psi^2(t) \geq \epsilon^2/8.$$

For all such t we have

$$\begin{aligned}
|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| &\leq |\varphi(t) - \psi(t)| \left[1 - \frac{\epsilon^2}{16}\right] \\
&\leq \left[1 - \frac{\epsilon^2}{16}\right] \|\varphi - \psi\|.
\end{aligned}$$

b) Suppose that for some t we have

$$|\varphi(t) - \psi(t)| \leq \epsilon/2,$$

then

$$|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \leq |\varphi(t) - \psi(t)| \leq (1/2)\|\varphi - \psi\|.$$

So, for all t we have

$$|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \leq \max\left\{1/2, 1 - \frac{\epsilon^2}{16}\right\} \|\varphi - \psi\|.$$

Hence, for each $\epsilon > 0$, if $\delta = \max\left\{1/2, 1 - \frac{\epsilon^2}{16}\right\} < 1$, then

$$\|\mathfrak{B}\varphi - \mathfrak{B}\psi\| \leq \delta \|\varphi - \psi\|.$$

Consequently, $\mathfrak{B}$ is a large contraction. □

3 Existence of periodic solutions

To apply Theorem 2.8, we need to define a Banach space $\mathbb{B}$, a bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings, one is a large contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and $\mathbb{M} = \{\varphi \in \mathbb{B} : \|\varphi\| \leq L\}$, where $L = \sqrt{3}/3$. We express equation (2.7) as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (H\varphi)(t),$$

where $\mathcal{A}, \mathcal{B} : \mathbb{M} \to \mathbb{B}$ are defined by

$$(\mathcal{A}\varphi)(t) = \int_t^{t+T} \varphi(s - \tau(s))[E(t,s) - R(s)G(t,s)] + G(t,s) f\left(s, \varphi^3(s - \tau(s))\right) ds, \tag{3.1}$$

and

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} G(t,s) q(s) \left[\varphi(s) - \varphi^3(s)\right] ds. \tag{3.2}$$

To simplify notations, we introduce the following constants

$$\alpha = \frac{T \exp\left(\int_0^T p(u)\, du\right)}{\left(e^l - 1\right)^2},\ \beta = \frac{e^m}{e^l - 1},\ \theta = \max_{t \in [0,T]} \left|\frac{c(t)}{1 - \tau'(t)}\right|,\ \sigma = \max_{t \in [0,T]} |q(t)|,$$
$$\lambda = \max_{t \in [0,T]} |b(t)|,\ \mu = \max_{t \in [0,T]} |R(s)|,\ \rho = \max_{t \in [0,T]} |f(t,0)|. \tag{3.3}$$

We need the following assumptions

$$\alpha\sigma T \leq 1, \tag{3.4}$$

$$JT\left[L(\beta\theta + \mu\alpha) + \alpha\left(kL^3 + \rho\right)\right] \leq L, \tag{3.5}$$

where J is constants with $J \geq 3$.

We shall prove that the mapping H has a fixed point which solves (1.1), whenever its derivative exists.

Lemma 3.1. *Suppose that conditions* (2.1) – (2.6), (2.12) *and* (3.5) *hold. Then* $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ *is compact.*

Proof. Let $\mathcal{A}$ be defined by (3.1). Obviously, $\mathcal{A}\varphi$ is continuous and it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. Observe that

$$|f(t,x)| \leq |f(t,x) - f(t,0)| + |f(t,0)| \leq k\|x\| + \rho.$$

So, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)| \\
&\leq \int_t^{t+T} |\varphi(s - \tau(s))| [|E(t,s)| + |R(s)||G(t,s)|] + |G(t,s)| \left|f\left(s, \varphi^3(s - \tau(s))\right)\right| ds \\
&\leq \int_t^{t+T} L(\beta\theta + \mu\alpha) + \alpha\left(kL^3 + \rho\right) ds \\
&\leq T\left[L(\beta\theta + \mu\alpha) + \alpha\left(kL^3 + \rho\right)\right] \leq \frac{L}{J} < L.
\end{aligned}$$

That is $\mathcal{A}\varphi \in \mathbb{M}$.

To see that $\mathcal{A}$ is continuous, we let $\varphi, \psi \in \mathbb{M}$. Given $\varepsilon > 0$, take $\eta = \varepsilon/N$ with $N = T\left[(\beta\theta + \mu\alpha) + 3\alpha k L^2\right]$ where k is given by (2.4). Now, for $\|\varphi - \psi\| < \eta$, we obtain

$$\begin{aligned}\|\mathcal{A}\varphi - \mathcal{A}\psi\| &\leq \int_t^{t+T} \left[(\beta\theta + \mu\alpha)\|\varphi - \psi\| + \alpha k \left\|\varphi^3 - \psi^3\right\|\right] ds \\ &\leq N\|\varphi - \psi\| < \varepsilon.\end{aligned}$$

This proves that $\mathcal{A}$ is continuous.
To show that the image of $\mathcal{A}$ is contained in a compact set. Let $\varphi_n \in \mathbb{M}$, where n is a positive integer. then, as above, we see that

$$\|\mathcal{A}\varphi_n\| \leq L.$$

Next we calculate $(\mathcal{A}\varphi_n)'(t)$ and show that it is uniformly bounded. By making use of (2.1), (2.2) and (2.3) we obtain by taking the derivative in (3.1) that

$$\begin{aligned}&(\mathcal{A}\varphi_n)'(t) \\ &= \tfrac{c(t)}{1-\tau'(t)} \tfrac{\exp\left(\int_t^{t+T} b(v)dv\right)-1}{\exp\left(\int_0^T b(v)dv\right)-1} \varphi_n(t - \tau(t)) \\ &+ \int_t^{t+T} \varphi_n(s - \tau(s)) \left[-b(t) E(t,s) - R(s)\left(-b(t) G(t,s) + \tfrac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right)-1}\right)\right] ds \\ &+ \int_t^{t+T} \left(-b(t) G(t,s) + \tfrac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right)-1}\right) f\left(s, \varphi_n^3(s - \tau(s))\right) ds.\end{aligned}$$

Consequently, by invoking (2.4) and (3.3) , we obtain

$$\begin{aligned}\left|(\mathcal{A}\varphi_n)'(t)\right| &\leq \theta\beta L + T\left[L(\lambda\beta\theta + \mu(\lambda\alpha + \beta)) + (\lambda\alpha + \beta)\left(kL^3 + \rho\right)\right] \\ &\leq D,\end{aligned}$$

for some positive constant D. Hence the sequence $(\mathcal{A}\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}\varphi_{n_k})$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact subset of $\mathbb{M}$. □

Lemma 3.2. *For $\mathcal{B}$ defined in* (3.2)*, suppose that* (3.4) *hold. then $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (3.2). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t + T) = (\mathcal{B}\varphi)(t)$. So, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned}|(\mathcal{B}\varphi)(t)| &\leq \int_t^{t+T} |G(t,s)|\,|q(s)|\,\left|\varphi(s) - \varphi^3(s)\right| ds \\ &\leq \alpha\sigma T\left\|\varphi - \varphi^3\right\|.\end{aligned}$$

Since $\|\varphi\| \leq L$, we have $\left\|\varphi - \varphi^3\right\| \leq \left(2\sqrt{3}\right)/9 < L$. So, for any $\varphi \in M$, we have

$$\|\mathcal{B}\varphi\| \leq L.$$

Thus $\mathcal{B}\varphi \in \mathbb{M}$. Consequently, we have $\mathcal{B} : \mathbb{M} \to \mathbb{M}$.
It remains to show that $\mathcal{B}$ is large contraction. From the proof of Proposition 2.9 we have for $\varphi, \psi \in \mathbb{M}$, with $\varphi \neq \psi$

$$|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \leq \alpha\sigma T\|\varphi - \psi\| \leq \|\varphi - \psi\|.$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \|\varphi - \psi\|$. Now, let $\epsilon \in (0,1)$ be given and let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi - \psi\| \geq \varepsilon$. From the proof of the Proposition 2.9 we have found $\delta < 1$ such that

$$|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| \leq \alpha\sigma T\delta\|\varphi - \psi\| \leq \delta\|\varphi - \psi\|.$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq \delta\|\varphi - \psi\|$. Consequently, $\mathcal{B}$ is a large contraction. □

Theorem 3.3. *Let $(P_T, \|.\|)$ be the Banach space of continuous T-periodic real valued functions and $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq L\}$, where $L = \sqrt{3}/3$. Suppose* (2.1) − (2.6)*,* (2.12)*,* (3.4) *and* (3.5) *hold. Then equation* (1.1) *has a T-periodic solution φ in the subset $\mathbb{M}$.*

Proof. By Lemma 3.1, the operator $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ is compact and continuous. Also, from Lemma 3.2, the operator $\mathcal{B} : \mathbb{M} \to \mathbb{M}$ is a large contraction. Moreover, if $\varphi, \psi \in M$, we see that

$$\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq \|\mathcal{A}\varphi\| + \|\mathcal{B}\psi\| \leq L/J + \left(2\sqrt{3}\right)/9 \leq L.$$

Thus $\mathcal{A}\varphi + \mathcal{B}\psi \in \mathbb{M}$.
Clearly, all the hypotheses of Burton-Krasnoselskii Theorem 2.8 are satisfied. Thus there exists a fixed point $\varphi \in \mathbb{M}$ such that $\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi$. By Lemma 2.4 this fixed point is a solution of (1.1) and the proof is complete. □

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