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Boundary Stabilization of Coupled Plate Equations

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Abstract. In this work, we study the indirect stabilization of a system of plate equations which are weakly coupled and boundary partially damped. One checks that exponential stability does not hold. Then, using an approach based on the growth of the resolvent on the imaginary axis, we show that the energy of smooth solutions of this system decays polynomially at infinity.

1 Introduction

In this paper we study the boundary stabilization of a coupled plates by means of a feedback acting on a part of the boundary. Let us first describe the open-loop control problem. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set of $\mathbb{R}^n$ representing the domain occupied by the plates. We denote by Γ the boundary of Ω and we assume that it is a smooth boundary of class C^4 such that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

With the above notation, we consider the following weakly coupled and partially damped plate equations:

$$\begin{cases} u_{tt} + \Delta^2 u + \alpha y = 0, \;\; \Omega \times (0, +\infty) \\ y_{tt} + \Delta^2 y + \alpha u = 0, \;\; \Omega \times (0, +\infty) \\ u = \partial_\nu u = 0, \quad y = \partial_\nu y = 0, \qquad \Gamma_0 \times (0, +\infty) \\ \Delta u = 0, \quad \partial_\nu(\Delta u) = \gamma u + u_t, \qquad y = \Delta y = 0, \qquad \Gamma_1 \times (0, +\infty) \\ u(x,0) = u^0, \; u_t(x,0) = u^1, \; y(x,0) = y^0, \; y_t(x,0) = y^1, \; \Omega. \end{cases} \tag{1.1}$$

Where $\gamma > 0$ and $\alpha \neq 0$ is a small constant, and ν is the unit-normal vector to Γ pointing toward the exterior of Ω and ∂_ν denotes the normal derivative. The damping u_t is only applied at Γ_1 part of Γ in the first equation. The second equation is indirectly damped through the coupling between the two equations.

The stabilization problem for coupled systems has been studied by several authors in recent years [1, 2, 4, 6, 7, 8, 9, 17].
The coupled wave-wave and wave-plate are studied in [2, 4, 8, 17] where the authors studied the polynomial decay rate of energy.

In [5], the author, have establish the polynomial energy estimate for the coupled plate-plate system with bounded feedback. The purpose of the present work is to study of the decay rate of the same system-type for the case of unbounded feedback. Therefore, the damping applied at the boundary of the first equation can be effectively transmitted through the coupling terms $y,\ u$ to the second equation.
The polynomial energy decay rate occurs in many control problems where the systems are strongly stable but not exponentially stable. To obtain this decay rate, several method exist in the literature.
An energy inequality was established in [19] as sufficient condition for polynomial decay rate $1/t$. A Riesz basis method was used in [15] which gives the polynomial decay rate based on the asymptotic relation of the real and imaginary part of the eigenvalues.
The frequency domain method (of interest in this paper) given, see [16], from the following growth of the resolvent of the infinitesimal on the imaginary axis

$$\sup_{|\beta|\geq 1} \frac{1}{\beta^l} \left\|(i\beta - \mathcal{A})^{-1}\right\| < +\infty, \quad \text{for some } l,$$

an energy estimate with the rate $\left(\dfrac{\ln t}{t}\right)^{\frac{1}{l}} \ln t$.
This rate is not optimal. In a recent work [10], the authors have to give an optimal energy estimate under the same assumptions. Which they eliminated the ln term in the decay rate. For this paper we will use this last result.

We define the energy of a solution (u, y) of (1.1) at time t as

$$E(t) = \frac{\gamma}{2}\int_{\Gamma_1} |u|^2 dx + \frac{1}{2}\int_{\Omega} \left\{|\Delta u|^2 + |u_t|^2 + |\Delta y|^2 + |y_t|^2 + 2\alpha\Re(u\overline{y})\right\}(x,t)dx. \tag{1.2}$$

By the integration by part's formula and using the boundary condition, we can easily check that every sufficiently smooth solution of (1.1) satisfies the energy identity

$$\frac{dE}{dt} = -\int_{\Gamma_1} |u_t(t,x)|^2 dx,$$

which in particular implies

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_{\Gamma_1} |u_t(t,x)|^2 dx\, dt, \tag{1.3}$$

for all $t_2 > t_1 \geq 0$. Therefore, the energy is a nonincreasing function of the time variable t and our system (1.1) is dissipative.

Introduce the Hilbert spaces over the field $\mathbb{C}$ of complex numbers

$$V_1 = \left\{u \in H^2(\Omega),\ \ u = \partial_\nu u = 0,\ \Gamma_0\right\},$$

$$V_2 = \left\{y \in H^2(\Omega),\ \ y = \partial_\nu y = 0,\ \Gamma_0 \text{ and } y = 0,\ \Gamma_1\right\}.$$

We define the energy space as following

$$\mathcal{H} = V_1 \times L^2(\Omega) \times V_2 \times L^2(\Omega).$$

For all $U_1 = (u_1, v_1, y_1, z_1) \in \mathcal{H}$ and $U_2 = (u_2, v_2, y_2, z_2) \in \mathcal{H}$, the inner product in $\mathcal{H}$ is defined by

$$< U_1, U_2 >= \gamma \int_{\Gamma_1} u_1 \overline{u}_2 \, d\Gamma + \int_\Omega \left[\Delta u_1 \Delta \overline{u}_2 + v_1 \overline{v}_2 + \Delta y_1 \Delta \overline{y}_2 + z_1 \overline{z}_2 + \alpha(u_1 \overline{y}_2 + y_1 \overline{u}_2)\right] dx.$$

Now we define a linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U = (u, y, v, z) \in \mathcal{H};\ u,\ y \in H^4(\Omega), \quad v,\ z \in H^2(\Omega) \\ u = v = \partial_\nu u = \partial_\nu v = 0,\ \ \Gamma_0 \\ \Delta u = 0, \quad \partial_\nu(\Delta u) = \gamma u + v,\ \ \Gamma_1 \\ y = z = \partial_\nu y = \partial_\nu z = 0,\ \ \Gamma_0 \\ y = z = \Delta y = 0,\ \ \Gamma_1. \end{array} \right\},$$

and

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ -\Delta^2 & 0 & -\alpha I & 0 \\ 0 & 0 & 0 & I \\ -\alpha I & 0 & -\Delta^2 & 0 \end{pmatrix}.$$

Then, setting $u = (u, u_t, y, y_t)$, we rewrite the system (1.1) into an evolution equation

$$\frac{dU}{dt} = \mathcal{A}U, \qquad U(0) = U_0 \in \mathcal{H}.$$

Proposition 1.1. *There is a real $\alpha_0 > 0$, such that for all $\alpha < \alpha_0$, $\mathcal{A}$ is a maximal dissipative operator on the energy space $\mathcal{H}$, for α be a small real number, therefore generates a C_0-semigroup of contractions on $\mathcal{H}$, which is denoted by $(S(t))_{t\geq 0}$.*

Proof. Let $U = (u, v, y, z) \in D(\mathcal{A})$. By an integration by parts and using the boundary conditions, we have

$$\begin{aligned} &(\mathcal{A}U, U) \\ = \ & \gamma \int_{\Gamma_1} v\overline{u} \, d\Gamma + \int_\Omega \left(\Delta v \Delta \overline{u} - \Delta^2 u \overline{v} - \alpha y \overline{u} + \Delta z \Delta \overline{y} - \Delta^2 y \overline{z} - \alpha u \overline{z} + \alpha(v\overline{y} + z\overline{u})\right) dx \\ = \ & \gamma \int_{\Gamma_1} v\overline{u} \, d\Gamma - \int_{\Gamma_1} \partial_\nu(\Delta u)\overline{v} \, d\Gamma \\ + \ & \int_\Omega (\Delta v \Delta \overline{u} - \Delta u \Delta \overline{v} + \Delta z \Delta \overline{y} - \Delta y \Delta \overline{z} - \alpha y \overline{v} - \alpha u \overline{z} + \alpha v \overline{y} + \alpha z \overline{u}) \, dx. \end{aligned}$$

Then, by the dissipation condition, we obtain

$$Re\,(\mathcal{A}U, U) = Re \int_{\Gamma_1} (\gamma v\overline{u} - \partial_\nu(\Delta u)\overline{v}) \, d\Gamma = -\int_{\Gamma_1} |v|^2 \, d\Gamma \leq 0. \tag{1.4}$$

Thus, $\mathcal{A}$ is a dissipative operator on $\mathcal{H}$.
Now, let $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$. We look for an element $U = (u, y, v, z) \in D(\mathcal{A})$ such that

$$(I - \mathcal{A})U = F. \tag{1.5}$$

Equivalently, we consider the following system

$$v = u - f_1, \quad z = y - f_3, \tag{1.6}$$

$$u + \Delta^2 u + \alpha y = f_1 + f_2, \tag{1.7}$$

$$y + \Delta^2 y + \alpha u = f_3 + f_4, \tag{1.8}$$

$$u = \partial_\nu u = 0 \quad \text{on } \Gamma_0, \tag{1.9}$$

$$\Delta u = 0, \quad \partial_\nu(\Delta u) = (\gamma + 1)u - f_1 \quad \text{on } \Gamma_1, \tag{1.10}$$

$$y = \partial_\nu y = 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad y = \Delta y = 0 \quad \text{on} \quad \Gamma_1. \tag{1.11}$$

Let $\varphi \in V_1$ and $\psi \in V_2$. Multiplying (1.7) by $\overline{\varphi}$ and (1.8) by $\overline{\psi}$, we get the following variational equation

$$\int_\Omega \left(u\overline{\varphi} + \Delta u \Delta\overline{\varphi} + y\overline{\psi} + \Delta y \Delta\overline{\psi} + \alpha y\overline{\varphi} + \alpha u\overline{\psi}\right) dx + (\gamma + 1)\int_{\Gamma_1} u\overline{\varphi}\, d\Gamma =$$

$$\int_\Omega \left((f_1 + f_2)\overline{\varphi} + (f_3 + f_4)\overline{\psi}\right) dx + \int_{\Gamma_1} f_1\overline{\varphi}\, d\Gamma. \tag{1.12}$$

It is easy to check that the left-hand side of (1.12) is a continuous and coercive bilinear form on the space $(V_1 \times V_2) \times (V_1 \times V_2)$ for $\alpha < \alpha_0$, and the right-hand side is a continuous linear form on the space $(V_1 \times V_2)$. Then thanks to Lax-Milgram Lemma, the variational equation (1.12) admits a unique solution $(u, y) \in (V_1 \times V_2)$.
Using some integrations by parts, we easily check that (u, y) satisfies

$$u + \Delta^2 u = f_1 + f_2 - \alpha y \in L^2(\Omega), \tag{1.13}$$

$$y + \Delta^2 y = f_3 + f_4 - \alpha u \in L^2(\Omega). \tag{1.14}$$

Then the classical elliptic theory, implie that the weak solution (u, y) of (1.13)-(1.14) associated with the boundary conditions (1.9)-(1.11) belongs to the space $H^4(\Omega) \times H^4(\Omega)$. Moreover, we have

$$\|u\|^2_{H^4(\Omega)} + \|v\|^2_{H^2(\Omega)} + \|y\|^2_{H^4(\Omega)} + \|z\|^2_{H^2(\Omega)} \leq C\,\|F\|^2_{\mathcal{H}},$$

where $C > 0$ is a positive constant. Therefore, $(u, v, y, z) \in D(\mathcal{A})$ and $(I - \mathcal{A})^{-1}$ is a compact in the energy space $\mathcal{H}$. Finally, thanks to Lumer-Philips Theorem [18, Theorem 1.4.3], we conclude that $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. The proof is thus completed. □

2 Stability results

Proposition 2.1. *The system* (1.1) *is not uniformly exponentially stable in the energy space* $\mathcal{H}$.

Proof. For that, we will use a result of Russell on compact perturbations of semigroup [20](see also [11] , [21] and [3]).

Let $BU = \begin{pmatrix} 0 \\ \alpha y \\ 0 \\ \alpha u \end{pmatrix}$ for each $U \in \mathcal{H}$, and $\mathcal{A}_0$ denotes the operator obtained from $\mathcal{A}$ by setting $\alpha = 0$. Then we have $\mathcal{A}_0 = \mathcal{A} + B$. Let us notice that the operator B is compact then $\mathcal{A}_0$ is a compact perturbation of $\mathcal{A}$. On the other hand the operator $\mathcal{A}_0$ is associated with uncoupled system obtained for $\alpha = 0$ in (1.1).

If we show that the semigroup generated by $\mathcal{A}_0$ is not exponential stable, then we will be able to conclude that the semigroup generated by $\mathcal{A}$ is not exponential stable.
Thus, let b be a nonzero real number. Let $\omega \in H_0^2(\Omega)$ solve the eigenvalue problem $\Delta^2\omega = b^2\omega$.
We consider $U = \begin{pmatrix} 0 \\ 0 \\ \omega \\ ib\omega \end{pmatrix}$, then $\mathcal{A}_0 U = ibU$; so $i\mathbb{R}$ is not subset of $\rho(\mathcal{A}_0)$. Consequently the semigroup generated by $\mathcal{A}_0$ is not exponential stable (in fact it is not even strong stable). Hence invoking Russell's result, one finds that the semigroup associated of our system (1.1) is not exponential stable. □

Since the energy of system (1.1) is no uniform decay rate, we will look for polynomial decay rate for smooth data. For that, we will use the result of Borichev-Tomilov (see [10]). In order to establish the polynomial energy decay rate, let us consider the usual geometrical control condition: there exists a point $x_0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \leq 0 \quad \text{on} \quad \Gamma_0, \qquad\qquad m \cdot \nu > 0 \quad \text{on} \quad \Gamma_1, \tag{2.1}$$

where $m = x - x_0$.

Theorem 2.2. *Let α be a nonzero real number such that $\alpha < \alpha_0$. Then,*

(i) The semigroup $(S(t))_{t\geq 0}$ is strongly stable

$$\lim_{t\to+\infty} \left\|S(t)U^0\right\| = 0, \qquad \forall\, U^0 \in \mathcal{H}.$$

(ii) For every positive integer m, there exists a constant $C_{\alpha,m} > 0$ such that, we have the following the decay estimate

$$\left\|S(t)U^0\right\| \leq \frac{C_{\alpha,m}}{t^{\frac{m}{3}}} \left\|U^0\right\|_{D(\mathcal{A}^m)}, \ \ \forall t > 0, \quad \forall\, U^0 \in D(\mathcal{A}^m).$$

Proof.

(i) The operator $\mathcal{A}$ has a compact resolvent; so the spectrum of $\mathcal{A}$ is discrete.
Firstly, we will show that $0 \in \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ stands for the resolvent set of $\mathcal{A}$. Suppose that 0 is an element of the spectrum of $\mathcal{A}$ and let $U \in D(\mathcal{A})$ be the associated normalized eigenfunction, i.e.,

$$\mathcal{A}U = 0. \tag{2.2}$$

Then, we obtain

$$v = 0, \qquad z = 0, \tag{2.3}$$

$$\Delta^2 u + \alpha y = 0, \tag{2.4}$$

$$\Delta^2 y + \alpha u = 0, \tag{2.5}$$

Multiplying (2.4) by $\overline{u}$ and (2.5) by $\overline{y}$, we get

$$\int_\Omega (|\Delta u|^2 + |\Delta y|^2)\, dx + \gamma \int_{\Gamma_1} |u|^2\, d\Gamma + 2\alpha \int_\Omega y\overline{u}\, dx = 0.$$

For α small enough, this equality gives $u = y = 0$.
Now, if we show that $\mathcal{A}$ has no purely imaginary eigenvalues, then it will follow from a

result in [9] that the semigroup $(S(t))_{t\geq 0}$ is strongly stable.
Let $b \neq 0$ be a real number. Suppose that there exists a unit element U in $D(\mathcal{A})$ such that

$$\mathcal{A}U = ibU. \tag{2.6}$$

We shall show that $U = 0$. Taking the inner product with U on both sides of (2.6), taking the real parts and using (1.4), we immediately find

$$\int_{\Gamma_1} |v|^2 d\Gamma = 0, \tag{2.7}$$

it follows that

$$v = 0 \text{ on } \Gamma_1. \tag{2.8}$$

Now, (2.6) can be written as

$$v = ibu, \quad z = iby, \tag{2.9}$$

$$b^2 u - \Delta^2 u - \alpha y = 0, \tag{2.10}$$

$$b^2 y - \Delta^2 y - \alpha u = 0, \tag{2.11}$$

$$u = \partial_\nu u = 0, \quad y = \partial_\nu y = 0 \quad \text{on} \quad \Gamma_0, \tag{2.12}$$

$$\Delta u = 0, \;\; \partial_\nu(\Delta u) = \gamma u, \quad y = \Delta y = 0 \quad \text{on} \quad \Gamma_1. \tag{2.13}$$

Since (2.8), (2.9), (2.13) and $b \neq 0$ yields

$$u = \partial_\nu(\Delta u) = 0 \text{ on } \Gamma_1. \tag{2.14}$$

Let us define the multiplier $Mu = n\overline{u} + 2m \cdot \nabla\overline{u}$. Multiplying (2.10) by Mu, then

$$nb^2 \int_\Omega |u|^2 \, dx + 2b^2 \int_\Omega u(m \cdot \nabla\overline{u}) \, dx - n \int_\Omega |\Delta u|^2 \, dx$$
$$-2 \int_\Omega \Delta^2 u m \cdot \nabla\overline{u} \, dx - \alpha \int_\Omega y M\overline{u} \, dx = 0.$$

For $u \in H^4(\Omega)$, we have the following Rellich's identity

$$2Re \int_\Omega \Delta^2 u(m \cdot \nabla\overline{u}) \, dx = -\int_\Omega div(m)|\Delta u|^2 dx + 2\int_\Omega \Delta m_k \frac{\partial u}{\partial x_k} \Delta\overline{u} dx$$
$$+4 \int_\Omega \nabla m_k \cdot \nabla \left(\frac{\partial u}{\partial x_k}\right) \Delta\overline{u} dx + \int_\Gamma m \cdot \nu |\Delta u|^2 d\Gamma - 2\int_\Gamma \Delta u \partial_\nu(m \cdot \nabla\overline{u}) d\Gamma$$
$$+2 \int_\Gamma m \cdot \nabla u \partial_\nu(\Delta\overline{u}) d\Gamma.$$

Hence, the boundary conditions (2.12), (2.13) and (2.14) gives

$$2Re \int_\Omega \Delta^2 u(m \cdot \nabla\overline{u}) \, dx = (4-n) \int_\Omega |\Delta u|^2 dx + \int_{\Gamma_0} m \cdot \nu |\Delta u|^2 d\Gamma$$
$$-2 \int_{\Gamma_0} \Delta u \partial_\nu(m \cdot \nabla\overline{u}) d\Gamma.$$

On the other hand, by integration by parts, we get

$$2Re\int_\Omega u(m\cdot\nabla\overline{u})\,dx = -n\int_\Omega |u|^2\,dx.$$

Then, we obtain

$$4\int_\Omega |\Delta u|^2dx = -\alpha\int_\Omega y(n\overline{u}+2m\cdot\nabla\overline{u})dx - \int_{\Gamma_0} m\cdot\nu|\Delta u|^2d\Gamma + 2\int_{\Gamma_0}\Delta u\partial_\nu(m\cdot\nabla\overline{u})d\Gamma. \tag{2.15}$$

On the part of boundary Γ_0, we claim that $\partial_\nu(m\cdot\nabla u) = m\cdot\nu\Delta u$ (see [13] and [12]). For this we remark that $u=\partial_\nu u = 0$ there. Hence, $\partial_i u = 0$ for $1\le i\le n$ on Γ_0, and we have

$$\partial_\nu(m\cdot\nabla u) = \sum_{j=1}^n \partial_j(\sum_{i=1}^n m_i\partial_i u)\nu_j = \sum_{i,j}\partial_j(m_i\partial_i u)\nu_j = \sum_{i,j} m_i\partial_{i,j}u\nu_j.$$

Setting $v=\partial_j u$, and recalling that $\nabla u = 0$ on Γ_0, we have $\nabla v = \partial_\nu v\nu$. Hence, $\partial_i v = \sum_{k=1}^n \partial_k v\nu_k\nu_i$ for all $1\le i\le n$. Coming back to $\partial_j u$, we obtain

$$\partial_{ij}u = \sum_k \partial_{kj}u\nu_k\nu_i.$$

Then, we deduce that

$$\partial_\nu(m\cdot\nabla u) = \sum_{i,j,k} m_i\partial_{kj}u\nu_k\nu_i\nu_j = \sum_i m_i\nu_i\sum_j\left[\sum_k(\partial_{kj}u\nu_k)\nu_j\right].$$

Hence, we obtain

$$\partial_\nu(m\cdot\nabla u) = \sum_i m_i\nu_i\sum_j\partial_{jj}u = m\cdot\nu\Delta u.$$

Then (2.15) becomes

$$4\int_\Omega |\Delta u|^2dx = -\alpha\int_\Omega y(n\overline{u}+2m\cdot\nabla\overline{u})dx + \int_{\Gamma_0} m\cdot\nu|\Delta u|^2d\Gamma. \tag{2.16}$$

Now, if we multiply (2.10) by $\overline{y}$ and (2.11) by $\overline{u}$. Integrating the sum and using the boundary condition, we get $\|u\|_{L^2(\Omega)} = \|y\|_{L^2(\Omega)}$.

Then using the geometrical condition (2.1), Cauchy-Schwartz, Poincare's inequality, and (2.16), we deduce that there exists a positive constant $C>0$, depending only on Ω, such that

$$C\int_\Omega |\Delta u|^2dx \le \alpha\int_\Omega |\Delta u|^2dx.$$

Finally, for α small enough, we obtain $u=0$. Then, from (2.10) we obtain $y=0$ which yields $U=0$.Hence, $i\mathbb{R}\subset\rho(\mathcal{A})$.

(ii) Now, we shall show the claimed decay estimate. This proof will rely on [10, Theorem 2.4] which establishes an equivalent between the polynomial decay estimate $\left\|S(t)\mathcal{A}^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = O(t^{-\frac{1}{l}})$ as t at in infinity, and the resolvent estimate

$$\left\|(ib-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = O(b^l) \text{ for some } l \text{ and as } b \text{ large .} \tag{2.17}$$

We will prove this result for $l = 3$. Assume that (2.17) is false, then there exist a sequence $b_n \longrightarrow +\infty$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ checking $\|U_n\| = 1$ such that

$$b_n^3 \|(ib_n I - \mathcal{A})U_n\| \longrightarrow 0. \tag{2.18}$$

To simplify the notations we will write U and b instead of U_n and b_n. Our goal is to obtain the contradiction $\|U\| \longrightarrow 0$ from (2.18).
We rewrite (2.18) as follows

$$b^3(ibu - v) = f_1 \longrightarrow 0 \text{ in } V_1, \tag{2.19}$$

$$b^3(ibv + \Delta^2 u + \alpha y) = f_2 \longrightarrow 0 \text{ in } L^2(\Omega), \tag{2.20}$$

$$b^3(iby - z) = g_1 \longrightarrow 0 \text{ in } V_2, \tag{2.21}$$

$$b^3(ibz + \Delta^2 y + \alpha u) = g_2 \longrightarrow 0 \text{ in } L^2(\Omega). \tag{2.22}$$

Then, with $\|U\| = 1$, (2.18) gives

$$ib\|U\|^2 - (\mathcal{A}U, U) = \frac{o(1)}{b^3}.$$

Hence, by (1.4) we obtain

$$\int_{\Gamma_1} |v|^2 \, d\Gamma = \frac{o(1)}{b^3}. \tag{2.23}$$

Substituting (2.23) in (2.19), and using that $U \in D(\mathcal{A})$ and the boundary conditions we deduce that

$$\|u\|_{L^2(\Gamma_1)} = \frac{o(1)}{b^{\frac{5}{2}}} \text{ and } \|\partial_\nu(\Delta u)\|_{L^2(\Gamma_1)} = \frac{o(1)}{b^{\frac{3}{2}}}. \tag{2.24}$$

Now, substituting (2.19) into (2.20), and (2.21) into (2.22), respectively, we get

$$b^2 u - \Delta^2 u - \alpha y = f \text{ in } L^2(\Omega), \tag{2.25}$$

$$b^2 y - \Delta^2 y - \alpha u = g \text{ in } L^2(\Omega), \tag{2.26}$$

where

$$\begin{aligned} \|f\|_{L^2(\Omega)} &= \left\| \frac{ibf_1 + f_2}{b^3} \right\|_{L^2(\Omega)} = \frac{o(1)}{b^2} \\ \|g\|_{L^2(\Omega)} &= \left\| \frac{ibg_1 + g_2}{b^3} \right\|_{L^2(\Omega)} = \frac{o(1)}{b^2}. \end{aligned} \tag{2.27}$$

Next, multiplying (2.25) by $\overline{y}$ and (2.26) by $\overline{u}$, and add the resulting equations, we find

$$\alpha \int_\Omega |u|^2 dx = \alpha \int_\Omega |y|^2 dx - \int_{\Gamma_1} \partial_\nu(\Delta y)\overline{u} dx + \int_\Omega (f\overline{y} - g\overline{u}) dx. \tag{2.28}$$

Then, we need to estimate the term $\displaystyle\int_{\Gamma_1} |\partial_\nu(\Delta y)|^2 d\Gamma$.

This is obtained by multiplying the equation (2.26) by $1/b$ which gives that $\frac{1}{b}\Delta^2 y$ is bounded in the $L^2(\Omega)$ space. Then using the continuity of the normal derivative we will

$$\int_{\Gamma_1} |\partial_\nu(\Delta y)|^2 d\Gamma \le Cb. \tag{2.29}$$

Moreover, using (2.21), (2.24), (2.27) and (2.29), the expression (2.28) gives that

$$b^2 \int_\Omega |u|^2 dx = b^2 \int_\Omega |y|^2 dx + o(1). \tag{2.30}$$

Multiplying (2.25) by $\overline{u}$ we obtain that

$$b^2 \int_\Omega |u|^2 dx - \int_\Omega |\Delta u|^2 dx = \int_\Omega (\alpha y + f)\overline{u} dx + \int_{\Gamma_1} \partial_\nu (\Delta u)\overline{u} dx. \tag{2.31}$$

Then, we deduce that

$$b^2 \int_\Omega |u|^2 dx - \int_\Omega |\Delta u|^2 dx = \frac{O(1)}{b^2}. \tag{2.32}$$

In what follows, we apply the standard multiplier technique to the plate equation. Multiplying (2.25) by $2m \cdot \nabla \overline{u}$ leads to

$$2b^2 \int_\Omega u(m \cdot \nabla \overline{u})dx - 2\int_\Omega \Delta^2 u(m \cdot \nabla \overline{u})dx = 2\int_\Omega (\alpha y + f)(m \cdot \nabla \overline{u})dx. \tag{2.33}$$

Using Rellich's identity given above and integrating by parts, we get

$$\begin{aligned} nb^2 \int_\Omega |u|^2 dx + (4-n)\int_\Omega |\Delta u|^2 dx &= -2Re \int_\Omega (\alpha y + f)(m \cdot \nabla \overline{u})dx + b^2 \int_{\Gamma_1} (m \cdot \nu)|u|^2 \, d\Gamma \\ &+ \int_{\Gamma_0} (m \cdot \nu)|\Delta u|^2 \, d\Gamma - 2\int_{\Gamma_1} (m \cdot \nu)\partial_\nu u \partial_\nu (\Delta \overline{u}) \, d\Gamma. \end{aligned} \tag{2.34}$$

In the other hand, using the continuous of the Neumann operator and the Sobolev injections, we can deduce that

$$\left| \int_{\Gamma_1} (m \cdot \nu)\partial_\nu u \partial_\nu (\Delta \overline{u}) \right| = \frac{o(1)}{b^{3/2}}.$$

And, with the geometric condition (2.1) such that, $m \cdot \nu \leq 0$ on Γ_0, we obtain

$$nb^2 \int_\Omega |u|^2 dx + (4-n)\int_\Omega |\Delta u|^2 dx \leq -2Re \int_\Omega (\alpha y + f)(m \cdot \nabla \overline{u})dx + \frac{o(1)}{b^{3/2}}.$$

Multiplying (2.32) by $(3-n)$ and summing the result relation with the above inequality, we get

$$\int_\Omega \left(b^2 |u|^2 + |\Delta u|^2\right) dx \leq -2Re \int_\Omega (\alpha y + f)(m \cdot \nabla \overline{u})dx + \frac{o(1)}{b^{3/2}}.$$

Thus, with (2.19), (2.21) and (2.27), the Young's inequality applied to the term $\int_\Omega (\alpha y + f)(m \cdot \nabla \overline{u})dx$ implies that

$$\int_\Omega \left(b^2 |u|^2 + |\Delta u|^2\right) dx \leq \frac{o(1)}{b^{3/2}}. \tag{2.35}$$

Since (2.30) and (2.35), it follows that $b^2 \int_\Omega |y|^2 \longrightarrow 0$.

And if we multiply (2.26) by $\overline{y}$, we get

$$b^2 \int_\Omega |y|^2 - \int_\Omega |\Delta y|^2 dx = \int_\Omega (\alpha u + g)\overline{y} dx.$$

Therefore $\int_\Omega |\Delta y|^2 dx \longrightarrow 0$. Finally, we obtain

$$\|\Delta u\|_{L^2(\Omega)} + \|bu\|_{L^2(\Omega)} + \|\Delta y\|_{L^2(\Omega)} + \|by\|_{L^2(\Omega)} \longrightarrow 0$$

which contradict the assumption that $\|U\| = 1$.

The proof is thus complete. □

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