Palestine Journal of Mathematics

Vol. 4(1) (2015) , 44–56

On the approximation by Cesáro submethod

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Communicated by Ayman Badawi

MSC 2010 Classifications: 40C05, 40G05, 41A25, 42A05, 42A10

Keywords and phrases: Trigonometric polynomials, almost monotone sequences, Cesáro submethod, degree of approximation.

This research is supported by Scientific Research and Application Center of Mersin University under Grant BAP-FBE MB (MK) 2012-2 YL

Abstract. In this paper the degree of approximation to functions belonging to Lipschitz class is estimated by C_λ-method obtained by deleting a set of rows from the Cesáro matrix C_1 under the some conditions.

1 Introduction

Assume that f is a $2\pi-$ periodic function and $f \in L_p := L_p[0, 2\pi]$ for $p \geq 1$ where L_p consists of all measurable functions for which the L_p- norm is defined as follows

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right\}^{\frac{1}{p}} < \infty.$$

On the other hand, the partial sum of the first $(n+1)$ terms of the Fourier series of $f \in L_p$ at a point x is denoted by

$$s_n(f;x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{n} A_k(f;x).$$

Furthermore, a function f belongs to the $Lip(\alpha, p)$ class if $\omega_p(\delta, f) = O(\delta^\alpha)$, where

$$\omega_p(\delta, f) = \sup_{|t| \leq \delta} ||f(\cdot + t) - f(\cdot)||_p \quad 0 < \alpha \leq 1; \quad p \geq 1,$$

is the integral modulus of continuity of $f \in L_p$.

One of the basic problems in the theory of approximation of functions and the theory of Fourier series is to examine the degree of approximation in given function spaces by some certain methods. In this sense, one of the important results encountered belongs to Quade in [8]. He solved a problem related with approximation by trigonometric polynomials on conjecture stated without proof by G. H. Hardy and J. E. Littlewood in 1928. In subsequent years, Chandra gave some attractive results including sharper estimates than some results of Quade by Nörlund and Riesz methods. In 2005, Leindler[5] weakened the conditions of monotonicity given by Chandra according to Nörlund and Riesz methods. We know that Nörlund and Riesz methods generalize the well known Cesáro method which has an important place in this theory. Naturally, there arises the question how we can generalize these approximation methods. There are two possibilities in this way. First it can be generalized by taking into account summability methods. The other one can be weakened the conditions of monotonicity. In this work we shall consider both of these conditions and move this direction. Accordingly, let F be an infinite subset of $\mathbb{N}$ as the range of a strictly increasing sequence of positive integers, with $F = (\lambda(n))_{n=1}^{\infty}$. The Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \ (n = 1, 2, ...),$$

where (x_k) is a sequence of a real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. Note that C_λ is obtained by deleting a set of rows from Cesáro matrix. The basic properties of C_λ-method can be found in [1] and [7]. By considering this method the following definitions was given in [3]:

$$N_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} s_m(f;x),$$

$$R_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m s_m(f;x),$$

where

$$s_n(f;x) = \frac{1}{\pi} \int_0^{2\pi} f\,(x+t)\, D_n(t)\, dt,$$

and

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\left(\frac{t}{2}\right)}.$$

Also,

$$P_{\lambda(n)} = p_0 + p_1 + p_2 + ... + p_{\lambda(n)} \neq 0 \; (n \geq 0),$$

and by convention $p_{-1} = P_{-1} = 0$.
In case $\lambda(n) = n$, the methods $N_n^\lambda(f;x)$ and $R_n^\lambda(f;x)$ give us classicaly known Nörlund and Riesz means. Provided that $p_n = 1$ for all $(n \geq 0)$ both of them yield

$$\sigma_n^\lambda(f;x) = \frac{1}{\lambda\,(n)+1} \sum_{m=0}^{\lambda(n)} s_m(f;x).$$

In addition to this, if $\lambda(n) = n$ for $\sigma_n^\lambda(f;x)$, then it coincides with Cesáro method C_1.

Moreover, let t_n be a trigonometrical polynomial of order n. Then, it is 2π- periodic and Lebesgue integrable. If $s_m(t_n;x)$ denotes partial sum of the first $(m+1)$ terms of the Fourier series of t_n at x, then

$$s_m(t_n;x) = \begin{cases} t_m(x), & if \quad m \leq n; \\ t_n(x), & if \quad m \geq n. \end{cases} \tag{1.1}$$

We shall also use the notations

$$\Delta a_n = a_n - a_{n+1}\ , \quad \Delta_m a(n,m) = a(n,m) - a(n,m+1).$$

While taking into account these methods, the monotonicity conditions on the sequence (p_n) are important. So, let's recall the definitions of some classes of numerical sequences discussed in detail in [4], [5] and [6]. Let $u := (u_n)$ be a nonnegative sequence and $C := (C_n) = \frac{1}{n+1}\sum\limits_{m=0}^{n} u_m$:

A sequence u is called almost monotone decreasing (briefly $u \in AMDS$) (increasing (briefly $u \in AMIS$)), if there exists a constant $K := K(u)$ which only depends on u such that

$$u_n \leq K u_m \qquad (K u_n \geq u_m)$$

for all $n \geq m$.

If $C \in AMDS$ ($C \in AMIS$), then we say that the sequence u is almost monotone decreasing (increasing) mean sequence and denoted by $C \in AMDMS$ ($C \in AMIMS$).

A sequence u tending to zero is called a rest bounded variation sequence ($RBVS$) (rest bounded variation mean sequence ($RBVMS$)), if it has the property

$$\sum_{m=k}^{\infty} |\Delta u_m| \leq K(u) u_k \quad (\sum_{m=k}^{\infty} |\Delta C_m| \leq K(u) C_k)$$

for all natural numbers k. Leindler first raised the rest bounded variation condition in [4].

A sequence u is called a head bounded variation sequence ($HBVS$) (head bounded variation mean sequence ($HBVMS$)), if it has the property

$$\sum_{m=0}^{k-1} |\Delta u_m| \leq K(u) u_k \quad (\sum_{m=0}^{k-1} |\Delta C_m| \leq K(u) C_k)$$

for all natural numbers k, or only for all $k \leq N$ if the sequence u has only finite nonzero terms and the last nonzero term u_N.

It is clear that the following inclusions are true for the above classes of numerical sequences:

$$RBVS \subset AMDS\ , \quad RBVMS \subset AMDMS$$

and

$$HBVS \subset AMIS\ , \quad HBVMS \subset AMIMS.$$

Moreover, Mohapatra and Szal showed that the following embedding relations are true in [6]:

$$AMDS \subset AMDMS$$

and

$$AMIS \subset AMIMS.$$

It is clear that the class of nonnegative and nondecreasing (nonincreasing) sequences is a subset of the class of almost monotone decreasing (increasing) sequences. Taking into these inclusions, both we will extend the results given in [5] by weakening the monotonicity conditions and we will give the degree of approximation of functions by C_λ-method of their Fourier series of functions that belong to the class L_p for $p \geq 1$. Especially, we consider the degree of approximation of $f \in L_p$ by trigonometrical polynomials $N_n^\lambda(f;x)$ and $R_n^\lambda(f;x)$ under the perspective of [5, 6]. We see that the results obtained in this paper strongly generalize the results in [2]-[5].

2 Main Results

The following results are important in the theory of Fourier series for both the creation and acceleration of convergence of a Fourier series and also for the acceleration of convergence in approximation theory.

Theorem 2.1. *Suppose that $f \in Lip(\alpha, p)$ and let (p_n) be positive. If one of the conditions,*

(i) $p > 1$, $0 < \alpha < 1$, $(p_n) \in AMIMS$ with

$$(\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)}), \tag{2.1}$$

(ii) $p > 1$, $0 < \alpha < 1$ and $(p_n) \in AMDMS$

satisfies, then

$$\left\|f - N_n^\lambda(f)\right\|_p = O(\lambda(n)^{-\alpha}).$$

Since $AMDS \subset AMDMS$ and $AMIS \subset AMIMS$, we can derive the following result from Theorem 2.1.

Corollary 2.2. *Suppose that $f \in Lip(\alpha, p)$ and let (p_n) be positive. If one of the conditions,*

(i) $p > 1$, $0 < \alpha < 1$, $(p_n) \in AMIS$ and (2.1) holds,

(ii) $p > 1$, $0 < \alpha < 1$ and $(p_n) \in AMDS$

satisfies, then

$$\left\|f - N_n^\lambda(f)\right\|_p = O(\lambda(n)^{-\alpha}).$$

This corollary also generalizes the cases (i) and (ii) of Theorem 1 given in [5] with respect to both monotonicity condition and Cesáro submethod C_λ. Therefore the results of Chandra [2] are generalized. Moreover, the last corollary can be also written in accordance with the classes $HBVMS$ and $RBVMS$.

A subsequent result is related to [5, Theorem 1] for sequences that are more general than monotone sequences in case $p > 1$, $\alpha = 1$. Accordingly, it is easy to see that if (p_n) is nondecreasing and (2.1) satisfies, then

$$\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$$

holds. On the other hand, if (p_n) is nonincreasing, then

$$\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)})$$

is also true. Therefore, the following result is implied under the weaker assumptions. In this way, we write the next theorem.

Theorem 2.3. *Let $f \in Lip(1,p)$ and let (p_n) be positive. If one of the following conditions is satisfied*

(i) $p > 1$, $\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$ and (2.1) holds,

(ii) $p > 1$ and $\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)})$,

then

$$\left\|f - N_n^{\lambda}(f)\right\|_p = O(n^{-1}). \tag{2.2}$$

Remark 2.4. Let $(p_n) \in RBVS$ with condition $(\lambda(n)+1) = O(P_{\lambda(n)})$. Then it is clear that

$$\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)})$$

is true. Therefore, keep in mind the Theorem 2.3-(ii), we can write the next corollary:

Corollary 2.5. *Let $f \in Lip(1,p)$, $p > 1$. If $(p_n) \in RBVS$ and the condition $(\lambda(n)+1) = O(P_{\lambda(n)})$ holds, then*

$$\left\|f - N_n^{\lambda}(f)\right\|_p = O(n^{-1}). \tag{2.3}$$

The following two results give us the results of Leindler for $p = 1$ and $0 < \alpha < 1$ in [5] in the event of $\lambda(n) = n$.

Theorem 2.6. *Let $f \in Lip(\alpha,1)$, $0 < \alpha < 1$, and let (p_n) be positive. If the condition*

$$\sum_{k=-1}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$$

holds, then

$$\left\|f - N_n^{\lambda}(f)\right\|_1 = O((\lambda(n))^{-\alpha}).$$

Remark 2.7. $N_n^{\lambda}(f,x)$ gives the method of $\sigma_n^{\lambda}(f,x)$ in the Theorem 2.6 in case $p_n = 1$. So, we have

$$\left\|f - \sigma_n^{\lambda}(f)\right\|_1 = O((\lambda(n))^{-\alpha}). \tag{2.4}$$

Theorem 2.8. *Let $f \in Lip(\alpha,1)$, $0 < \alpha < 1$, and let (p_n) be positive. If (p_n) satisfies (2.1) and the condition*

$$\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$$

holds, then

$$\left\|f - R_n^{\lambda}(f)\right\|_1 = O(\lambda(n)^{-\alpha}). \tag{2.5}$$

Remark 2.9. Let $(p_n) \in HBVS$ with condition (2.1). Then it is easy to observe that

$$\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O((\lambda(n))^{-1}P_{\lambda(n)}).$$

Hence, we can write the following corollary due to Theorem 2.8:

Corollary 2.10. *Let $f \in Lip(\alpha,1)$, $0 < \alpha < 1$. If $(p_n) \in HBVS$ with the condition (2.1), then (2.5) holds.*

Moreover, we can write with the following way the analogy of Theorem 2.1 for generalized Riesz method.

Theorem 2.11. *Assume that $f \in Lip(\alpha,p)$ and let (p_n) be positive. If one of the conditions is satisfied*

(i) $p > 1$, $0 < \alpha < 1$, $(p_n) \in AMDMS$ and $(\lambda(n)+1) = O(P_{\lambda(n)})$ holds,

(ii) $p > 1$, $0 < \alpha < 1$ and $(p_n) \in AMIMS$,

then

$$\left\|f - R_n^{\lambda}(f)\right\|_p = O(\lambda(n)^{-\alpha}).$$

In [6], we know that class of head bounded variation mean sequences(HBVS) includes the class of non-decreasing sequences(NDS). Therefore, taking into account this fact, we write the following result that generalizes Theorem 3 given in [2] by motivation in [6, Theorem 5].

Theorem 2.12. *Let $f \in Lip(1,1)$ and (p_n) with (2.1) be positive. If $((n+1)^{-\eta}p_n) \in HBVS$ for some $\eta > 0$, then*

$$\left\|f - R_n^{\lambda}(f)\right\|_1 = O(n^{-1}). \tag{2.6}$$

3 Some Auxiliary Results

In this section, we shall give some auxiliary results requiring to prove the theorems given in Section 2.

Lemma 3.1. *Let*

$$(p_n) \in AMDMS$$

or

$$(p_n) \in AMIMS \text{ and satisfy (2.1).}$$

Then, for $0 < \alpha < 1$,

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_{\lambda(n)-m} = O((\lambda(n)+1)^{-\alpha} P_{\lambda(n)}).$$

Proof. We shall use the analogy technique in [6] in proof of this lemma. Let us denote integer part of $\frac{\lambda(n)}{2}$ by r. So,

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_{\lambda(n)-m} = \sum_{m=0}^{r} (m+1)^{-\alpha} p_{\lambda(n)-m} + \sum_{m=r+1}^{\lambda(n)} (m+1)^{-\alpha} p_{\lambda(n)-m}$$

$$\leq \sum_{m=0}^{r} (m+1)^{-\alpha} p_{\lambda(n)-m} + (r+1)^{-\alpha} P_{\lambda(n)}.$$

First by using Abel's transformation for the sum in the right sight of inequality in above and then by applying Lagrange's mean value theorem to function $f(x) = x^{-\alpha}$, $0 < \alpha < 1$ on interval $(m+1, m+2)$, we conclude that

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_{\lambda(n)-m} \leq \sum_{m=0}^{r-1} \{(m+1)^{-\alpha} - (m+2)^{-\alpha}\} \sum_{k=0}^{m} p_{\lambda(n)-k} + (r+1)^{-\alpha} P_{\lambda(n)}$$

$$= \sum_{m=0}^{r-1} \frac{\alpha(m+1)^{\alpha-1}}{(m+1)^{\alpha}(m+2)^{\alpha}} \sum_{k=0}^{m} p_{\lambda(n)-k} + (r+1)^{-\alpha} P_{\lambda(n)}$$

$$\leq \sum_{m=0}^{r-1} \frac{1}{(m+2)^{\alpha}} \left(\frac{1}{(m+1)} \sum_{k=0}^{m} p_{\lambda(n)-k} \right) + (r+1)^{-\alpha} P_{\lambda(n)}.$$

If $(p_n) \in AMIMS$ and satisfies (2.1), then we have

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_{\lambda(n)-m} \leq p_{\lambda(n)} \sum_{m=0}^{r-1} \frac{1}{(m+2)^{\alpha}} + (r+1)^{-\alpha} P_{\lambda(n)}$$

$$= O(\frac{p_{\lambda(n)}}{1+\lambda(n)})(1+\lambda(n))^{1-\alpha} + (r+1)^{-\alpha} P_{\lambda(n)} = O((\lambda(n)+1)^{-\alpha} P_{\lambda(n)}). \tag{3.1}$$

If $(p_n) \in AMDMS$, then

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_{\lambda(n)-m} \leq \left(\frac{1}{(r+1)} \sum_{k=0}^{r} p_{\lambda(n)-k} \right) \sum_{m=0}^{r-1} \frac{1}{(m+2)^{\alpha}} + (r+1)^{-\alpha} P_{\lambda(n)}$$

$$\leq (r+1)^{-\alpha} \sum_{k=0}^{r} p_{\lambda(n)-k} + (r+1)^{-\alpha} P_{\lambda(n)} = O((\lambda(n)+1)^{-\alpha} P_{\lambda(n)}). \tag{3.2}$$

Hence, the required result is obtained from (3.1) and (3.2). □

Lemma 3.2. *Let*

$$(p_n) \in AMIMS$$

or

$$(p_n) \in AMDMS \text{ and satisfy } (\lambda(n)+1) = O(P_{\lambda(n)})\ .$$

Then, for $0 < \alpha < 1$,

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_m = O((\lambda(n)+1)^{-\alpha} P_{\lambda(n)}).$$

We will omit here the proof of Lemma 3.2 because of the fact that its proof is similar with the proof of Lemma 3.1.

The proof of Corollary 2.2 is obvious from Theorem 2.1. However, we can also prove this corollary by using the following Lemma whose proof is slightly different from the proof of the above lemma.

Lemma 3.3. *Let*

$$(p_n) \in AMDS$$

or

$$(p_n) \in AMIS \text{ and satisfy (2.1).}$$

Then, for $0 < \alpha < 1$,

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} = O(\lambda(n)^{-\alpha} P_{\lambda(n)}). \tag{3.3}$$

Proof. Let us denote integer part of $\frac{\lambda(n)}{2}$ by r. So,

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} = \sum_{m=1}^{r} m^{-\alpha} p_{\lambda(n)-m} + \sum_{m=r+1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m}. \tag{3.4}$$

If $(p_n) \in AMDS$, then

$$\sum_{m=1}^{r} m^{-\alpha} p_{\lambda(n)-m} \leq K p_{\lambda(n)-r} \sum_{m=1}^{\lambda(n)} m^{-\alpha} = O(n^{1-\alpha}) p_{\lambda(n)-r} \tag{3.5}$$

and

$$\sum_{m=r+1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} \leq (r+1)^{-\alpha} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} = (r+1)^{-\alpha} P_{\lambda(n)}. \tag{3.6}$$

From (3.4), (3.5) and (3.6), we obtain (3.3). If $(p_n) \in AMIS$ and satisfies (2.1), then

$$\sum_{m=1}^{r} m^{-\alpha} p_{\lambda(n)-m} \leq K p_{\lambda(n)} \sum_{m=1}^{\lambda(n)} m^{-\alpha} = O(\frac{P_{\lambda(n)}}{\lambda(n)} \lambda(n)^{1-\alpha}). \tag{3.7}$$

Taking into account (3.6) and (3.7), therefore we get (3.3). □

Lemma 3.4. *The following inequalities are valid:*

$$A_n^{\lambda} := \sum_{m=1}^{\lambda(n)} \left|\Delta_m \left\{ m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m}) \right\}\right| = O(1) \sum_{m=0}^{\lambda(n)-1} |\Delta p_m| \tag{3.8}$$

and if

$$\sum_{m=1}^{\lambda(n)-1} m|\Delta p_m| = O(P_{\lambda(n)})$$

then

$$A_n^{\lambda} = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right). \tag{3.9}$$

Proof. With a simple analysis, we get

$$\begin{aligned}\left|\Delta_m \left\{m^{-1}U_m^{\lambda(n)}\right\}\right| &= \left|\frac{P_{\lambda(n)} - P_{\lambda(n)-m-1}}{m(m+1)} + \frac{P_{\lambda(n)-m-1} - P_{\lambda(n)-m}}{m}\right| \\ &= \left|\frac{1}{m(m+1)}\left\{\sum_{k=\lambda(n)-m}^{\lambda(n)} p_k - (m+1)p_{\lambda(n)-m}\right\}\right|\end{aligned}$$

where $U_m^{\lambda(n)} := P_{\lambda(n)} - P_{\lambda(n)-m}$. It is easy to show by induction, similar to [4, p.134], that

$$\left|\left\{\sum_{k=\lambda(n)-m}^{\lambda(n)} p_k - (m+1)p_{\lambda(n)-m}\right\}\right| \leq \sum_{k=1}^{m} k|p_{\lambda(n)-k+1} - p_{\lambda(n)-k}|.$$

Hence, we have

$$\begin{aligned}\sum_{m=1}^{\lambda(n)}\left|\Delta_m\left(\frac{U_m^{\lambda(n)}}{m}\right)\right| &\leq \sum_{m=1}^{\lambda(n)}\frac{1}{m(m+1)}\sum_{k=1}^{m} k|p_{\lambda(n)-k+1} - p_{\lambda(n)-k}| \\ &\leq \sum_{k=1}^{\lambda(n)} k|p_{\lambda(n)-k+1} - p_{\lambda(n)-k}|\left(\sum_{m=k}^{\infty}\frac{1}{m(m+1)}\right) \\ &= \sum_{k=0}^{\lambda(n)-1} |\Delta p_k|.\end{aligned}$$

Therefore the first part of the lemma has been proven. Now let's verify the second part of the lemma. We know that

$$\begin{aligned}A_n^{\lambda} &\leq \sum_{m=1}^{\lambda(n)}\frac{1}{m(m+1)}\sum_{k=1}^{m} k|p_{\lambda(n)-k+1} - p_{\lambda(n)-k}| \\ &= \left(\sum_{m=1}^{r} + \sum_{m=r+1}^{\lambda(n)}\right)\frac{1}{m(m+1)}\sum_{k=1}^{m} k|\Delta_k p_{\lambda(n)-k}| =: I + J \qquad (3.10)\end{aligned}$$

where r is integer part of $\frac{\lambda(n)}{2}$. Firstly let's estimate I.

$$\begin{aligned}I &\leq \sum_{k=1}^{r} k|\Delta_k p_{\lambda(n)-k}|\sum_{m=k}^{\infty}\frac{1}{m(m+1)} = \sum_{k=1}^{r}|\Delta_k p_{\lambda(n)-k}| \\ &\leq \frac{1}{\lambda(n)-r}\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O\left(\frac{1}{\lambda(n)}\right)\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right) \qquad (3.11)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}J &\leq \sum_{m=r}^{\lambda(n)}\frac{1}{m(m+1)}\sum_{k=1}^{m} k|\Delta_k p_{\lambda(n)-k}| \\ &\leq \sum_{m=r}^{\lambda(n)}\frac{1}{m(m+1)}\left(\sum_{k=1}^{r} + \sum_{k=r+1}^{m}\right) k|\Delta_k p_{\lambda(n)-k}| =: J_1 + J_2. \qquad (3.12)\end{aligned}$$

Taking into account our assumption for J_1 and J_2, respectively, we get

$$\begin{aligned}J_1 &\leq \sum_{m=r}^{\lambda(n)}\frac{1}{m(m+1)}\sum_{k=1}^{r} k|\Delta_k p_{\lambda(n)-k}| \\ &\leq \sum_{m=r}^{\lambda(n)}\frac{1}{(m+1)}\sum_{k=1}^{r}|\Delta_k p_{\lambda(n)-k}| = O\left(\frac{1}{\lambda(n)-r}\right)\sum_{k=\lambda(n)-r}^{\lambda(n)-1} k|\Delta p_k| \\ &= O\left((\lambda(n))^{-1}\right)\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right) \qquad (3.13)\end{aligned}$$

and

$$
\begin{aligned}
J_2 &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{m(m+1)} \left\{ \sum_{k=r}^{m} k|\Delta_k p_{\lambda(n)-k}| \right\} \\
&= O\left(\frac{1}{\lambda(n)}\right) \sum_{m=r}^{\lambda(n)} \sum_{k=r}^{m} |\Delta_k p_{\lambda(n)-k}| \\
&= O\left((\lambda(n))^{-1}\right) \sum_{k=0}^{\lambda(n)-1} (k+1)|\Delta p_k| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right). \qquad (3.14)
\end{aligned}
$$

By combining the (3.10)-(3.14), we confirm (3.9). □

The following auxiliary results have been given by Quade in [8].

Lemma 3.5. . *If* $f \in Lip(\alpha, 1)$, $0 < \alpha < 1$, *then*

$$\|f - \sigma_n(f)\|_1 = O(n^{-\alpha}).$$

Lemma 3.6. . *Let* $f \in Lip(\alpha, p)$ *for* $0 < \alpha \leq 1$ *and* $p > 1$. *Then*

$$\|f - s_n(f)\|_p = O(n^{-\alpha}).$$

Lemma 3.7. . *If* $f \in Lip(\alpha, p)$ *for* $0 < \alpha \leq 1$ *and* $p \geq 1$, *then for any positive integer* n, f *may be approximated in* L_p *- space by a trigonometrical polynomial* t_n *order* n *such that*

$$\|f - t_n\|_p = O(n^{-\alpha}).$$

Lemma 3.8. . *If* $f \in Lip(1, p)$ *for* $p > 1$, *then*

$$\|\sigma_n(f) - s_n(f)\|_p = O(n^{-1}).$$

In the next part, we shall present the proofs of the theorems by using the analogy technique given in some references such as [5]-[6].

4 Proofs of the Main Results

Proof of Theorem 2.1. By the definition of $N_n^\lambda(f,x)$, we have

$$N_n^\lambda(f,x) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \left\{ s_m(f,x) - f(x) \right\}. \qquad (4.1)$$

From hypothesis, Lemma 3.1 and Lemma 3.6, we obtain

$$
\begin{aligned}
\left\|N_n^\lambda(f) - f\right\|_p &\leq \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \left\|s_m(f) - f\right\|_p \\
&= \frac{O(1)}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} (m+1)^{-\alpha} = O(\lambda(n)^{-\alpha}).
\end{aligned}
$$

Therefore, the proofs of the cases (i) and (ii) have been completed together. □

Proof of Corollary 2.2. Proceeding as above, from Lemma 3.3 and Lemma 3.6, we get

$$\left\|N_n^\lambda(f) - f\right\|_p \leq \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} \left\|s_m(f) - f\right\|_p + \frac{p_{\lambda(n)}}{P_{\lambda(n)}} \left\|s_0(f) - f\right\|_p$$

$$= \frac{1}{P_{\lambda(n)}} O(\lambda(n)^{-\alpha} P_{\lambda(n)}) + O(\lambda(n)^{-\alpha}) = O(\lambda(n)^{-\alpha}).$$

□

Proof of Theorem 2.3. First we consider the case (i). Let $p > 1$ *and* $\alpha = 1$. It is clear

$$N_n^{\lambda}(f,x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} P_{\lambda(n)-m} A_m$$

that

$$s_n(f,x) - N_n^{\lambda}(f,x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{n} U_m^{\lambda(n)} A_m - \frac{1}{P_{\lambda(n)}} \sum_{m=n+1}^{\lambda(n)} P_{\lambda(n)-m} A_m.$$

Denoting $\eta_m := P_{\lambda(n)-m}$ and $I := \frac{1}{P_{\lambda(n)}} \sum_{m=n+1}^{\lambda(n)} \eta_m A_m$. By Abel's Transformation we have

$$\begin{aligned} s_n(f,x) - N_n^{\lambda}(f,x) &= \frac{1}{P_{\lambda(n)}} \left[\sum_{m=1}^{n-1} \Delta_m (\frac{U_m^{\lambda(n)}}{m}) \sum_{k=0}^{m} kA_k + \frac{U_n^{\lambda(n)}}{n} \sum_{k=0}^{m} kA_k \right] - I \\ &= \frac{1}{P_{\lambda(n)}} \left[\sum_{m=1}^{n} \Delta_m (\frac{U_m^{\lambda(n)}}{m}) \sum_{k=0}^{m} kA_k + \frac{U_{n+1}^{\lambda(n)}}{n+1} \sum_{k=0}^{m} kA_k \right] - I \end{aligned}$$

and hence

$$\begin{aligned} \left\| s_n(f) - N_n^{\lambda}(f) \right\|_p &\leq \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{n} \left| \Delta_m (\frac{U_m^{\lambda(n)}}{m}) \right| \sum_{k=0}^{m} kA_k \\ &+ \frac{U_{n+1}^{\lambda(n)}}{P_{\lambda(n)}} \frac{1}{n+1} \left\| \sum_{k=0}^{m} kA_k \right\|_p + \|I\|_p \end{aligned} \tag{4.2}$$

By using Abel's transformation we have

$$\begin{aligned} I &= \frac{1}{P_{\lambda(n)}} (\sum_{m=n+1}^{\lambda(n)} \Delta_m (\frac{\eta_m}{m}) \sum_{k=1}^{m} kA_k \\ &+ \left[\frac{\eta_{\lambda(n)}}{\lambda(n)} - \Delta (\frac{\eta_{\lambda(n)}}{\lambda(n)}) \right] \sum_{k=1}^{\lambda(n)} kA_k - \frac{\eta_{n+1}}{n+1} \sum_{k=1}^{n} kA_k) \\ &= \frac{1}{P_{\lambda(n)}} (\sum_{m=n+1}^{\lambda(n)} \Delta_m (\frac{\eta_m}{m}) \sum_{k=1}^{m} kA_k - \frac{\eta_{n+1}}{n+1} \sum_{k=1}^{n} kA_k) \end{aligned}$$

and hence

$$\|I\|_p \leq \frac{1}{P_{\lambda(n)}} \sum_{m=n+1}^{\lambda(n)} \left| \Delta_m (\frac{\eta_m}{m}) \right| \left\| \sum_{k=1}^{m} kA_k \right\|_p + \frac{\eta_{n+1}}{P_{\lambda(n)}} \frac{1}{n+1} \left\| \sum_{k=1}^{n} kA_k \right\|_p \tag{4.3}$$

Since

$$s_n(f,x) - \sigma_n(f,x) = \frac{1}{n+1} \sum_{k=1}^{n} kA_k$$

by Lemma 3.8 we get

$$(n+1) \left\| s_n(f) - \sigma_n(f) \right\|_p = \left\| \sum_{k=1}^{n} kA_k \right\|_p = O(1) \tag{4.4}$$

So, combining (4.2),(4.3) and (4.4) we have

$$\begin{aligned} \left\| s_n(f) - N_n^{\lambda}(f) \right\|_p &= O(\frac{1}{P_{\lambda(n)}}) \sum_{m=1}^{n} \left| \Delta_m (\frac{U_m^{\lambda(n)}}{m}) \right| + \frac{U_{n+1}^{\lambda(n)}}{P_{\lambda(n)}} O(n^{-1}) \\ &+ O(\frac{1}{P_{\lambda(n)}}) \sum_{m=n+1}^{\lambda(n)} \left| \Delta_m (\frac{\eta_m}{m}) \right| + O(n^{-1}) \end{aligned} \tag{4.5}$$

If we consider $\Delta_m(\frac{\eta_m}{m}) = \Delta_m(\frac{-U_m^{\lambda(n)}}{m}) + P_{\lambda(n)}\Delta_m(\frac{1}{m})$ then

$$\begin{aligned}\sum_{m=n+1}^{\lambda(n)} \left|\Delta_m(\frac{\eta_m}{m})\right| &= \sum_{m=n+1}^{\lambda(n)} \left|\Delta_m(\frac{U_m^{\lambda(n)}}{m})\right| + P_{\lambda(n)} \sum_{m=n+1}^{\lambda(n)} \Delta_m(\frac{1}{m}) \\ &= \sum_{m=n+1}^{\lambda(n)} \left|\Delta_m(\frac{U_m^{\lambda(n)}}{m})\right| + O(\frac{P_{\lambda(n)}}{n}) \end{aligned} \tag{4.6}$$

Due to (3.8) of Lemma 3.4, we know that

$$\sum_{m=n+1}^{\lambda(n)} \left|\Delta_m(\frac{U_m^{\lambda(n)}}{m})\right| = O(1) \sum_{k=0}^{\lambda(n)-1} |\Delta p_k| \tag{4.7}$$

and

$$\sum_{m=1}^{n} \left|\Delta_m(\frac{U_m^{\lambda(n)}}{m})\right| = O(1) \sum_{k=0}^{\lambda(n)-1} |\Delta p_k|. \tag{4.8}$$

Taking into account (4.5)-(4.8) and the condition (i) of Theorem 2.3 we obtain

$$\left\|s_n(f) - N_n^{\lambda}(f)\right\|_p = O(n^{-1}). \tag{4.9}$$

Therefore, by using (4.9) and Lemma 3.6 we get (2.2) for the case (i).

Similarly, we prove the case (ii). Namely, by considering Lemma 3.6, (3.9) of Lemma 3.4, (4.5) and (4.6) we obtain (2.2) under the condition (ii) of Theorem 2.3. Accordingly, the proof of Theorem 2.3 is completed. □

Proof of Theorem 2.6. By using (4.1) and Abel's transformation, we have

$$\begin{aligned} N_n^{\lambda}(f,x) - f(x) &= \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \left\{s_m - f(x)\right\} \\ &= \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (m+1)\Delta_m(p_{\lambda(n)-m}) \frac{1}{m+1} \sum_{k=0}^{m} \left\{s_k - f(x)\right\} \\ &= \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (m+1)\Delta_m(p_{\lambda(n)-m}) \left(\sigma_m(f,x) - f(x)\right). \end{aligned}$$

If the norm $\|\cdot\|_1$ of each side is taken and by using the result of Quade [8] for $p = 1$ and $0 < \alpha < 1$, Lemma 3.5, we get

$$\begin{aligned} \left\|N_n^{\lambda}(f) - f\right\|_1 &\le \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (m+1) \left|\Delta_m(p_{\lambda(n)-m})\right| \left\|\sigma_m(f) - f\right\|_1 \\ &\le \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (m+1) m^{-\alpha} \left|\Delta_m(p_{\lambda(n)-m})\right| \\ &= O\left(\frac{(\lambda(n)+1)^{1-\alpha}}{P_{\lambda(n)}}\right) \sum_{m=0}^{\lambda(n)} \left|\Delta_m(p_{\lambda(n)-m})\right| \\ &= O\left(\frac{(\lambda(n))^{1-\alpha}}{P_{\lambda(n)}}\right) \sum_{m=-1}^{\lambda(n)-1} |\Delta p_m| = O((\lambda(n))^{-\alpha}) \end{aligned}$$

by suitability $p_{-1} = 0$ under the condition of theorem. □

Proof of Theorem 2.8. Since

$$f(x) - R_n^{\lambda}(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m(f(x) - s_m(f,x)), \tag{4.10}$$

therefore, in view of Abel's transformation, we obtain

$$\sum_{m=0}^{\lambda(n)} p_m(f(x)-s_m(f,x)) = \sum_{m=0}^{\lambda(n)-1}(m+1)(f(x)-\sigma_m(f;x))\Delta p_m \sum_{k=0}^{m}(f(x)-s_m(f,x))$$

$$+(\lambda(n)+1)p_{\lambda(n)}(f(x)-\sigma_m^{\lambda}(f;x)).$$

After continuing with the norm from here, by considering (2.4) and Lemma 3.5, we have

$$\|f-R_n^{\lambda}(f)\|_1 \leq \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)-1}(m+1)|\Delta(p_m)|\|f-\sigma_m(f)\|_1$$

$$+\frac{(1+\lambda(n))p_{\lambda(n)}}{P_{\lambda(n)}}\|f-\sigma_n^{\lambda}(f)\|_1 = \frac{1}{P_{\lambda(n)}}|\Delta(p_0)|\|f-\sigma_0(f)\|_1$$

$$+\frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)-1}(m+1)|\Delta(p_m)|\|f-\sigma_m(f)\|_1+\frac{(1+\lambda(n))p_{\lambda(n)}}{P_{\lambda(n)}}\|f-\sigma_n^{\lambda}(f)\|_1$$

$$= O(\frac{(\lambda(n))^{1-\alpha}}{P_{\lambda(n)}})\sum_{m=0}^{\lambda(n)-1}|\Delta(p_m)| + O(\lambda(n)^{-\alpha}) = O(\lambda(n)^{-\alpha}).$$

Thus, this yields (2.5). □

Proof of Theorem 2.11. Let $p>1$ and $0<\alpha<1$. We have

$$f(x)-R_n^{\lambda}(f;x) = \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} p_m(f(x)-s_m(f,x)).$$

Accordingly, we get the expected result from Lemma 3.2 and Lemma 3.6 by combining the conditions (i) and (ii):

$$\|f-R_n^{\lambda}(f)\|_p \leq \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} p_m\|f-s_m(f)\|_p$$

$$= \frac{O(1)}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} p_m(m+1)^{-\alpha} = O(\lambda(n)^{-\alpha}).$$

□

Proof of Theorem 2.12. If $t_n(x)$ is a trigonometric polynomials we have $s_m(t_n,x) = t_m(x)$ when $m \leq n$. Hence,

$$s_m(f,x)-t_m(x) = s_m(f,x)-s_m(t_n,x) = s_m(f-t_n,x).$$

From integral representation of partial sum of Fourier series we have

$$s_m(f-t_n,x) = \frac{1}{\pi}\int_0^{2\pi}\{f(x+u)-t_n(x+u)\}D_m(u)du$$

and

$$R_n^{\lambda}(f,x) - \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} p_m t_m(x) = \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} p_m s_m(f-t_n,x)$$

$$= \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} p_m \frac{1}{\pi}\int_0^{2\pi}\{f(x+u)-t_n(x+u)\}D_m(u)du$$

$$= \frac{1}{\pi}\int_0^{2\pi}\{f(x+u)-t_n(x+u)\}K_n^{\lambda}(u)du \tag{4.11}$$

where $D_m(u)$ is Dirichlet kernel and

$$K_n^{\lambda}(u) := \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m D_m(u).$$

Let us take $\|.\|_1$ norm of (4.11) and by general form of Minkowskii inequality we have

$$\left\| R_n^{\lambda}(f) - \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m t_m \right\|_1 = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{\pi} \int_0^{2\pi} \{ f(x+u) - t_n(x+u) \} K_n^{\lambda}(u) du \right| dx$$

$$\leq \left(\frac{1}{\pi} \int_0^{2\pi} \left| K_n^{\lambda}(u) \right| du \right) \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x) - t_n(x)| dx \right) = \frac{1}{\pi} \|f - t_n\|_1 \int_0^{2\pi} \left| K_n^{\lambda}(u) \right| du$$

$$= \frac{2}{\pi} \|f - t_n\|_1 \left(\int_0^{\pi/\lambda(n)} \left| K_n^{\lambda}(u) \right| du + \int_{\pi/\lambda(n)}^{\pi} \left| K_n^{\lambda}(u) \right| du \right) =: \frac{2}{\pi} \|f - t_n\|_1 (I_1 + J_1).$$

After this, by considering the method in [6, p. 13], we will complete the proof. Let's estimate I_1. For $0 < t \leq \pi/\lambda(n) \leq \pi/n$, from Jordan's inequality, $(sin(t/2))^{-1} \leq \pi/t$, and $sin(n+1)t \leq (n+1)t$, we have

$$I_1 = \int_0^{\pi/\lambda(n)} \left| K_n^{\lambda}(u) \right| du = \frac{O(1)}{P_{\lambda(n)}} \int_0^{\pi/\lambda(n)} \left| \sum_{m=0}^{\lambda(n)} p_m (m+1) \right| du = O(1). \tag{4.12}$$

We know that if $((n+1)^{-\eta} p_n) \in HBVS$, then $((n+1)^{-\eta} p_n) \in AMIS$ and hence $(p_n) \in AMIS$. Therefore, by using again Jordan inequality and taking into account (2.1), we get

$$J_1 = \int_{\pi/\lambda(n)}^{\pi} \left| K_n^{\lambda}(u) \right| du = \frac{O(1)}{P_{\lambda(n)}} \int_{\pi/\lambda(n)}^{\pi} \frac{1}{u} \left| \sum_{m=0}^{\lambda(n)} p_m \sin(m + \frac{1}{2}) u \right| du$$

$$= \frac{O(1)}{P_{\lambda(n)}} \int_{\pi/\lambda(n)}^{\pi} \frac{1}{u} p_{\lambda(n)} u^{-1} du = O(1). \tag{4.13}$$

On collecting the above results (4.12) and (4.13) we thus get

$$\left\| R_n^{\lambda}(f) - \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m t_m \right\|_1 = O(1) \|f - t_n\|_1 . \tag{4.14}$$

By using (4.14) and Lemma 3.7 in the case $p = \alpha = 1$ we have

$$\left\| f - R_n^{\lambda}(f) \right\|_1 = O(n^{-1}) + \left\| f - \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m t_m \right\|_1 . \tag{4.15}$$

In view of Lemma 3.7, $(p_n) \in AMIS$ and (2.1), we see that the norm on the right of (4.15) is

$$\left\| f - \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m t_m \right\|_1 \leq \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m \|f - t_m\|_1$$

$$= O(\frac{1}{P_{\lambda(n)}}) \sum_{m=1}^{\lambda(n)} p_m \|f - t_m\|_1 + O(\frac{p_0}{P_{\lambda(n)}}) \|f - t_0\|_1$$

$$= O(\frac{1}{P_{\lambda(n)}}) \sum_{m=0}^{\lambda(n)} (m+1)^{-1} p_m + O(n^{-1}).$$

By applying Abel's transformation on the last sum, we observe that

$$\left\|f-R_n^{\lambda}(f)\right\|_1=O(n^{-1})+\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)-1}\left|\Delta_m(\frac{p_m}{(m+1)^{\eta}})\right|\sum_{k=0}^{m}(k+1)^{\eta-1}$$

$$+\frac{p_{\lambda(n)}}{P_{\lambda(n)}(\lambda(n)+1)^{\eta}}\sum_{m=0}^{\lambda(n)}(m+1)^{\eta-1}$$

$$\leq\frac{(\lambda(n)+1)^{\eta}}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)-1}\left|\Delta_m(\frac{p_m}{(m+1)^{\eta}})\right|+\frac{p_{\lambda(n)}}{P_{\lambda(n)}}+O(n^{-1}). \tag{4.16}$$

Since $((n+1)^{-\eta}p_n)\in HBVS$, it follows that

$$\sum_{m=0}^{\lambda(n)-1}\left|\Delta_m(\frac{p_m}{(m+1)^{\eta}})\right|=O\left((\lambda(n)+1)^{-\eta}p_{\lambda(n)}\right)$$

and owing to (2.1), from (4.16), we get (2.6). Therefore, Theorem 2.12 is proved. □

The proofs of Corollary 2.5 and Corollary 2.10 are clear from Theorem 2.3 and Theorem 2.8, respectively.

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Received: Match 30, 2013.

Accepted: October 13, 2013.