Palestine Journal of Mathematics

Vol. 3(Spec 1) (2014) , 388–394

About some properties of algebras obtained by the Cayley-Dickson process

Cristina Flaut

Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays

.

Communicated by Ayman Badawi

MSC 2010 Classifications: 17A35.

Keywords and phrases: Cayley-Dickson process, Fibonacci Quaternions, Multiplication table for algebras obtained by the Cayley-Dickson process.

Abstract. This paper is a short survey about some properties of algebras obtained by the Cayley-Dickson process and some of their applications

1. Introduction

It is well known that in October 1843, William Rowan Hamilton made a great discovery finding quaternion algebra, a 4-dimensional algebra over $\mathbb{R}$ which is an associative and a noncommutative algebra. In December 1843, John Graves discovered the octonions, an 8-dimensional algebra over $\mathbb{R}$ which is nonassociative and noncommutative algebra. These algebras were later rediscovered by Arthur Cayley in 1845 and are also known sometimes as the *Cayley numbers.* This process, of passing from $\mathbb{R}$ to $\mathbb{C}$, from $\mathbb{C}$ to $\mathbb{H}$ and from $\mathbb{H}$ to $\mathbb{O}$ has been generalized to algebras over fields and over rings. It is called the *Cayley-Dickson doubling process* or the *Cayley–Dickson process.*

Even if are old, Quaternion and Octonion algebras have at present many applications, especially in physics, coding theory, computer science, etc. For example, reliable high rate of transmission can be obtained using Space-Time coding. For constructing Space-Time codes, Quaternion division algebras were chosen as a new tool, as for example the Alamouti code, which can be built from a quaternion division algebra (see [Al; 98]).

The classical Cayley-Dickson process is briefly presented in the following. For details about this, the reader is referred to [Sc; 66] and [Sc; 54]. From now on, in the whole paper, we will consider K a field with $charK \neq 2$.

Let A be an algebra over the field K. A unitary algebra $A \neq K$ such that we have $x^2 + \alpha(x)x + \beta(x) = 0$, for each $x \in A$, with $\alpha(x), \beta(x) \in K$, is called a *quadratic algebra*.

Let A be a finite dimensional unitary algebra over a field K with a *scalar involution*

$$^{-} : A \to A, a \to \overline{a},$$

i.e. a linear map satisfying the following relations:

$$\overline{ab} = \overline{b}\overline{a}, \overline{\overline{a}} = a,$$

and

$$a + \overline{a}, a\overline{a} \in K \cdot 1 \text{ for all } a, b \in A.$$

An element $\overline{a}$ is called the *conjugate* of the element a, the linear form

$$t : A \to K\ ,\ t(a) = a + \overline{a}$$

and the quadratic form

$$n : A \to K,\ n(a) = a\overline{a}$$

are called the *trace* and the *norm* of the element a, respectively. Therefore, such an algebra A with a scalar involution is quadratic.

Let $\gamma \in K$ be a fixed non-zero element. On the vector space $A \oplus A$, we define the following algebra multiplication:

$$A \oplus A : (a_1, a_2)(b_1, b_2) = \left(a_1 b_1 + \gamma \overline{b_2} a_2, a_2 \overline{b_1} + b_2 a_1\right). \tag{1.1}$$

We obtain an algebra structure over $A \oplus A$, denoted by (A, γ) and called the *algebra obtained from A by the Cayley-Dickson process.* It results that dim $(A, \gamma) = 2 \dim A$.

For $x \in (A, \gamma)$, $x = (a_1, a_2)$, the map

$$^{-} : (A, \gamma) \to (A, \gamma)\ ,\ x \to \bar{x} = (\overline{a}_1, \text{-}a_2)\,, \tag{1.2}$$

is a scalar involution of the algebra (A, γ), extending the involution $^{-}$ of the algebra A. Let

$$t\,(x) = t(a_1)$$

and

$$n\,(x) = n\,(a_1) - \gamma n(a_2)$$

be the *trace* and the *norm* of the element $x \in (A, \gamma)$, respectively.

If we take $A = K$ and apply this process t times, $t \geq 1$, we obtain an algebra over K,

$$A_t = \left(\frac{\alpha_1, ..., \alpha_t}{K}\right). \tag{1.3}$$

By induction in this algebra, the set $\{1, e_2, ..., e_n\}, n = 2^t$, generates a basis with the properties:

$$e_i^2 = \alpha_i 1,\ \alpha_i \in K, \alpha_i \neq 0,\ i = 2, ..., n \tag{1.4}$$

and

$$e_i e_j = -e_j e_i = \beta_{ij} e_k,\ \beta_{ij} \in K,\ \beta_{ij} \neq 0, i \neq j, i, j = 2, ...n, \tag{1.5}$$

β_{ij} and e_k being uniquely determined by e_i and e_j.

From [Sc; 54], Lemma 4, it results that in any algebra A_t with the basis $\{1, e_2, ..., e_n\}$ satisfying the above relations we have:

$$e_i\,(e_i x) = \alpha_i^2 = (x e_i) e_i, \tag{1.6}$$

for every $x \in A$ and for all $i \in \{1, 2, ..., n\}$.

A finite-dimensional algebra A is a division algebra if and only if A does not contain zero divisors (see [Sc;66]).

An algebra A is called *central simple* if the algebra $A_F = F \otimes_K A$ is simple for every extension F of K. An algebra A is called *alternative* if $x^2 y = x\,(xy)$ and $xy^2 = (xy)\,y$, for all $x, y \in A$. An algebra A is called *flexible* if $x\,(yx) = (xy)\,x = xyx$, for all $x, y \in A$ and *power associative* if the subalgebra $< x >$ of A generated by any element $x \in A$ is associative. Each alternative algebra is a flexible algebra and a power associative algebra.

Algebras A_t of dimension 2^t obtained by the Cayley-Dickson process, described above, are central-simple, flexible and power associative for all $t \geq 1$ and, in general, are not division algebras for all $t \geq 1$. But there are fields (for example, the rational function field) on which, if we apply the Cayley-Dickson process, the resulting algebras A_t are division algebras for all $t \geq 1$. (See [Br; 67] and [Fl; 13]).

2. About Fibonacci Quaternions

Since the above described algebras are usually without division, finding quickly examples of invertible elements in an arbitrary algebra obtained by the Cayley-Dickson process appear to be a not easy problem. A partial solution for generalized real Quaternion algebras can be found using Fibonacci quaternions.

Let $\mathbb{H}\,(\alpha_1, \alpha_2)$ be the generalized real quaternion algebra. In this algebra, every element has the form $a = a_1 \cdot 1 + a_2 e_2 + a_3 e_3 + a_4 e_4$, where $a_i \in \mathbb{R}, i \in \{1, 2, 3, 4\}$.

In [Ho; 61], the Fibonacci quaternions were defined to be the quaternions on the form

$$F_n = f_n \cdot 1 + f_{n+1} e_2 + f_{n+2} e_3 + f_{n+3} e_4, \tag{2.1}$$

called the nth Fibonacci quaternions, where

$$f_n = f_{n-1} + f_{n-2,}\ n \geq 2, \tag{2.2}$$

with $f_0 = 0, f_1 = 1$, are Fibonacci numbers.

The norm formula for the nth Fibonacci quaternions is:

$$\boldsymbol{n}(F_n) = F_n\overline{F}_n = 3f_{2n+3}, \tag{2.3}$$

where $\overline{F}_n = f_n \cdot 1 - f_{n+1}e_2 - f_{n+2}e_3 - f_{n+3}e_4$ is the conjugate of the F_n in the algebra $\mathbb{H}(\alpha_1, \alpha_2)$ (see [Ho; 61]). There are many authors which studied Fibonacci quaternions in the real division quaternion algebra giving more and surprising new properties (see [Sw; 73], [Sa-Mu; 82] and [Ha; 12], [Fl, Sh; 13], [Fl, Sh; 13(1)]).

Theorem 2.1. ([Fl, Sh; 13] Theorem 2.4.) *The norm of the* nth *Fibonacci quaternion* F_n *in a generalized quaternion algebra is*

$$\boldsymbol{n}(F_n) = h_{2n+2}^{1+2\alpha_2,3\alpha_2} + (\alpha_1 - 1)h_{2n+3}^{1+2\alpha_2,\alpha_2} - 2(\alpha_1 - 1)(1+\alpha_2) f_n f_{n+1}. \tag{2.4}$$

We know that the expression for the nth term of a Fibonacci element is

$$f_n = \frac{1}{\sqrt{5}}[a^n - b^n] = \frac{a^n}{\sqrt{5}}[1 - \frac{b^n}{\alpha^n}], \tag{2.5}$$

where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$.

Using the above notations, we can compute the following limit

$$\lim_{n\to\infty} \boldsymbol{n}(F_n) = \lim_{n\to\infty}(f_n^2 + \alpha_1 f_{n+1}^2 + \alpha_2 f_{n+2}^2 + \alpha_1\alpha_2 f_{n+3}^2) =$$

$$= \lim_{n\to\infty}(\frac{a^{2n}}{5} + \alpha_1\frac{a^{2n+2}}{5} + \alpha_2\frac{a^{2n+4}}{5} + \alpha_1\alpha_2\frac{a^{2n+6}}{5}) =$$

$$= sgnE(\alpha_1, \alpha_2) \cdot \infty.$$

Since $a^2 = a + 1$, we have $E(\alpha_1, \alpha_2) = (\frac{1}{5} + \frac{\alpha_1}{5}a^2 + \frac{\alpha_2}{5}a^4 + \frac{\alpha_1\alpha_2}{5}a^6) =$
$= \frac{1}{5}(1 + \alpha_1(a+1) + \alpha_2(3a+2) + \alpha_1\alpha_2(8a+5)) =$
$= \frac{1}{5}[1 + \alpha_1 + 2\alpha_2 + 5\alpha_1\alpha_2 + a(\alpha_1 + 3\alpha_2 + 8\alpha_1\alpha_2)]$.

If $E(\alpha_1, \alpha_2) > 0$, there exist a number $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have

$$h_{2n+2}^{1+2\alpha_2,3\alpha_2} + (\alpha_1 - 1)h_{2n+3}^{1+2\alpha_2,\alpha_2} - 2(\alpha_1 - 1)(1+\alpha_2) f_n f_{n+1} > 0.$$

If $E(\alpha_1, \alpha_2) < 0$, there exist a number $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ we have

$$h_{2n+2}^{1+2\alpha_2,3\alpha_2} + (\alpha_1 - 1)h_{2n+3}^{1+2\alpha_2,\alpha_2} - 2(\alpha_1 - 1)(1+\alpha_2) f_n f_{n+1} < 0.$$

It results that for all $\alpha_1, \alpha_2 \in \mathbb{R}$ with $E(\alpha_1, \alpha_2) \neq 0$, in the algebra $\mathbb{H}(\alpha_1, \alpha_2)$ there is a natural number $n_0 = \max\{n_1, n_2\}$ such that $\boldsymbol{n}(F_n) \neq 0$, hence F_n is an invertible element, for all $n \geq n_0$.

In this way, Fibonacci Quaternion elements can provide us many important information in the algebra $\mathbb{H}(\alpha_1, \alpha_2)$ providing sets of invertible elements in algebraic structures without division. For other details, see [Fl, Sh; 13].

3. Multiplication table in Cayley-Dickson algebras

Multiplication table for algebras obtained by the Cayley-Dickson process over the real field was studied in [Ba; 09]. In this paper, the author gave an algorithm to find quickly product of two elements in these algebras. In the following, we shortly present this algorithm. In [Ba; 13], the author gave all 32 possibilities to define a "Cayley-Dickson product" used in the Cayley-Dickson doubling process, such that the obtained algebras are isomorphic.

If we consider multiplication (1.1) under the form

$$A \oplus A : (a_1, a_2)(b_1, b_2) := (a_1b_1 + \gamma b_2\overline{a_2}, \overline{a_1}b_2 + b_1a_2), \tag{3.1}$$

the obtained algebras are isomorphic with those obtained with multiplication (1.1).

For $\alpha_1 = ... = \alpha_t = -1$ and $K = \mathbb{R}$, in [Ba; 09] the author described how we can multiply the basis vectors in the algebra $A_t, \dim A_t = 2^t = n$. He used the binary decomposition for the subscript indices.

Let e_p, e_q be two vectors in the basis B with p, q representing the binary decomposition for the indices of the vectors, that means p, q are in $\mathbb{Z}_2^n$. We have that $e_p e_q = \gamma_n(p,q)\, e_{p\otimes q}$, where:
i) $p \otimes q$ are the "*exclusive or*" for the binary numbers p and q (the sum of p and q in the group $\mathbb{Z}_2^n$);
ii) $\gamma_n : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \{-1, 1\}$ is a map, called the *twist map*.
In this section, we will consider $K = \mathbb{R}$. Using the same notations as in the Bales's paper, we consider the following matrices:

$$A_0 = A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}. \tag{3.2}$$

In the same paper [Ba; 09], the author find the properties of the twist map γ_n and put the signs of this map in a table. He partitioned the twist table for $\mathbb{Z}_2^n$ into 2×2 matrices and obtained the following result:

Theorem 3.1. **(**[Ba; 09], Theorem 2.2., p. 88-91**)** *For* $n > 0$, *the Cayley-Dickson twist table* γ_n *can be partitioned in quadratic matrices of dimension* 2 *of the form* $A, B, C, -B, -C$, *defined in the relation* (3.2).

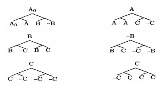

Fig. 1: Twist trees([Ba; 09], Table 9)

Definition 3.2. Let $x = x_0, x_1, x_2, ...$ and $y = y_0, y_1, y_2, ...$ be two sequences of real numbers. The ordered pair

$$(x, y) = x_0, y_0, x_1, y_1, x_2, y_2, ...$$

is a sequence obtained by *shuffling* the sequences x and y.

Proposition 3.3. *Let* $A_t = \left(\frac{-1,...,-1}{\mathbb{R}}\right)$ *be an algebra obtained by the Cayley-Dickson process with multiplication given by relation* (3.1) *and* $\{e_0 = 1, e_1, ..., e_{n-1}\}$,
$n = 2^t$ *a basis in* A_t. *Let* $r \geq 1,\ r < k \leq i < t$. *We have*

$$\begin{array}{c|cc} \cdot & e_T & e_{T+1} \\ \hline e_{2^{k-r+1}} & (-1)^{r+2} e_M & -(-1)^{r+2} e_{M+1} \\ e_{2^{k-r+1}+1} & -(-1)^{r+2} e_{M+1} & -(-1)^{r+2} e_M \end{array}, \tag{3.3}$$

where the binary decomposition of M *is* $M_2 = 2^k \otimes T$, *whith* $T = 2^r + 2^{r+1} + ... + 2^k + 2^i$.

Proof. We compute $e_{2^{k-r+1}} e_T$. We have $e_{2^{k-r+1}} e_T = \gamma(s,q)\, e_M$, where the binary decomposition of M is $M_2 = 2^{k-r+1} \otimes T$ and s is the binary decomposition for 2^{k-r+1} and q is the binary decomposition for T,

$$s = \underbrace{00...0}_{i-k+r-1}\ \underbrace{100...0}_{k-r+2}, q = 1\underbrace{00...0}_{i-k-1}\underbrace{111..1}_{k-r+1}\underbrace{0...0}_{r}.$$

By "shuffling" $s \otimes q$, it results

$$\underbrace{01\ 00\ 00...00}_{i-k}\ \underbrace{01\ 01\ 01\ ...01}_{k-2r-1}\ \underbrace{11\ 01\ 01\ ...01}_{r+2}\ \underbrace{00\ 00\ ...00\ 00}_{r}.$$

Starting with A_0,we get:

$$\underbrace{A_0 \stackrel{01}{\to} A \stackrel{00}{\to} ... \stackrel{00}{\to}}_{i-k} \underbrace{A \stackrel{01}{\to} A \stackrel{01}{\to} ... \stackrel{01}{\to}}_{k-2r-1} \underbrace{A \stackrel{11}{\to} -C \stackrel{01}{\to} C \stackrel{01}{\to} -C \stackrel{01}{\to} C ... \stackrel{01}{\to} (-1)^{r+2} C}_{r+2} \underbrace{\stackrel{00}{\to} ... \stackrel{00}{\to} (-1)^{r+2} C}_{r}.$$

Therefore $\gamma(s,q) = (-1)^{k-r+1}$.

Now, we compute $e_{2^{k-r+1}}e_{T+1}$. For this, we will "shuffling" $\underbrace{00...0}_{i-k+r-1}\underbrace{100...0}_{k-r+2}$ with $1\underbrace{00...0}_{i-k-1}\underbrace{111..1}_{k-r+1}\underbrace{0...1}_{r}$.

It results

$$\underbrace{01\ 00\ 00...00}_{i-k}\underbrace{01\ 01\ 01\ ...01}_{k-2r-1}\underbrace{11\ 01\ 01\ ...01}_{r+2}\underbrace{00\ 00\ ...00\ 01}_{r}.$$

Starting with A_0,we get:

$$\underbrace{A_0 \overset{01}{\to} A \overset{00}{\to} ... \overset{00}{\to}}_{i-k}\underbrace{A \overset{01}{\to} A \overset{01}{\to} ... \overset{01}{\to}}_{k-2r-1}\underbrace{A \overset{11}{\to} -C \overset{01}{\to} C \overset{01}{\to} -C \overset{01}{\to} C... \overset{01}{\to} (-1)^{r+2} C}_{r+2}\underbrace{\overset{00}{\to} ... \overset{01}{\to} (-1)^{r+3} C}_{r}.$$

For $e_{2^{k-r+1}+1}e_T$, "shuffling" $\underbrace{00...0}_{i-k+r-1}\underbrace{100...1}_{k-r+2}$ with $1\underbrace{00...0}_{i-k-1}\underbrace{111..1}_{k-r+1}\underbrace{0...0}_{r}$, it results

$$\underbrace{01\ 00\ 00...00}_{i-k}\underbrace{01\ 01\ 01\ ...01}_{k-2r-1}\underbrace{11\ 01\ 01\ ...01}_{r+2}\underbrace{00\ 00\ ...00\ 10}_{r}.$$

Starting with A_0,we get:

$$\underbrace{A_0 \overset{01}{\to} A \overset{00}{\to} ... \overset{00}{\to}}_{i-k}\underbrace{A \overset{01}{\to} A \overset{01}{\to} ... \overset{01}{\to}}_{k-2r-1}\underbrace{A \overset{11}{\to} -C \overset{01}{\to} C \overset{01}{\to} -C \overset{01}{\to} C... \overset{01}{\to} (-1)^{r+2} C}_{r+2}\underbrace{\overset{00}{\to} ... \overset{10}{\to} (-1)^{r+3} C}_{r}.$$

For $e_{2^{k-r+1}+1}e_{T+1}$, we compute first $\left(2^{k-r+1}+1\right) \otimes (T+1)$. We obtain:

$$\left(2^{k-r+1}+1\right) \otimes (T+1) =$$
$$= \left(\underbrace{00...0}_{i-k+r-1}\underbrace{100...1}_{k-r+2}\right) \otimes \left(1\underbrace{00...0}_{i-k-1}\underbrace{111..1}_{k-r+1}\underbrace{0...1}_{r}\right) =$$
$$= \underbrace{10...0}_{i-k}\underbrace{11..10}_{r-1}\ \underbrace{1...1}_{k-2r+1}\underbrace{0...0}_{r} = 2^{k-r+1} \otimes T = M.$$

Now, "shuffling" $\underbrace{00...0}_{i-k+r-1}\underbrace{100...1}_{k-r+2}$ with $1\underbrace{00...0}_{i-k-1}\underbrace{111..1}_{k-r+1}\underbrace{0...1}_{r}$, it results

$$\underbrace{01\ 00\ 00...00}_{i-k}\underbrace{01\ 01\ 01\ ...01}_{k-2r-1}\underbrace{11\ 01\ 01\ ...01}_{r+2}\underbrace{00\ 00\ ...00\ 11}_{r}$$

Starting with A_0,we get:

$$\underbrace{A_0 \overset{01}{\to} A \overset{00}{\to} ... \overset{00}{\to}}_{i-k}\underbrace{A \overset{01}{\to} A \overset{01}{\to} ... \overset{01}{\to}}_{k-2r-1}\underbrace{A \overset{11}{\to} -C \overset{01}{\to} C \overset{01}{\to} -C \overset{01}{\to} C... \overset{01}{\to} (-1)^{r+2} C}_{r+2}\underbrace{\overset{00}{\to} ... \overset{11}{\to} (-1)^{r+3} C}_{r}.$$

□

4. Some applications in Algebra and Coding Theory

Let A_t be an algebra obtained by the Cayley-Dickso process over the field $\mathbb{R}$, with the basis $\{1, e_2, ..., e_n\}, n = 2^t$. The unit elements in A_t are $\{\pm 1, \pm e_2, ..., \pm e_n\}$. In [Ma, Be, Ga; 09], the authors defined *the integers* of the A_t to be the set

$$A_t[\mathbb{Z}] = \{x_1 \cdot 1 + \sum_{i=2}^{2^n} x_i \cdot e_i,\ x_1, x_i \in \mathbb{Z}, i \in \{2, ..., n\}\}.$$

$A_t[\mathbb{Z}]$ is a non-associative and non-commutative ring on which the following equivalence relation can be defined.

Definition 4.1. Let $a, x, y \in A_t[\mathbb{Z}]$. We say that x, y *are right(left) congruent modulo* a if and only if there is the element $b \in A_t[\mathbb{Z}]$ such that

$$x - y = ba \text{ (or } x - y = ab). \tag{4.1}$$

We denote this relation with $x \equiv_r y \text{ mod } a$ (or $x \equiv_s y \text{ mod } a$) and this relation is well defined. We will consider the quotient ring

$$A_t[\mathbb{Z}]_a = \{x \text{ mod } a \ / \ x \in A_t[\mathbb{Z}]\}.$$

If $a \neq 0$ is not a zero divisor, then $A_t[\mathbb{Z}]_a$ has $N(a)^{2^{n-1}}$ elements (see [Ma, Be, Ga; 09] for other details).

Since algebras A_t are poor in properties, due to the power-associativity, if we take $w \in A_t[\mathbb{Z}]$, then the set $\mathbb{U} = \{a + bw \ / \ a, b \in \mathbb{Z}\}$ become an associative and a commutative ring with $\mathbb{U} \subset A_t[\mathbb{Z}]$.

Let $\mathbb{U}$ be the ring defined above, included in $A_t[\mathbb{Z}]$, with $t \in \{2, 3\}$.

Definition 4.2. An element $x \in \mathbb{U}$ *is prime* in $\mathbb{U}$ if x is not an invertible element in $\mathbb{U}$ and if $x = ab$, then a or b is an invertible element in $\mathbb{U}$.

It is obvious that if $\pi \in \mathbb{U}$ is a prime element, then $n(\pi)$ is a prime element in $\mathbb{Z}$.

If we consider relation (4.1) on $\mathbb{U}$, due to commutativity, "the left" is the same with "the right" and if π is a prime element in $\mathbb{U}$, therefore $\mathbb{U}_\pi$ is a field isomorphic with $\mathbb{Z}_p$, where $n(\pi) = p, p$ a prime element in $\mathbb{Z}$, as we can see from the above statements.

Proposition 4.3. ([Fl; 14], [Gu; 13], [Hu; 94])
i) If $x, y \in \mathbb{V}$, then there are $z, v \in \mathbb{V}$ such that $x = zy + v$, with $N(v) < N(y)$.
ii) With the above notation, we have that the remainder v has the formula

$$v = x - \left[\frac{x\overline{y}}{y\overline{y}}\right] y, \tag{4.2}$$

where the symbol $[\,,]$ is the rounding to the closest integer. For the octonions, the rounding of an octonion integer can be found by rounding the coefficients of the basis, separately, to the closest integer.

Proposition 4.4. ([Fl; 14], [Gu; 13], [Hu; 94])
i) The above relation is an equivalence relation on $\mathbb{U}$. The set of equivalence class is denoted by $\mathbb{U}_\pi$ and is called the residue class of $\mathbb{U}$ modulo π.
ii) The modulo function $\mu : \mathbb{U} \to \mathbb{U}_\pi$ is $\mu(x) = v \bmod \pi = x - \left[\frac{x\overline{y}}{y\overline{y}}\right] y$, where $x = z\pi + v$, *with $N(\pi) < N(y)$.*
iii) $\mathbb{U}_\pi$ is a field isomorphic with $\mathbb{Z}_p, p = N(\pi), p$ a prime number.

Remark 4.6. **(**[Ne, In,Fa, El, Pa; 01]**)** From the above, we have that for $v_i, v_j \in \mathbb{U}_\pi, i, j \in \{1, 2, ..., p-1\}$, $u_i + u_j = u_k$ if and only if $k = i + j \text{ mod } p$ and $u_i \cdot u_j = u_k$ if and only if $k = i \cdot j \text{ mod } p$. From here, we have the following labelling procedure:

1) Let $\pi \in \mathbb{U}$ be a prime, with $n(\pi) = p, p$ a prime number, $\pi = a + bw, a, b \in \mathbb{Z}$.
2) Let $s \in \mathbb{Z}$ be the only solution of the equation $a + bx \text{ mod } p,\ x \in \{0, 1, 2, ..., p-1\}$.
3) The element $k \in \mathbb{Z}_p$ is the label of the element $u = m + nw \in \mathbb{U}$ if $m + ns = k \text{ mod } p$ and $n(u)$ is minimum.

In this way, we obtain the map

$$\alpha : \mathbb{Z}_p \to \mathbb{U}_\pi,\ \alpha(\mathbf{m}) = \mu(m + \pi) = (m + \pi) \text{ mod } \pi.$$

Example 4.5.
Let $t = 2, w = 1 + e_2 + e_3 + e_4, p = 13, \pi = -1 + 2w$. We remark that $n(\pi) = 13$ and $w^2 - 2w + 4 = 0$. The field $\mathbb{U}_\pi$ isomorphic with $\mathbb{Z}_{13}$ is

$$\mathbb{U}_\pi = \{0, 1, 2, 3, -3 + w, -2 + w, -1 + w, 1 - w, 2 - w, 3 - w, -3, -2, -1\}$$

Indeed, using relations $w^2 = 2w - 4$ and $\overline{w} = 2 - w$, we have:
$\mathbf{4} = 4 + \pi = 3 + 2w = -3 + w$, since $3 + 2w = (-1 + 2w)\overline{w} + w - 3$, with $n(w - 3) = 7 < 13 = n(\pi)$;
$\mathbf{5} = 5 + \pi = 4 + 2w = -2 + w$;
$\mathbf{6} = 6 + \pi = 5 + 2w = -1 + w$.
Using the above labelling procedure, we have
$\alpha : \mathbb{Z}_p \to \mathbb{U}_\pi, \alpha(0) = 0, \alpha(1) = 1, \alpha(2) = 2, \alpha(3) = 3,$
$\alpha(4) = -3 + w, \alpha(5) = -2 + w, \alpha(6) = -1 + w, \alpha(7) = 1 - w,$
$\alpha(8) = 2 - w.\alpha(9) = 3 - w, \alpha(10) = -3, \alpha(11) = -2, \alpha(12) = -1.$

Remark 4.6. Since each natural number can be write as a sum of four squares, if $m \in \mathbb{N}$, such that $m = a_1^2 + a_2^2 + a_3^2 + a_4^2, a_i \in \mathbb{N}, i \in \{1, 2, 3\}$ and if $q = 2a_1$, therefore the equation

$$x^2 - qx + m = 0, \tag{4.3}$$

has always solutions in A_t, for all t. Indeed, let $z = a_1 \cdot 1 + a_2 \cdot e_i + a_3 \cdot e_j + a_4 \cdot e_k$, where $i \neq j \neq k$ and $e_i, e_j, e_k \in \{e_2, ..., e_n\}, n = 2^t$. The element z is always a solution of the equation (4.3), since $t(z) = 2a_1 = q$ and $n(x) = a_1^2 + a_2^2 + a_3^2 + a_4^2 = m$.

Remark 4.7. Such kind a field obtained above has many applications in Coding Theory, since on these fields can be constructed good codes which can detected and corrected some error patterns which occur most frequently (see [Fl; 14], [Gu; 13], [Hu; 94], [Ma, Be, Ga; 09], [Ne, In,Fa, El, Pa; 01]).

References

[Al; 98] S.M. Alamouti, *A simple transmit diversity technique for wireless communications*, IEEE J. Selected Areas Communications, **16(1998)**, 1451-1458.

[Ba; 13] J.W. Bales, *A Catalog of Cayley-Dickson-like Products*, http://arxiv.org/pdf/1107.1301v4.pdf.

[Ba; 09] J. W. Bales, *A Tree for Computing the Cayley-Dickson Twist*, Missouri J. Math. Sci., **21(2)(2009)**, 83–93.

[Br; 67] R. B. Brown, *On generalized Cayley-Dickson algebras*, Pacific J. of Math., **20(3)(1967)**, 415-422.

[Fl; 14] C. Flaut, *Codes over a subset of Octonion Integers*, http://arxiv.org/pdf/1401.7828.pdf

[Fl; 13] C. Flaut, *Levels and sublevels of algebras obtained by the Cayley-Dickson process*, Ann Mat Pur Appl **192(2013)**, 1099-1114.

[Fl, Sh; 13] C. Flaut, V. Shpakivskyi, *On Generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions*, Adv. Appl. Cliff ord Algebras **23(2013)**, 673–688.

[Fl, Sh; 13(1)] C. Flaut, V. Shpakivskyi, *Real Matrix Representations for the Complex Quaternions*, Adv. Appl. Cliff ord Algebras **23(2013)**, 657–671.

[Gu; 13] M. Güzeltepe, *Codes over Hurwitz integers*, DiscreteMath, **313(5)(2013)**, 704-714.

[Ha; 12] S. Halici, *On Fibonacci Quaternions*, Adv. in Appl. Clifford Algebras, **22(2)(2012)**, 321-327.

[Ho; 61] A. F. Horadam, *A Generalized Fibonacci Sequence*, Amer. Math. Monthly, **68(1961)**, 455-459.

[Hu; 94] K. Huber, *Codes over Gaussian integers*, IEEE Trans. Inform. Theory, **40(1994)**, 207–216.

[Ma, Be, Ga; 09] C. Martinez, R. Beivide, E. Gabidulin, *Perfect codes from Cayley graphs over Lipschitz integers*, IEEE Trans. Inform. Theory **55(8)(2009),** 3552–3562.

[Ne, In,Fa, El, Pa; 01] T.P. da N. Neto, J.C. Interlando, M.O. Favareto, M. Elia, R. Palazzo Jr., *Lattice constellation and codes from quadratic number fields*, IEEE Trans. Inform.Theory **47(4)(2001)** 1514–1527.

[Sa-Mu; 82] P. V. Satyanarayana Murthy, *Fibonacci-Cayley Numbers*, The Fibonacci Quarterly, **20(1)(1982)**, 59-64.

[Sc; 66] R. D. Schafer, *An Introduction to Nonassociative Algebras,* Academic Press, New-York, 1966.

[Sc; 54] R. D. Schafer, *On the algebras formed by the Cayley-Dickson process,* Amer. J. Math., **76(1954)**, 435-446.

[Sw; 73] M. N. S. Swamy, *On generalized Fibonacci Quaternions*, The Fibonacci Quaterly, **11(5)(1973)**, 547-549.

Author information

Cristina Flaut, Faculty of Mathematics and Computer Science, Ovidius University, Bd. Mamaia 124, 900527, CONSTANTA, Romania.
E-mail: `cflaut@univ-ovidius.ro` and `cristina_flaut@yahoo.com`

Received: February 15, 2014.

Accepted: March 28, 2014.