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A NOTE ON ECS-MODULES

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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

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Abstract. A module M is said to satisfy the ECS condition if every ec-closed submodule of M is a direct summand. It is known that the class of ECS-modules is not closed under direct sums. In this paper, we studied when a direct sum of two modules is an ECS-module and when an ECS-module has a decomposition into uniform submodules.

1 Introduction

Throughout this paper, all rings are associative with unity and R denotes such a ring. All modules are unital right R-modules. A right R-module M has *finite uniform (Goldie) dimension* if M does not contain an infinite direct sum of non-zero submodules. It is well known that a module M has finite uniform dimension if and only if there exists a positive integer n and uniform submodules U_i $(1 \leq i \leq n)$ of M such that $U_1 \oplus U_2 \oplus ... \oplus U_n$ is an essential submodule of M and in this case n is an invariant of the module called the *uniform dimension* of M, (see, for example, [1, p. 294, ex. 2]).

Recall that a module M is said to be *extending* or *CS* if every complement (or closed) submodule of M is a direct summand. Equivalently, every submodule of M is essential in a direct summand of M (see [6], [10]). Following [9], we call a (closed) submodule N of M as *ec-(closed) submodule* if N contains essentially a cyclic submodule, i.e., there exists $x \in N$ such that xR is essential in N. Note that every direct summand of an ec-closed submodule of M is ec-closed. A module M is said to be *principally extending (or P-extending)* if every cyclic submodule of M is essential in a direct summand. Following [5], a module is said to be ECS if every ec-closed submodule is a direct summand. Among examples of ECS-modules, we could mention that extending modules and von Neumann regular rings. Furthermore, it can be seen easily that for a module of finite uniform dimension CS and ECS concepts coincide. ECS-modules were investigated in [5] and [9]. In this paper, we continue the study of ECS-modules. To this end, we studied when a P-extending module and also a direct sum of two modules are ECS-modules. Moreover we generalize a well known result on CS-modules to ECS-modules which provides a decomposition into uniform submodules.

Let R be a ring and M a right R-module. If $X \subseteq M$, then $X \leq M$ denotes X is a submodule of M. Moreover $End(M_R)$, $Z(M)$, $E(M)$ and $r(m)$ $(m \in M)$ symbolize the ring of endomorphisms of M, the singular submodule of M, the injective hull of M and the right annihilator of m in R, i.e., $r(m) = \{r \in R : mr = 0\}$, respectively. Recall from [2], $S_l(R) = \{e^2 = e \in R : xe = exe \text{ for all } x \in R\}$. A ring is called *Abelian* if every idempotent is central. Other terminology and notation can be found in [6] and [7].

2 Preliminary Results

In this section, we study relationships between the P-extending and ECS conditions. In particular, we make it clear that ec-closed and complement submodules are different from each other. The next Lemma is taken from [5, Proposition 1.1] and we state here without proof.

Lemma 2.1. Let M be a module. Consider the following statements.
(i) M is CS
(ii) M is ECS
(iii) M is P-extending

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$. In general, the converses to these implications do not hold.

Corollary 2.2. Let M be a nonzero indecomposable module. Then the following statements are equivalent.
(i) M is CS
(ii) M is ECS
(iii) M is P-extending
(iv) M is uniform

Proof. Immediate by Lemma 2.1. □

Our next result and thereafter its companion Proposition show that ec-closed and complement submodules are not the same, in general. First note that there are uniform rings R for which $Z(R_R) \neq 0$, i.e., R is not nonsingular (see [4]).

Lemma 2.3. Let R be a right uniform nonsingular ring and P_R be a projective module. Then there exists a right R-module M such that P_R is a complement but not an ec-submodule of M.

Proof. Since P_R is projective there exists a free R-module say M, such that P_R is a direct summand of M. It is clear that P is a complement in M and M_R is nonsingular. Assume P is ec-submodule of M. There exists $0 \neq x \in P$ such that xR is essential in P. However, $r(x)$ is essential in R_R, by assumption. Hence $x \in Z(P_R) = 0$, a contradiction. So P_R is not ec-submodule of M. □

Proposition 2.4. Let $n \geq 3$ be any odd integer. Let $\mathbb{R}$ be the real field and S the polynomial ring $\mathbb{R}[x_1, x_2, ..., x_n]$. Then the ring $R = S/Ss$, where $s = \sum_{i=1}^{n} x_i^2 - 1$, is a commutative Noetherian domain. Moreover the free R-module $M = \bigoplus_{i=1}^{n} R$ contains a complement K which is not ec-closed.

Proof. It is easy to check that R is a commutative Noetherian domain. Note that R is uniform and M_R is nonsingular.
Let $\phi : M \to R$ be the homomorphism defined by $\phi(a_1 + Ss, a_2 + Ss, ..., a_n + Ss) = a_1x_1 + a_2x_2 + ... + a_nx_n + Ss$ for all a_i in S $(1 \leq i \leq n)$. Clearly, ϕ is an epimorphism, and hence, its kernel K is a direct summand of M, i.e., $M = K \oplus K'$ for some submodule K'. Obviously, $K' \cong R$ and K is a complement submodule of M. Assume that K is an ec-closed submodule of M. Then there exists $0 \neq x \in K_R$ such that xR is essential in K_R. However $r(x)$ is essential in R_R. So $x \in Z(K_R) = 0$, a contradiction. It follows that K_R is not an ec-closed submodule of M. □

Note that the module K_R in the proof of Proposition 2.4 is indecomposable projective of uniform dimension $n - 1$ (see [12]). Thus K_R is not included in Lemma 2.1 but it is included in Lemma 2.3. In conjuction with Proposition 2.4, we have the following easy result.

Proposition 2.5. Let M be an ECS right R-module and $N \leq M$. Assume M contains a cyclic essential submodule. Then N is a direct summand if and only if N is an ec-closed.

Proof. Let $Y = xR$ for some $0 \neq x \in M$ such that Y is essential in M_R. If N is ec-closed then by hypothesis, N is a direct summand. Conversely, assume that N is a direct summand. Then $M = N \oplus N'$ for some $N' \leq M$. Let $\pi : M \to N$ be the projection homomorphism. Then $Y \cap K = xR \cap K \leq \pi(Y) = \pi(x)R \leq K$ and $\pi(x)R$ is essential in K. Hence K is ec-closed. □

Since the ECS property lies strictly between the CS and P-extending properties, it is natural to seek conditions which ensure that a P-extending module is ECS or an ECS-module is CS. Such conditions were illustrated in [5, Proposition 1.2]. Now, we prove another result which makes a P-extending module is ECS.

Theorem 2.6. Let M be a R-module such that $End(M_R)$ is Abelian and $X \leq M$ implies $X = \sum_{i \in I} h_i(M)$, where $h_i \in End(M_R)$. Then M is P-extending if and only if M is ECS.

Proof. Assume M is P-extending and X is an ec-closed in M. There exists $x \in X$ such that xR is essential in X. Then $X = \sum_{i \in I} h_i(M)$, where each $h_i \in End(M_R)$. So, by hypothesis, xR is essential in $eM = D$ where $e^2 = e \in End(M_R)$. Thus $M = eM \oplus D'$ where $D' = (1-e)M$. It is clear that $X \oplus D'$ is essential in M. Let $0 \neq y \in X$. Then $y = ey + (1-e)y$. But $y = \sum_{i \in I} h_i(m_i)$ where $m_i \in M$. Thus $(1 - e)y = (1 - e)\sum_{i \in I} h_i(m_i) = \sum_{i \in I} h_i((1 - e)m_i) \in X \cap D' = 0$, i.e., $y = ey$. Hence $X \leq D$. It follows that X is essential in D. So $X = D$. Hence M_R is ECS. The converse follows from Lemma 2.1. □

Recall that an R-module M is said to be a *multiplication module* if for each $X \leq M$ there exists $A_R \leq R_R$ such that $X = MA$.

Corollary 2.7. If M is an R-module satisfying any of the following conditions, then M is P-extending if and only if M is ECS.
(i) $M_R = R_R$ and R is Abelian.
(ii) M is cyclic and R is commutative.
(iii) M is a multiplication module and R is commutative.

Proof. By Theorem 2.6 the result is true for condition (i). Now assume that M is cyclic and R is commutative. There exists $B_R \leq R_R$ such that M_R is isomorphic to R/B. Let Y/B be an R-submodule of R/B. So, $Y/B = (\sum_{i\in I} y_i R) + B = (\sum_{i\in I} y_i R + B)R$, where each $y_i \in Y$. Define $h_i : R/B \to R/B$ by $h_i(r+B) = y_i + B$. Then $h_i \in End((R/B)_R)$. Hence $Y/B = \sum_{i\in I} h_i(R/B)$. Since R is commutative, $End((R/B)_R)$ is commutative. Thus Theorem 2.6 yields the result for condition (ii).
Finally, assume that M is a multiplication module and R is commutative. Let $X = MA$, where $A_R \leq R_R$. For each $a \in A$ define $h_a : M \to M$ by $h_a(m) = ma$ for $m \in M$. Then $X = MA = \sum_{a\in A} h_a(M)$. Observe that every submodule of a multiplication module is fully invariant. By [3, Lemma 1.9], if $e^2 = e \in End(M_R)$, then e and $1 - e \in S_l(End(M_R))$. Hence e is central. So $End(M_R)$ is Abelian. Again, Theorem 2.6 yields the result. □

3 Direct Sums of ECS-Modules

In this section, we deal with when a direct sum of two modules is an ECS-module. Recall that the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}/\mathbb{Z}p$, where p is any prime integer, is not CS-module by [11, Example 10]. Since M has finite uniform dimension, M is not ECS-module. Moreover $M_{\mathbb{Z}}$ is a direct sum of two uniform (and hence ECS) modules and even if $\mathbb{Q}$ is $\mathbb{Z}/\mathbb{Z}p$-injective, $\mathbb{Z}/\mathbb{Z}p$ is not $\mathbb{Q}$-injective.

In [9], the authors assumed that ECS and P-extending conditions are the same and proved results for P-extending modules. However these two conditions are different by Lemma 2.1. So we give the corrected forms of some results in [9] which are stated for a direct sum of modules being P-extending. First we have the following result.

Proposition 3.1. Let $M = M_1 \oplus M_2$ be a module, where the M_i are uniform and $End(M_i)$ local for $i = 1, 2$. Then the following conditions are equivalent:
(i) M is a CS-module, and monomorphisms $M_i \to M_j$ are isomorphisms; $i \neq j$.
(ii) M is an ECS-module, and monomorphisms $M_i \to M_j$ are isomorphisms; $i \neq j$.
(iii) M_i are M_j-injective; $i \neq j$.

Proof. $(i) \Rightarrow (ii)$. Clear by Lemma 2.1.
$(ii) \Rightarrow (iii)$. Let $f : E(M_i) \to E(M_j)$ be an arbitrary homomorphism, where $i \neq j$. Let $X = \{x \in M_i : f(x) \in M_j\}$. Then $A = \{x + f(x) : x \in X\}$ is a closed and uniform submodule of M, by [8, Lemma 1]. Hence A is an ec-closed in M. By hypothesis, $M = A \oplus M_i$ or $M = A \oplus M_j$. If $M = A \oplus M_i$, then $M_j = f(X)$, and hence $f^{-1} : M_j \to X \subseteq M_i$ is, by assumption, an isomorphism, i.e., $X = M_i$. On the other hand, if $M = A \oplus M_j$, then $X = M_i$.
$(iii) \Rightarrow (i)$. Obvious. □

Proposition 3.2. Let $M = M_1 \oplus M_2$, and let $C \cap M_1$ be an ec-submodule of M, for every ec-closed submodule C of M. Then M is ECS if and only if every ec-closed submodule C with $C \cap M_1 = 0$ or $C \cap M_2 = 0$ is a direct summand.

Proof. The necessity is clear. For the sufficiency, let C be an ec-closed submodule of M with cR is essential in C. If $C \cap M_1 = 0$, then we are done. Otherwise, $C \cap M_1$ is an ec-submodule of M, by assumption. Let C_1 be the closure of $C \cap M_1$ in C, then C_1 is an ec-closed submodule of M, with $C_1 \cap M_2 = 0$. By hypothesis, C_1 is a direct summand of M. Hence $M = C_1 \oplus C_2$ for some submodule C_2 of M. Thus $C = C_1 \oplus (C \cap C_2)$. So $C \cap C_2$ is an ec-closed submodule of M with $(C \cap C_2) \cap M_1 = 0$, and therefore $C \cap C_2$ is a direct summand of M. Hence C is a direct summand of M. It follows that M is an ECS-module. □

Theorem 3.3. Let $M = M_1 \oplus M_2$ where M_1 is of finite uniform dimension. Then M is ECS if and only if every ec-closed submodule C of M, with $C \cap M_1 = 0$, or C is of finite uniform dimension, is a direct summand.

Proof. The necessary condition is obvious. For the sufficient condition, let C be an ec-closed submodule of M with mR is essential in C. If $C \cap M_1 = 0$, then we are done. Now, let $0 \neq c \in C \cap M_1$ and C_1 be the closure of cR in C. Note that C_1 has finite uniform dimension. By hypothesis, C_1 is a direct summand of M. Thus $M = C_1 \oplus K$ for some submodule K of M. Hence $C = C_1 \oplus D$, where $D = K \cap C$ is closed in M. Since D is a direct summand of an ec-closed submodule C, then D is ec-closed. If $D \cap M_1 = 0$, then by assumption D is a direct summand, and hence C is a direct summand of M. If $D \cap M_1 \neq 0$, then by repeating the previous steps, we have $D = C_2 \oplus C_3$, where C_2 is a direct summand and has a nonzero intersection with M_1. Continuing in this manner, we obtain $C = C_1 \oplus C_2 \oplus ... \oplus C_n$, where C_i is a direct summand of M $(i = 1, 2, .., n-1)$ and C_n contains an essential cyclic submodule with $C_n \cap M_1 = 0$. By hypothesis, C_n is a direct summand of M and therefore C is a direct summand of M. □

Corollary 3.4. Let $M = M_1 \oplus M_2$. Then every ec-closed submodule of M with finite uniform dimension, is a direct summand if and only if every ec-closed submodule C of M with finite uniform dimension such that $C \cap M_1 = 0$ or $C \cap M_2 = 0$, is a direct summand.

Proof. Similar to the proof of Theorem 3.3. □

Proposition 3.5. Let $M = M_1 \oplus M_2$, where M_1 is a semisimple module. Then M is ECS if and only if every ec-closed submodule C of M with $C \cap M_1 = 0$, is a direct summand.

Proof. The necessity is obvious. For the sufficiency, let C be an ec-closed submodule of M. If $C \cap M_1 = 0$, then we are done. So assume that $C \cap M_1 \neq 0$. Thus $C \cap M_1$ is a direct summand of M_1. It follows that $C = (C \cap M_1) \oplus D$ for some submodule D of C. Since D is an ec-closed submodule of M and $D \cap M_1 = 0$, then D is a direct summand of M. Thus C is a direct summand of M. □

4 A Decomposition into Uniform Submodules

Finally we prove a result which decomposes an ECS-module as a direct sum of uniform submodules. For analogy result in the CS case, we refer to [6] (see, also [10]). Since we will use it in our result, we need to give the following definition. Let M be a module and let $N = \bigoplus_{i \in I} N_i$ be a direct sum of submodules N_i $(i \in I)$ of M. Then N is called a *local direct summand* of M if $\bigoplus_{i \in I'} N_i$ is a direct summand of M for every finite subset I' of I. It is well known that a local direct summand is a complement. Now, we have the following result.

Theorem 4.1. Let R be a ring and let M be an R-module such that R satisfies ACC on right ideals of the form $r(m)$ $(m \in M)$. If every direct summand of M is P-extending and every local direct summand of M is a direct summand then M is a direct sum of uniform submodules.

Proof. Let $0 \neq m \in M$ such that $r(m)$ is maximal in $\{r(x) : 0 \neq x \in M\}$. There exists a direct summand K of M such that mR is essential in K. Suppose that K is not indecomposable. Then there exist non-zero submodules K_1 and K_2 of K such that $K = K_1 \oplus K_2$. There exist $m_i \in K_i$ $(i = 1, 2)$ such that $m = m_1 + m_2$. If $m_1 = 0$ then $m = m_2 \in K_2$, and $mR \cap K_1 = 0$ gives $K_1 = 0$, a contradiction. Thus $m_1 \neq 0$. Clearly $r(m) \subseteq r(m_1)$. Hence $r(m) = r(m_1)$, by the choice of m. Similarly $m_2 \neq 0$ and $r(m) = r(m_2)$. Because $m_1 \neq 0$, there exist $r_1, r_2 \in R$ such that $0 \neq m_1 r_1 = m r_2 = (m_1 + m_2) r_2 = m_1 r_2 + m_2 r_2$. Thus $m_2 r_2 = 0$, and hence $r_2 \in r(m_2) \backslash r(m)$, a contradiction. Thus K is indecomposable. By hypothesis, K is a P-extending module and so Corollary 2.2 yields that K is uniform.

By Zorn's Lemma, M contains a maximal local direct summand $N = \bigoplus_{i \in I} N_i$, where N_i is a uniform submodule of M for each $i \in I$. By assumption, $M = N \oplus N'$ for some $N' \leq M$. If $N' \neq 0$ then, by the above argument, $N' = U \oplus U'$ for some $U, U' \leq M$ with U uniform. Then $N \oplus U$ is a local direct summand, contradicting the choice of N. Thus $N' = 0$. It follows that $M = \bigoplus_{i \in I} N_i$ is a direct sum of uniform submodules. □

Corollary 4.2. Let R be a ring and let M be an R-module such that R satisfies ACC on right ideals of the form $r(m)$ $(m \in M)$. If M is an ECS-module and every local direct summand of M is a direct summand then M is a direct sum of uniform submodules.

Proof. By Lemma 2.1 and Theorem 4.1. □

As direct consequences of Corollary 4.2, we have next corollaries.

Corollary 4.3. Let R be a right Noetherian ring and let M be an R-module. If M is an ECS-module and every local direct summand of M is a direct summand then M is a direct sum of uniform submodules.

Corollary 4.4. Let R be a ring and let M be an R-module such that R satisfies ACC on right ideals of the form $r(m)$ $(m \in M)$. If M is a CS-module then M is a direct sum of uniform submodules.

Observe that the indecomposable nonuniform module K_R in the proof of Proposition 2.4 is not included in the above Corollaries of Theorem 4.1.

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