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SOLUTION OF INTEGRAL EQUATIONS AND LAPLACE - STIELTJES TRANSFORM

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Abstract We have obtained solutions of integral equations such as Volterra convolution type of first and second type and Abel integral equation using Laplace - Stieltjes transform and applications are mentioned through examples. The convolution property for Laplace - Stieltjes transform is obtained. The solution obtained is considered in distributional sense.

1 Introduction

Integral transforms, viz. Fourier, Laplace, Hankel, Sumudu, Elzaki, Aboodh, and the wavelet transform among other, have been used [1, 3, 4, 5, 6, 7, 8] to obtain solutions of various types of differential and integral equations. The Laplace transform can be studied and researched from years ago [1, 9] In this paper, Laplace - Stieltjes transform is employed in evaluating solutions of certain integral equations that is aided by the convolution. It is remarked that the solution of integral equations obtained by using Laplace - Stieltjes transform, can also be defined on distribution spaces.

Let $F(t)$ is a well defined function of t, for $t \geq 0$ and s be a complex number. The Stieltjes integral is

$$(L_s F)(s) = F^*(s) = \int_0^\infty e^{-st} dF(t), \tag{1.1}$$

which converges on some $s_{0,}$ the Stieltjes integral (1.1) converges on s such that $Re(s) > Re(s_0)$. The integral (1.1) is called Laplace - Stieltjes transform of $F(t)$. Its inversion formula is given by

$$F(t) = \lim_{c\to\infty} \int_{b-ic}^{b+ic} \frac{e^{st}}{s} F^*(s) ds. \tag{1.2}$$

where i is imaginary unit, $b > \max(\sigma, 0)$ and σ is the radius of convergence.

The inversion theorem is given by Saltz [11] and further an improved form of it is proved by Ditzian and Jakimovzvi [2]. Laplace - Stieltjes transform is defined for generalized functions by [3] . Growth for analytic function of Laplace - Stieltjes transform and some other properties are proved by [13, 14]. Convolution theorem and the Parseval equation is defined by [4].

1. Laplace - Stieltjes transform of derivative is defined by

$$(L_s(F'(t)) = s[F^*(s) - F(0)] \tag{1.3}$$

2. When $F(t) = t^n$,the same yields

$$F^*[t^n] = \frac{n!}{s^n} = \frac{\Gamma(n+1)}{s^n}. \tag{1.4}$$

3. If α, β are any constants and $F(t)$ and $G(t)$ are functions of t, then the linear property for Laplace - Stieltjes transform is defined by

$$F^*[\alpha f(t) + \beta g(t)] = \alpha F^*(s) + \beta G^*(s) \quad . \tag{1.5}$$

4. The relation between the Laplace - Stieltjes transform and Laplace transform is given by

$$F^*(s) = \int_0^\infty e^{-st} dF(t) = s \int_0^\infty e^{-st} F(t) \tag{1.6}$$

Theorem 1.1: *Let $F(t)$ and $G(t)$ are Laplace - Stieltjes transform of $F^*(s)$ and $G^*(s)$, respectively. Then the convolution is*

$$L_S[(F * G)] = \frac{1}{s} F^*(s) G^*(s)$$

Proof . The convolution of two function $F(t)$ and $G(t)$ is

$$F(t) * G(t) = \int_0^\infty F(t-\tau) G(t) d\tau$$

Using Laplace - Stieltjes transform (1.6), we get

$$\begin{aligned} L_s[F(t) * G(t)] &= L_s \left[\int_0^\infty F(t-\tau) G(\tau) d\tau \right] \\ &= s \int_0^\infty \int_0^\infty e^{-st} F(t-\tau) G(\tau) d\tau dt \\ &= s \int_0^\infty G(t) d\tau \int_0^\infty e^{-st} F(t-\tau) dt \end{aligned}$$

Now setting $t - \tau = \xi$,we have

$$\begin{aligned} L_s[F(t) * G(t)] &= s \int_0^\infty G(\tau) d\tau \int_{-\tau}^\infty e^{-s(\tau+\xi)} F(\xi) d\xi \\ &= s \int_0^\infty e^{-s\tau} G(\tau) d\tau \int_0^\infty e^{-s\xi} F(\xi) d\xi \\ &= \frac{1}{s} \left[s^2 \int_0^\infty e^{-s\tau} G(\tau) d\tau \int_0^\infty e^{-s\xi} F(\xi) d\xi \right] \end{aligned}$$

i.e.

$$L_s[F(t) * G(t)] = \frac{1}{s} F^*(s) G^*(s) \tag{1.7}$$

This proves the theorem of convolution.

2 Solution of Integral Equations by Laplace - Stieltjes Transform

Solution of different types of integral equations are given by using different types of integral transforms [1, 6, 7, 8]. In this section we use Laplace - Stieltjes to obtain solution of certain integral equation.

1. Consider the Volterra integral equation of first kind with a convolution type kernel

$$f(x) = \int_0^x k(s-t)g(t)dt \qquad , \tag{2.1}$$

where $k(s-t)$ depends only on the difference $(x-t)$. Invoking the Laplace - Stieltjes transform (1.1) and convolution (1.7), the expression (2.1) yields

$$F^*(s) = \frac{1}{s}K^*(s)G^*(s)$$

i.e.

$$G^*(s) = \frac{sF^*(s)}{K^*(s)}. \tag{2.2}$$

Taking inverse Laplace - Stieltjes transform, we obtain

$$g(x) = G^{-1}\left[\frac{sF^*(s)}{K^*(s)}\right], \tag{2.3}$$

as the solution to (2.1).

2. Consider the Volterra integral equation of second kind with a convolution type kernel

$$g(x) = f(x) + \int_0^x k(s-t)g(t)dt \qquad . \tag{2.4}$$

On applying the Laplace - Stieltjes transform (1.1) to both the sides, and using convolution formula (1.7), equation (2.4) gives

$$G^*(s) = F^*(s) + \frac{1}{s}K^*(s)G^*(s)$$

i.e.

$$G^*(s)[s - K^*(s)] = sF^*(s)$$

i.e.

$$G^*(s) = \frac{sF^*(s)}{[s - K^*(s)]}$$

and the inverse Laplace - Stieltjes transform, when invoked, gives

$$g(x) = G^{-1}\left[\frac{sF^*(s)}{[s - K^*(s)]}\right], \tag{2.5}$$

which is the required solution of (2.4).

Similarly, considering Fredhlom integral equation of first and second kind of convolution type and using the Laplace - Stieltjes transform and its convolution, under similar analysis, solutions for these can be obtained.

3. Now, let we consider the Abel integral equation [5, 7]

$$f(t) = \int_0^x \frac{g(x)}{(t-x)^\alpha}dx \;, 0 < \alpha < 1 \tag{2.6}$$

i.e.

$$f = g * t_+^{-\alpha} \qquad , \tag{2.7}$$

where $t_+^{-\alpha} = t^{-\alpha}H(t)$, and $t_+^{-\alpha}$ is Heaviside unit step function . When $F^*[t^{-\alpha}]$ is known and the convolution of the Laplace - Stieltjes transform is employed, the solution of (2.6) will be as follows

$$F^*[f(t)] = \frac{1}{s}F^*[g(t)]\; F^*[t_+^{-\alpha}] \qquad .$$

When $f(t) = t^n$, the Laplace - Stieltjes transform is $F^*(s) = \frac{\Gamma(n+1)}{s^n}$. Similarly, we prove that if $f(t) = t^{-\alpha}$, then the Laplace - Stieltjes transform is $F^*(s) = \Gamma(1-\alpha)s^{\alpha}$, where $H(t) = 1, t \geq 0$. Putting the value of $F^*[t^{-\alpha}]$ in above equation, we have

$$F^*(s) = \frac{1}{s} F^*[g(t)] \cdot \Gamma(1-\alpha)s^{\alpha} \quad . \tag{2.8}$$

$$F^*[g(t)] = \frac{F^*(s)}{\Gamma(1-\alpha)} \cdot \frac{1}{s^{\alpha-1}} \quad .$$

i.e.

$$= \frac{F^*(s)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} \cdot \frac{1}{s^{\alpha-1}} \ ; \quad \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$$

$$F^*[g(t)] = \frac{\sin \pi\alpha}{\pi} [\Gamma(\alpha) \frac{F^*(s)}{s^{\alpha-1}}]$$

$$= \frac{\sin \pi\alpha}{\pi} \cdot s \ \ [t^{\alpha-1} * F^*(s)] \quad ; \quad F^*[t^{n-1}] = \frac{\Gamma(n)}{s^{n-1}}$$

$$F^*[g(t)] = \frac{\sin \pi\alpha}{\pi} \cdot s \cdot F^* \left[\int_0^t (t-x)^{\alpha-1} f(x) dx \right] \tag{2.9}$$

i.e.

$$F^*[g(t)] = \frac{\sin \pi\alpha}{\pi} \cdot s \cdot F^*[G(t)] \quad , \tag{2.10}$$

where $G(t) = \int_0^t (t-x)^{\alpha-1} f(x) dx, G(0) = 0$.

By virtue of (1.3), $F^*[G'(t)] = sG^*(s) - sG(0) = sG^*(s)$. Invoking it in (2.10), we have

$$F^*[g(t)] = \frac{\sin \pi\alpha}{\pi} \cdot s [\frac{F^*[G'(t)}{s}].$$

Therefore, the complete solution of (2.10) is obtained as

$$g(t) = \frac{\sin(\pi\alpha)}{\pi} \frac{d}{dt} \left[\int_0^t (t-x)^{\alpha-1} f(x) dx \right] \quad . \tag{2.11}$$

.

We need to specify the space (or spaces) of generalized function in order to define integral equations on distribution spaces. Then, we need to give an interpretation of the equation in terms of an operator defined in that space of distributions. This interpretation should be such that when applied to ordinary functions, integral equation can be recovered. One is the space $D'_{41}[a, \infty)$, which is known as mixed distribution space [cf. [5]] that can be identified with the space of distribution $D'(R)$ whose support is $[a, \infty)$. Another distribution space is $D'_{43}[a, \infty)$, which can be identified with the space $S'(R)$ (tempered distribution space) whose support is contained in $[a, \infty)$.

The interpretation of integral equation can be achieved by using the concept of convolution of distributions. If both u and v have supports bounded on the left, then $u * v$ is always defined. Actually, if supp $u \subseteq [a, \infty)$ and supp $v \subseteq [b, \infty)$, then supp $u * v \subseteq [a + b, \infty)$.Thus, the convolution can be considered as a bilinear operation $* : D'_{41}[a, \infty) \times D'_{41}[b, \infty) \to D'_{41}[a+b, \infty)$. If $u \in D'_{41}[a, \infty)$ and $v \in D'_{41}[b, \infty)$ are locally integrable functions, then we have

$$(u * v)(t) = \int_a^{t-b} u(\tau) v(t-\tau) d\tau \quad , \quad t > a + b \quad . \tag{2.12}$$

When $b = 0$ and $v \in D'_{41}[0, \infty)$,we have $u * v \in D'_{41}[a, \infty)$. Thus the convolution, with v, defines an operator of the space $D'_{41}[a, \infty)$, which is given by

$$(u * v)(t) = \int_a^t u(\tau)v(t-\tau)d\tau \quad , \quad t > a \quad , \tag{2.13}$$

where u and v are locally integrable functions. The integral equation and its solution can be interpreted in the distributional sense.

To define the integral equations on distribution spaces, we need to specify the spaces we intend to use. Such are given in terms of distribution spaces [4] and mixed distribution spaces [5]. Moreover, the solutions of integral equations (viz. Volterra integral equations (2.1) and (2.4), Abel integral equation (2.6)) obtained above by Laplace - Stieltjes transform, can be interpreted in distributional sense, either by considering integral equations on distributions spaces where $f(t), g(t)$ and $t^{-\alpha}$ are locally integrable or defining the Laplace - Stieltjes transform on distribution spaces.

3 Applications of Laplace - Stieltjes Transform

In this section we give certain exclusive examples to illustrate the use of the Laplace - Stieltjes transform in solving certain integral equations.

Example 1 : Find the function $g(x)$ which satisfies the equation

$$g(x) = x + \int_0^x g(t) \sin(x-t)dt \tag{3.1}$$

Solution : Invoking the Laplace - Stieltjes transform on both the sides of (3.1) and using (1.7), we obtain

$$G^*(s) = \frac{1}{s} + G^*(s)\,\frac{1}{s}\cdot\frac{s}{(s^2+1)}$$

i.e.

$$G^*(s) = \frac{s^2+1}{s^3}$$

i.e.

$$G^*(s) = \frac{1}{s} + \frac{1}{s^3}.$$

Taking inverse Laplace - Stieltjes transform, we get

$$g(x) = x + \frac{x^3}{3!} \quad , \tag{3.2}$$

which is the required solution.

Example 2 : Solve the integral equation

$$y(x) = x + \frac{1}{6}\int_0^x (x-t)^3 y(t)dt \tag{3.3}$$

Solution : Applying the Laplace - Stieltjes transform on both sides of (3.3) , we have

$$F^*(s) = \frac{1}{s} + \frac{1}{6}\frac{1}{s}\left[\frac{3!}{s^3}\right] F^*(s) \ .$$

$$F^*(s) = \left(\frac{s^3}{s^4-1}\right)$$

i.e.

$$F^*(s) = \frac{1}{2}\left[\frac{s}{(s^2+1)} + \frac{s}{(s^2-1)}\right] \ .$$

Taking inverse Laplace - Stieltjes transform of both the sides, we have

$$y(t) = \frac{1}{2}(\sin x + \sinh x)$$

This is the required solution.

Example 3 : Solve the integral equation

$$x = \int_0^x e^{x-t} f(t) dt \tag{3.4}$$

Solution : Equation (3.4) can be written as

$$x = f(x) * e^x \tag{3.5}$$

Taking Laplace - Stieltjes transform on both sides of (3.5) , we have

$$F^*[x] = F^*[f(x)] * F^*[e^x] \ .$$

Using convolution of Laplace - Stieltjes transform (1.7) and values of functions as given, it yields

$$\begin{aligned} \frac{1}{s} &= \frac{1}{s} F^*[f(x)] \cdot \frac{s}{(s-1)} \\ F^*(s) &= 1 - \frac{1}{s} \end{aligned}$$

Taking inverse Laplace - Stieltjes transform, we have

$$f(x) = 1 - x \ . \tag{3.6}$$

Conclusion : Laplace - Stieltjes is introduced in this paper, where convolution property for the transform is proved and used to solve certain types of integral equations and suggested interpretation of integral equations on distribution spaces, which specifies that the solution obtained using Laplace - Stieltjes transform for integral equations can be defined on the distribution spaces. Few examples are given for solving integral equations using the Laplace - Stieltjes transform.

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