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A SUBCLASS OF MULTIVALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH ITERATIONS OF THE CHO-KWON-SRIVASTAVA OPERATOR

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Abstract. Let $\sum_p$ denote the class of meromorphically multivalent functions $f(z)$ of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \cdots\})$$

which are analytic in the *punctured open unit disk* $\mathbb{U}^* = \{z : 0 < |z| < 1\}$. In this paper, by making use of a meromorphic analogue of the Cho-Kwon-Srivastava operator and its iterations, a new subclass of meromorphic p-valent functions is introduced. Inclusion theorems and other properties of these function class are studied.

1 Introduction and Definition

Let $\sum_p$ denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the *punctured open unit disk*

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}.$$

For functions $f \in \sum_p$ given by (1.1) and $g \in \sum_p$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (z \in \mathbb{U}^*),$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = \frac{z^p f(z) \star z^p g(z)}{z^p} := \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z) \quad (z \in \mathbb{U}^*),$$

where $\star$ denotes the usual Hadamard product (or convolution) of analytic functions.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that the function $f(z)$ is subordinate to $g(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In such a case, we write $f(z) \prec g(z) \quad (z \in \mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then (see [6, 12, 20])

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Liu and Srivastava [10] studied meromorphic analogue of the Carlson-Shaffer operator [4] by introducing the function $\phi_p(a, c; z)$ given by

$$\phi_p(a, c; z) := \frac{{}_2F_1(a, 1; c; z)}{z^p} =: \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p}$$
$$(z \in \mathbb{U}^*, a \in \mathbb{C}, c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- := \{0, -1, -2, \cdots\}) \tag{1.2}$$

where ${}_2F_1(a,1;c;z)$ is the *Gauss hypergeometric series* and $(\lambda)_k$ is the Pochhammer symbol (or shifted factorial) given by

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0) \\ \lambda(\lambda+1)....(\lambda+k-1) & (k \in \mathbb{N}). \end{cases}$$

Recently, Mishra et al. [13] (see also [16]) considered the function $\phi_p^{\dagger}(a,c;z)$, the generalized multiplicative inverse of $\phi_p(a,c;z)$ given by the relation

$$\phi_p(a,c;z) * \phi_p^{\dagger}(a,c;z) = \frac{1}{z^p(1-z)^{\lambda+p}} \quad (a,c \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ \lambda > -p;\ z \in \mathbb{U}^*). \tag{1.3}$$

Note that if $\lambda = -p+1$, then $\phi_p^{\dagger}(a,c;z)$ is the inverse of $\phi_p(a,c;z)$ with respect to the Hadamard product $*$. Using this function they introduced the following operator $\mathcal{I}_{\lambda,p}^{n,m}(a,c) : \sum_p \longrightarrow \sum_p$ defined by

$$\mathcal{I}_{\lambda,p}^{n,m}(a,c)f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left[\frac{(\lambda+p)_k(c)_k}{(a)_k(1)_k}\right]^n \left[\frac{p-kt}{p}\right]^m a_{k-p}z^{k-p}$$
$$(z \in \mathbb{U}^*,\ t \geq 0,\ m,\ n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}). \tag{1.4}$$

The operator $\mathcal{I}_{\lambda,p}^{n,m}(a,c)$ is obtained by taking compositions of m-iterations of the combinations operator

$$C^t f(z) = (1-t)f(z) + \frac{tz}{p}(-f(z))'$$

with n-iterations of the operator

$$\mathcal{L}_p^{\lambda}(a,c)f(z) = \phi_p^{\dagger}(a,c;z) * f(z).$$

The operator $\mathcal{I}_{\lambda,p}^{n,m}(a,c)$ generalizes several previously studied familiar operators (for details, see [13, 16]).

It is easily verify from (1.4) that

$$z(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z) = \frac{p}{t}(1-t)\mathcal{I}_{\lambda,p}^{n,m}(a,c)f(z) - \frac{p}{t}\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f(z). \tag{1.5}$$

Here we recall that the holomorphic analogue of the function $\phi_p^{\dagger}(a,c;z)$ if the function $\phi_p^{\dagger}(a,c;z)$ given by the relation

$$z^p\ {}_2F_1(a,1;c;z) * \phi_p^{\dagger}(a,c;z) := \frac{z^p}{(1-z)^{\lambda+p}} \quad (a,c \in \mathbb{C}\setminus\mathbb{Z}_0^-, \lambda > -p; z \in \mathbb{U})$$

and the corresponding transform defined by

$$\mathcal{L}_p^{\lambda}(a,c)f(z) = \phi_p^{\dagger}(a,c;z) * f(z)$$

were studied by Cho, Kwon and Srivastava [5]. The transform $\mathcal{L}_p^{\lambda}(a,c)$ is popularly known as the Cho-Kwon-Srivastava operator (see, for detail [7, 18, 21]).
Few literature is available on systematic study of successive iterations of certain transforms on classes of meromorphic as well as analytic functions (see e.g., [1, 2, 13, 16, 19]). Furthermore, using the operator $\mathcal{I}_{\lambda,p}^{n,m}(a,c)$, Panigrahi [17] and Mishra and Soren [14] have investigated its various interesting properties (for recent expository work on meromorphic functions see [3, 8, 9, 22]).

Motivated by the aforementioned work, in this paper we introduce a new subclass of meromorphic functions and investigate inclusion theorems and other properties of a certain class of meromorphically p-valent functions, which are defined by making use of a meromorphic analogue of the Cho-Kwon-Srivastava operator and its iterations given by (1.4).

Throughout this paper, we assume that $p,\ l \in \mathbb{N},\ \epsilon_l = e^{\frac{2\pi i}{l}}$, and for $f \in \sum_p$, we have

$$\begin{aligned} f_{p,l}^{n,m}(\lambda,a,c;z) =& \frac{1}{l}\sum_{j=0}^{l-1} \epsilon_l^{jp} \left(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f\right)(\epsilon_l^j z) \\ =& \frac{1}{z^p} + \left[\frac{(\lambda+p)_l(c)_l}{(a)_l(1)_l}\right]^n \left[\frac{p-lt}{p}\right]^m a_{l-p}z^{l-p} + \cdots . \end{aligned} \tag{1.6}$$

Note that the series we consider is a gap series, each nonzero coefficient appearing after l gaps. For $l = 1$, it follows from (1.6) that

$$f_{p,1}^{n,m}(\lambda, a, c; z) = \mathcal{I}_{\lambda,p}^{n,m}(a, c)f(z).$$

Let $\mathcal{P}$ denote the class of functions of the form:

$$p(z) = 1 + b_1 z + b_2 z^2 + \cdots ,$$

which are analytic and convex in $\mathbb{U}$ satisfying the condition $\Re(p(z)) > 0 \ \ (z \in \mathbb{U})$.

By making use of the operator $\mathcal{I}_{\lambda,p}^{n,m}(a, c)$, we now define a new subclass of $\sum_p$ as follows:

Definition 1.1. A function $f(z) \in \sum_p$ is said to be in the class $\mathcal{T}_{p,l}^{n,m}(\lambda, a, c, \alpha, \beta; h)$ if it satisfies the following subordination conditions:

$$-\beta \frac{z\left[(1+\alpha)(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z) + \alpha(\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f)'(z)\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda,a,c;z) + \alpha f_{p,l}^{n,m+1}(\lambda,a,c;z)\right]} - (1-\beta)\frac{z(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z)}{pf_{p,l}^{n,m}(\lambda,a,c;z)} \prec h(z), \tag{1.7}$$

$$(a, c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \ \lambda > -p, \ n, \ m \in \mathbb{N}_0, \ \alpha > 0, \ \beta \geq 0, \ h \in \mathcal{P}; \ z \in \mathbb{U}).$$

When $n = 1$ we use the following notation :

$\mathcal{T}_{p,l}^{1,m}(\lambda, a, c, \alpha, \beta; h) := \mathcal{T}_{p,l}^{m}(\lambda, a, c, \alpha, \beta; h)$.

In particular for $l = 1, \ \beta = 0$ and $h(z) = \frac{1+Az}{1+Bz} \ \ (-1 \leq B < A \leq 1)$ in (1.7), we get the following function class.

$$\mathcal{T}_p^{n,m}(\lambda, a, c, \alpha, A, B) = \left\{ f \in \sum_p : -\frac{z(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z)}{p\mathcal{I}_{\lambda,p}^{n,m}(a,c)f(z)} \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathbb{U}) \right\}. \tag{1.8}$$

2 Preliminaries

We need the following lemmas for our present investigation:

Lemma 2.1. (see [11]) Let $\beta, \ \gamma \in \mathbb{C}$. Suppose that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with

$$\phi(0) = 1, \quad \Re(\beta\phi(z) + \gamma) > 0 \quad (z \in \mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then the following subordination:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \phi(z) \quad (z \in \mathbb{U}),$$

implies that $p(z) \prec \phi(z)$.

Lemma 2.2. (see [15]) Let $\beta, \gamma \in \mathbb{C}$. Suppose that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with

$$\phi(0) = 1, \quad \Re(\beta\phi(z) + \gamma) > 0 \quad (z \in \mathbb{U}).$$

Also let

$$q(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

If $p(z) \in \mathcal{P}$ and satisfies the following subordination:

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \phi(z)$$

then $p(z) \prec \phi(z)$.

Lemma 2.3. *Let* $f \in \mathcal{T}_{p,l}^{n,m}(\lambda, a, c, \alpha, \beta; \phi)$. *Then*

$$-\beta \frac{z\left[(1+\alpha)\left(f_{p,l}^{n,m}(\lambda,a,c;z)\right)' + \alpha\left(f_{p,l}^{n,m+1}(\lambda,a,c;z)\right)'\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda,a,c;z) + \alpha f_{p,l}^{n,m+1}(\lambda,a,c;z)\right]} - (1-\beta)\frac{z\left(f_{p,l}^{n,m}(\lambda,a,c;z)\right)'}{pf_{p,l}^{n,m}(\lambda,a,c;z)} \prec \phi(z). \tag{2.1}$$

Furthermore, if $\phi(z) \in \mathcal{P}$ *with*

$$\Re\left\{\frac{1}{\beta}\left(p - \frac{p}{\alpha t} - \frac{2p}{t} - p\phi(z)\right)\right\} > 0 \quad (\alpha,\ \beta,\ t > 0;\ z \in \mathbb{U}), \tag{2.2}$$

then

$$-\frac{z\left(f_{p,l}^{n,m}(\lambda, a, c; z)\right)'}{pf_{p,l}^{n,m}(\lambda, a, c; z)} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{2.3}$$

Proof. From (1.6), we have

$$\begin{aligned} f_{p,l}^{n,m}(\lambda, a, c, ; \epsilon_l^j z) =& \frac{1}{l}\sum_{k=0}^{l-1} \epsilon_l^{kp}\left(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f\right)(\epsilon_l^{k+j} z) \\ =& \epsilon_l^{-jp} f_{p,l}^{n,m}(\lambda, a, c; z) \quad (j = 0, 1, ...l-1), \end{aligned} \tag{2.4}$$

and

$$\left(f_{p,l}^{n,m}(\lambda, a, c; z)\right)' = \frac{1}{l}\sum_{k=0}^{l-1} \epsilon_l^{(p+1)k}\left(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f\right)'(\epsilon_l^k z). \tag{2.5}$$

Replacing m by $m+1$ in (2.4) and (2.5) respectively, we can get

$$f_{p,l}^{n,m+1}(\lambda, a, c; \epsilon_l^j z) = \epsilon_l^{-jp} f_{p,l}^{n,m+1}(a, c; z) \tag{2.6}$$

and

$$\left(f_{p,l}^{n,m+1}(\lambda, a, c; z)\right)' = \frac{1}{l}\sum_{k=0}^{l-1} \epsilon_l^{(p+1)k}\left(\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f\right)'(\epsilon_l^k z). \tag{2.7}$$

From (2.4) to (2.7) we can get

$$-\beta\frac{z\left[(1+\alpha)\left(f_{p,l}^{n,m}(\lambda, a, c; z)\right)' + \alpha\left(f_{p,l}^{n,m+1}(\lambda, a, c; z)\right)'\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda, a, c; z) + \alpha f_{p,l}^{n,m+1}(\lambda, a, c; z)\right]} - (1-\beta)\frac{z\left(f_{p,l}^{n,m}(\lambda, a, c; z)\right)'}{pf_{p,l}^{n,m}(\lambda, a, c; z)}$$

$$\begin{aligned} =& -\frac{1}{l}\sum_{k=0}^{l-1}\beta\frac{\epsilon_l^k z\left[(1+\alpha)\left(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f\right)'(\epsilon_l^k z) + \alpha\left(\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f\right)'(\epsilon_l^k z)\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda, a, c; \epsilon_l^k z) + \alpha f_{p,l}^{n,m+1}(\lambda, a, c; \epsilon_l^k z)\right]} \\ & -\frac{(1-\beta)}{l}\sum_{k=0}^{l-1}\frac{\epsilon_l^k z\left(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f\right)'(\epsilon_l^k z)}{pf_{p,l}^{n,m}(\lambda, a, c; \epsilon_l^k z)}. \end{aligned} \tag{2.8}$$

Since $f \in \mathcal{T}_{p,l}^{n,m}(\lambda, a, c, \alpha, \beta; \phi)$, it follows that

$$-\beta\frac{\epsilon_l^k z\left[(1+\alpha)\left(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f\right)'(\epsilon_l^k z) + \alpha\left(\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f\right)'(\epsilon_l^k z)\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda, a, c; \epsilon_l^k z) + \alpha f_{p,l}^{n,m+1}(\lambda, a, c; \epsilon_l^k z)\right]} - (1-\beta)\frac{\epsilon_l^k z\left(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f\right)'(\epsilon_l^k z)}{pf_{p,l}^{n,m}(\lambda, a, c; \epsilon_l^k z)} \prec \phi(z). \tag{2.9}$$

Since $\phi(z)$ is convex and univalent in $\mathbb{U}$, the assertion (2.1) of Lemma 2.3 follows from (2.8) and (2.9).

From (1.5) and (1.6) we obtain

$$z\left(f_{p,l}^{n,m}(\lambda, a, c; z)\right)' + \frac{p}{t}f_{p,l}^{n,m+1}(\lambda, a, c; z) = \frac{\frac{p}{t}(1-t)}{l}\sum_{k=0}^{l-1}\epsilon_l^{pk}\left(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f\right)(\epsilon_l^k z) = \frac{p(1-t)}{t}f_{p,l}^{n,m}(\lambda, a, c; z). \tag{2.10}$$

Let $f \in \mathcal{T}_{p,l}^{n,m}(\lambda, a, c, \alpha, \beta; \phi)$ and suppose that

$$\psi(z) = -\frac{z\left(f_{p,l}^{n,m}(\lambda, a, c; z)\right)'}{pf_{p,l}^{n,m}(\lambda, a, c; z)} \quad (z \in \mathbb{U}). \tag{2.11}$$

Clearly $\psi(z)$ is analytic in $\mathbb{U}$ and $\psi(0) = 1$. It follows from (2.10) and (2.11) that

$$1 - t + t\psi(z) = \frac{f_{p,l}^{n,m+1}(\lambda, a, c; z)}{f_{p,l}^{n,m}(\lambda, a, c; z)}. \tag{2.12}$$

Taking logarithmic differentiation on both sides of (2.12) and making use of (2.10) and (2.11) in the resulting equation, we get

$$z\left(f_{p,l}^{n,m+1}(\lambda, a, c; z)\right)' = -\left([p - pt + pt\psi(z)]\psi(z) - tz\psi'(z)\right) f_{p,l}^{n,m}(\lambda, a, c; z). \tag{2.13}$$

Now it follows from (2.1) and (2.11) to (2.13) that

$$\begin{aligned}
-\beta\frac{z\left[(1+\alpha)\left(f_{p,l}^{n,m}(\lambda, a, c; z)\right)' + \alpha\left(f_{p,l}^{n,m+1}(\lambda, a, c; z)\right)'\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda, a, c; z) + \alpha f_{p,l}^{n,m+1}(\lambda, a, c; z)\right]} &- (1-\beta)\frac{z\left(f_{p,l}^{n,m}(\lambda, a, c; z)\right)'}{pf_{p,l}^{n,m}(\lambda, a, c; z)} \\
= \beta\frac{(1+\alpha)\psi(z) + \alpha\left(\{-t + t\psi(z)\}\psi(z) - \frac{t}{p}z\psi'(z)\right)}{(1+\alpha) + \alpha(1 - t + t\psi(z))} &+ (1-\beta)\psi(z) \\
= \psi(z) + \frac{z\psi'(z)}{\frac{1}{\beta}\left(p - \frac{p}{\alpha t} - \frac{2p}{t} - p\psi(z)\right)} &\prec \phi(z) \quad (z \in \mathbb{U}).
\end{aligned} \tag{2.14}$$

Since

$$\Re\left\{\frac{1}{\beta}\left(p - \frac{p}{\alpha t} - \frac{2p}{t} - p\phi(z)\right)\right\} > 0 \quad (\alpha, \beta, t > 0, z \in \mathbb{U}),$$

the assertion (2.3) of Lemma 2.3 follows by virtue of (2.14) and Lemma 2.1. This completes the proof of Lemma 2.3.

3 Main Results

Theorem 3.1. *Let $\phi(z) \in \mathcal{P}$ be such that*

$$\Re\left\{\frac{1}{\beta}\left(p - \frac{p}{\alpha t} - \frac{2p}{t} - p\phi(z)\right)\right\} > 0 \quad (\alpha, \beta, t > 0, z \in \mathbb{U}).$$

Then

$$\mathcal{T}_{p,l}^{n,m}(\lambda, a, c, \alpha, \beta; \phi(z)) \subset \mathcal{T}_{p,l}^{n,m}(\lambda, a, c, \alpha; \phi(z))$$

Proof. Let $f \in \mathcal{T}_{p,l}^{n,m}(\lambda, a, c, \alpha, \beta; \phi(z))$ and suppose that

$$q(z) = -\frac{z\left(\mathcal{I}_{\lambda,p}^{n,m}(a, c)f\right)'(z)}{pf_{p,l}^{n,m}(\lambda, a, c; z)} \quad (z \in \mathbb{U}). \tag{3.1}$$

Clearly $q(z)$ is analytic in $\mathbb{U}$ and $q(0) = 1$. It follows from (1.5) and (3.1) that

$$q(z)f_{p,l}^{n,m}(\lambda, a, c; z) = -\frac{1}{t}(1-t)\mathcal{I}_{\lambda,p}^{n,m}(a, c)f(z) + \frac{1}{t}\mathcal{T}_{\lambda,p}^{n,m+1}(a, c)f(z). \tag{3.2}$$

Differentiating both sides of (3.2) with respect to z and using (3.1) in the resulting equation, we obtain

$$zq'(z) + \left[\frac{z\left(f_{p,l}^{n,m}(\lambda, a, c; z)\right)'}{f_{p,l}^{n,m}(\lambda, a, c; z)} - \frac{p}{t}(1-t)\right]q(z) = \frac{p}{t}\frac{z\left(\mathcal{I}_{\lambda,p}^{n,m+1}(a, c)f\right)'(z)}{pf_{p,l}^{n,m}(\lambda, a, c; z)}. \tag{3.3}$$

Making use of (2.11), (2.12), (3.1) and (3.2) in (1.7) yield

$$-\beta\frac{z\left[(1+\alpha)(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z)+\alpha(\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f)'(z)\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda,a,c;z)+\alpha f_{p,l}^{n,m+1}(\lambda,a,c;z)\right]}-(1-\beta)\frac{z(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z)}{pf_{p,l}^{n,m}(\lambda,a,c;z)}$$

$$=\beta\frac{(1+\alpha)q(z)-\frac{\alpha t}{p}\left[zq'(z)+\left(p-\frac{p}{t}-p\psi(z)\right)q(z)\right]}{(1+\alpha)+\alpha(1-t+t\psi(z))}+(1-\beta)q(z)$$

$$=q(z)+\frac{zq'(z)}{\frac{1}{\beta}\left(p-\frac{p}{\alpha t}-\frac{2p}{t}-p\psi(z)\right)}\prec\phi(z)\quad(z\in\mathbb{U}).\tag{3.4}$$

Since

$$\Re\left\{\frac{1}{\beta}\left(p-\frac{p}{\alpha t}-\frac{2p}{t}-p\phi(z)\right)\right\}>0\quad(\alpha,\beta,t>0,z\in\mathbb{U}),$$

by virtue of Lemma 2.3, we have

$$\psi(z)=-\frac{z\left(f_{p,l}^{n,m}(\lambda,a,c;z)\right)'}{pf_{p,l}^{n,m}(\lambda,a,c;z)}\prec\phi(z)\quad(z\in\mathbb{U}).$$

Thus, by (3.4) and Lemma 2.2, we find that

$$q(z)\prec\phi(z)\quad(z\in\mathbb{U}),$$

which implies

$$\mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha,\beta;\phi(z))\subset\mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha;\phi(z))$$

The proof of Theorem 3.1 is thus completed.

For $n=1$, Theorem 3.1 takes the following form:

Corollary 3.2. *Let $\phi(z)\in\mathcal{P}$ be such that*

$$\Re\left\{\frac{1}{\beta}\left(p-\frac{p}{\alpha t}-\frac{2p}{t}-p\phi(z)\right)\right\}>0\quad(\alpha,\beta,t>0;z\in\mathbb{U}).$$

Then

$$\mathcal{T}_{p,l}^{m}(\lambda,a,c,\alpha,\beta,\phi)\subset\mathcal{T}_{p,l}^{m}(\lambda,a,c,\alpha,\phi).$$

Taking $\phi(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 3.1, we get the following result.

Corollary 3.3. *Let $-1\le B<A\le 1$ and*

$$\frac{1+A}{1+B}<\left(1-\frac{1}{\alpha t}-\frac{2}{t}\right)\quad(\alpha,\ t>0).$$

Then

$$\mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha,\beta,A,B)\subset\mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha,A,B).$$

Theorem 3.4. *Let $h(z)\in\mathcal{P}$ and $0\le\beta_1<\beta_2$ be such that*

$$\Re\left\{\frac{1}{\beta_2}\left(p-\frac{p}{\alpha t}-\frac{2p}{t}-ph(z)\right)\right\}>0\quad(\alpha,\beta,t>0,z\in\mathbb{U}).$$

Then

$$\mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha,\beta_2;h(z))\subset\mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha,\beta_1;h(z)).$$

Proof. Let $f\in\mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha,\beta_2;h(z))$. Then by Definition 1.1, we have

$$-\beta_2\frac{z\left[(1+\alpha)(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z)+\alpha(\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f)'(z)\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda,a,c;z)+\alpha f_{p,l}^{n,m+1}(\lambda,a,c;z)\right]}-(1-\beta_2)\frac{z(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z)}{pf_{p,l}^{n,m}(\lambda,a,c;z)}\prec h(z).\tag{3.5}$$

We define the function $q(z)$ by the following:

$$q(z) = -\frac{z\left(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f\right)'(z)}{pf_{p,l}^{n,m}(\lambda,a,c;z)} \quad (z \in \mathbb{U}).$$

Therefore by Theorem 3.1, we get

$$\mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha,\beta_2;h(z)) \subset \mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha,\beta_1;h(z)).$$

Hence,

$$q(z) \prec h(z) \quad (z \in \mathbb{U}). \tag{3.6}$$

We also observe that the following identity holds:

$$-\beta_1 \frac{z\left[(1+\alpha)(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z) + \alpha(\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f)'(z)\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda,a,c;z) + \alpha f_{p,l}^{n,m+1}(\lambda,a,c;z)\right]} - (1-\beta_1)\frac{z(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z)}{pf_{p,l}^{n,m}(\lambda,a,c;z)}$$

$$= \frac{\beta_1}{\beta_2}\left[-\beta_2 \frac{z\left[(1+\alpha)(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z) + \alpha(\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f)'(z)\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda,a,c;z) + \alpha f_{p,l}^{n,m+1}(\lambda,a,c;z)\right]} - (1-\beta_2)\frac{z(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z)}{pf_{p,l}^{n,m}(\lambda,a,c;z)}\right] + \left(1 - \frac{\beta_1}{\beta_2}\right)q(z).$$

Since $0 \le \frac{\beta_1}{\beta_2} < 1$, and $h(z)$ is convex univalent in $\mathbb{U}$, we conclude from (3.5) and (3.6) that

$$-\beta_1 \frac{z\left[(1+\alpha)(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z) + \alpha(\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f)'(z)\right]}{p\left[(1+\alpha)f_{p,l}^{n,m}(\lambda,a,c;z) + \alpha f_{p,l}^{n,m+1}(\lambda,a,c;z)\right]} - (1-\beta_1)\frac{z(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f)'(z)}{pf_{p,l}^{n,m}(\lambda,a,c;z)} \prec h(z) \quad (z \in \mathbb{U}).$$

Thus

$$f(z) \in \mathcal{T}_{p,l}^{n,m}(\lambda,a,c,\alpha,\beta_1;h).$$

The proof of Theorem 3.4 is completed.

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