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On Generalized Statistical Convergence

Maya Altınok, Zehra Kurtdişi and Mehmet Küçükaslan

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Abstract In this paper, statistical convergence is generalized by using regular Nörlund mean N(p) where $p = (p_n)$ is a positive sequence of natural numbers. It is called statistical Nörlund convergence and denoted by the symbol st-N(p).

Besides convergence properties of st-N(p), some inclusion results have been given between st-N(p) convergence and strongly N(p) and statistical convergence.

Also, st-N(p) and st-N(q) convergences are compared under some certain restrictions.

1 Introduction and Backround

Statistical convergence of real (or complex) valued sequences was first introduced by Fast H. [6] and Steinhaus I. J. [18] independently in the 1951 as a generalization of ordinary convergence. This subject has been applied various field of mathematics by many authors such as Erdös P.- Terenbaum G. [5], Freeedman A. R.- Sember J. J. - Raphael M. [7], etc. Besides, Connor J.[3, 4], Fridy J. A. [8], Fridy J. A.- Orhan C. [9], Salat T. [16], Schenberg I. J. [17]. Statistical convergence is closely related to the natural density of the subset K of natural numbers $\mathbb{N}$ (see more in [2]).

For $n \in \mathbb{N}$, let $K(n) := \{k \mid k \leq n, k \in K\}$ for $K \subset \mathbb{N}$. Then, the natural density (or asymptotic density) of $K \subseteq \mathbb{N}$ is denoted by $\delta(K)$, and

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{K(n)}(k) \tag{1.1}$$

if this limit exists. In (1.1) the symbol $\chi_{K(n)}(.)$ denotes the characteristic function of the set $K(n)$.

A real (or complex) sequence $x = (x_n)$ is said to be statistical convergent to $l \in \mathbb{R}$ $(\in \mathbb{C})$, if the set $K(\varepsilon) := \{k \mid k \leq n, |x_k - l| \geq \varepsilon\}$ has natural density zero for every $\varepsilon > 0$, i. e., $\delta(K(\varepsilon)) = 0$. This limit is denoted by $x_n \to l$(st).

Throughout this paper, let $p = (p_n)$ be a sequence of nonnegative natural numbers with $p_0 = 0$ and $p_n > 0$ for all $n \in \mathbb{N}$ and $P_n := \sum_{k=0}^{n} p_k$.

The Nörlund mean of the sequence $x = (x_n)$ is defined by $t_n := \frac{1}{P_n} \sum_{k=1}^{n} p_{n-k+1} x_k$.

The sequence $x = (x_n)$ is said to be $N(p)$ convergent to $l \in \mathbb{R}$ if the sequence $(t_n)_{n\in\mathbb{N}}$ convergent to $l \in \mathbb{R}$, and strongly N(p) convergence to l if

$$\lim_{n\to\infty} \frac{1}{P_n} \sum_{k=1}^{n} p_{n-k+1} |x_k - l| = 0.$$

and it is denoted by $x_n \to l$ (N(p)).

Definition 1.1. The sequence $x = (x_n)$ is said to be statistically Nörlund convergent to l if

$$\lim_{n\to\infty} \frac{1}{P_n} \sum_{k=1}^{n} p_{n-k+1} \chi_{K(\varepsilon)}(k) = 0, \tag{1.2}$$

holds. It is denoted by $x_n \to l$ (st-N(p)).

The case $p_n = 1$ in Definition 1.1 is coincide with usual statistical convergence [8]. This kind of generalization of statistical convergence has been given by Fredman and Sember in [7] by using any regular matrix summability method.

Let us recall that if the summability method transforms convergent sequence to convergent sequence with the same limit, then it is called regular.

It is clear from [15] that the method N(p) is regular if and only if $\frac{p_n}{P_n} \to 0$, $n \to \infty$.

Especially, by using regular Nörlund mean different kind of generalization of statistical convergence has been given in literature such as [1], [11], [12], [13], [14] and etc.

The next lemma plays key role in the proofs of Theorems 2.9 and 2.10.

Lemma 1.2. *[15] If N(p) is regular, then the series $p(x) = \sum_{n=1}^{\infty} p_n x^{n-1}$ and $P(x) = \sum_{n=1}^{\infty} P_n x^{n-1}$ are convergent for all $|x| < 1$.*

It is easy to see that the following series are convergent for all $|x| < 1$,

$$k(x) = \sum_{n=1}^{\infty} k_n x^{n-1} = \frac{q(x)}{p(x)} = \frac{Q(x)}{P(x)}$$

and

$$h(x) = \sum_{n=1}^{\infty} h_n x^{n-1} = \frac{p(x)}{q(x)} = \frac{P(x)}{Q(x)}$$

where $q(x) = \sum_{n=1}^{\infty} q_n x^{n-1}$, $Q(x) = \sum_{n=1}^{\infty} Q_n x^{n-1}$. Also, by comparing their coefficients

$$k_1 p_n + ... + k_n p_1 = q_n, \quad k_1 P_n + ... + k_n P_1 = Q_n, \tag{1.3}$$

$$h_1 q_n + ... + h_n q_1 = p_n, \quad h_1 Q_n + ... + h_n Q_1 = P_n. \tag{1.4}$$

is obtained.

2 New Results

Let us consider the function $d_p : \mathbb{C}^{\mathbb{N}} \times \mathbb{C}^{\mathbb{N}} \to [0, \infty)$ where $\mathbb{C}^{\mathbb{N}}$ denotes the set of all complex variable sequences, as follows:

$$d_p(x, y) := \limsup_{n\to\infty} \frac{1}{P_n} \sum_{k \le n} p_{n-k+1} \varphi(|x_k - y_k|)$$

for $x = (x_n), y = (y_n) \in \mathbb{C}^{\mathbb{N}}$ and φ is the function $\varphi : [0, \infty) \to [0, \infty)$ where

$$\varphi(t) := \begin{cases} t, & t \le 1, \\ 1, & t > 1. \end{cases}$$

The function d_p is a semi-metric and it is called p-semi-metric on $\mathbb{C}^{\mathbb{N}}$ (for more information [10]).

Theorem 2.1. *The sequence $x = (x_n)$ is st-N(p) convergent to $l \in \mathbb{R}$ if and only if $d_p(x, y) = 0$ where $y_n = l$ for all $n \in \mathbb{N}$.*

Proof. Let us assume $d_p(x, y) = 0$ where $y_n = l$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{P_n} \sum_{\substack{k \leqslant n \\ |x_k - l| \geqslant \varepsilon}} p_{n-k+1} &\le \begin{cases} \frac{1}{\varepsilon} \limsup\limits_{n\to\infty} \frac{1}{P_n} \sum\limits_{k\leqslant n} p_{n-k+1}\varphi(|x_k - l|), & \varepsilon \le |x_k - l| \le l, \\ \limsup\limits_{n\to\infty} \frac{1}{P_n} \sum\limits_{k\leqslant n} p_{n-k+1}\varphi(|x_k - l|), & |x_k - l| > 1. \end{cases} \\
&\leqslant \max\{1, \frac{1}{\varepsilon}\} \limsup_{n\to\infty} \frac{1}{P_n} \sum_{k\leqslant n} p_{n-k+1}\varphi(|x_k - l|) \\
&= \max\{1, \frac{1}{\varepsilon}\} d_p(x, l)
\end{aligned}$$

for every $\varepsilon > 0$.

This last inequality implies that the sequence (x_n) is st-N(p) convergent to l.
Now, assume that $x=(x_n)$ is st-N(p) convergent to $l\in\mathbb{R}$. Then, for every $\varepsilon>0$ we have

$$
\begin{aligned}
\frac{1}{P_n}\sum_{k\leqslant n} p_{n-k+1}\,\varphi(|x_k-y_k|) &= \frac{1}{P_n}\sum_{\substack{k\leqslant n\\ |x_k-l|<\varepsilon}} p_{n-k+1}\varphi(|x_k-l|)\\
&+ \frac{1}{P_n}\sum_{\substack{k\leqslant n\\ |x_k-l|\geqslant\varepsilon}} p_{n-k+1}\varphi(|x_k-l|)\\
&\leqslant \varepsilon\frac{1}{P_n}\sum_{k\leqslant n}p_{n-k+1}+\frac{1}{P_n}\sum_{\substack{k\leqslant n\\ |x_k-l|\geqslant\varepsilon}} p_{n-k+1}
\end{aligned}
$$

Since N(p) is regular, then the following inequality

$$
\begin{aligned}
d_p(x,y) &= \limsup_{n\to\infty}\frac{1}{P_n}\sum_{k\leqslant n}p_{n-k+1}\varphi(|x_k-y_k|)\\
&\leqslant \varepsilon\limsup_{n\to\infty}\frac{1}{P_n}\sum_{k\leqslant n}p_{n-k+1}+\limsup_{n\to\infty}\frac{1}{P_n}\sum_{\substack{k\leqslant n\\ |x_k-L|\geqslant\varepsilon}} p_{n-k+1}\leq\varepsilon
\end{aligned}
$$

holds. This calculation implies that $d_p(x,y)\leq\varepsilon$ for any $\varepsilon>0$ where $y_n=l$ $(n\in\mathbb{N})$. So, the proof is completed. □

Corollary 2.2. *If the sequence $x=(x_k)$ is convergent to l then $d_p(x,y)=0$, where $y_n=l$ for all $n\in\mathbb{N}$.*

Proof. Let us assume $x=(x_k)$ is convergent to l. It means that for every $\varepsilon>0$, there exists an $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that $|x_n-l|<\frac{\varepsilon}{2}$ holds for all $n>n_0$. Therefore, the following inequality

$$
\begin{aligned}
d_p(x,y) &= \limsup_{n\to\infty}\frac{1}{P_n}\sum_{k\leqslant n}p_{n-k+1}\varphi(|x_k-l|)\\
&= \limsup_{n\to\infty}\left[\frac{1}{P_n}\sum_{k\leqslant n_0}p_{n-k+1}\varphi(|x_k-l|)+\frac{1}{P_n}\sum_{n_0+1\leqslant k\leqslant n}p_{n-k+1}\varphi(|x_k-l|)\right]\\
&\leqslant \limsup_{n\to\infty}\frac{1}{P_n}\sum_{k\leqslant n_0}p_{n-k+1}+\limsup_{n\to\infty}\frac{1}{P_n}\sum_{k=n_0+1}^{n}p_{n-k+1}|x_k-l|\\
&\leqslant n_0\limsup_{n\to\infty}\frac{p_n}{P_n}+\varepsilon\limsup_{n\to\infty}\frac{1}{P_n}\sum_{k=n_0+1}^{n}p_{n-k+1}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}
$$

holds. It gives that $d_p(x,y)=0$. □

Following theorem shows that strongly N(p) convergence implies st-N(p) convergence:

Theorem 2.3. *If the sequence $x=(x_n)$ is strongly N(p) convergent to l then $x=(x_n)$ is st-N(p) convergent to l.*

Proof. Assume that (x_n) is strongly N(p) convergent to l. That is

$$\lim_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}|x_k-l|=0.$$

Also, the inequality

$$\begin{aligned}\frac{1}{P_n} & \sum_{k=1}^{n} p_k|x_k-l| = \frac{1}{P_n}\left(\sum_{k\in K(\varepsilon)}+\sum_{k\notin K(\varepsilon)}\right)p_{n-k+1}|x_k-l| \\ \geq & \frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}|x_k-l|\chi_{K(\varepsilon)}(k) \geq \varepsilon\frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}\chi_{K(\varepsilon)}(k) \\ = & \varepsilon\frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}\chi_{K(\varepsilon)}(k).\end{aligned}$$

holds. After taking limit each side of above inequality when $n \to \infty$, then the desired result is obtained. □

Corollary 2.4. *If $x_n \to l$ $(n \to \infty)$, then $x_n \to l$ (st-N(p)).*

Remark 2.5. The converse of Theorem 2.3 and Corollary 2.4 are not true in general.

For simplicity let $\alpha = 1$. Consider the sequence $x = (x_n)$ and $p = (p_n)$ as follows:

$$x_n := \begin{cases} m^3, & n = m^2, \quad m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

and $p_n = 1$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_{K(\varepsilon)}(k) = \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{[|\sqrt{n}|]}1 = 0$$

is hold. But,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}1|x_k-0| &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{[|\sqrt{n}|]}m^3 \\ &= \lim_{n\to\infty}\frac{[|\sqrt{n}|]([|\sqrt{n}|]+1)(2[|\sqrt{n}|]+1)}{6n} = \infty.\end{aligned}$$

Since the sequence is unbounded, then it is not convergence in usual case.

In the next theorem, it is proved if the sequence is bounded, then st-N(p) convergence implies strongly N(p) convergence.

Theorem 2.6. *If the sequence $x = (x_n)$ is bounded and st-N(p) convergent to l, then $x_n \to l$ (N(p)).*

Proof. Assume $x = (x_n)$ bounded and

$$\lim_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}\chi_{K(\varepsilon)}(k) = 0$$

holds for any $\varepsilon > 0$. Therefore, from the boundedness of (x_n), there exists a positive $M > 0$ such that $|x_n - l| < M$ holds for all $n \in \mathbb{N}$. By using this fact, the following inequality

$$\begin{aligned}\frac{1}{P_n} & \sum_{k=1}^{n}p_{n-k+1}|x_k-l| = \left(\sum_{k\in K(\varepsilon)}+\sum_{k\notin K(\varepsilon)}\right)p_{n-k+1}|x_k-l| \\ = & \frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}|x_k-l|\chi_{K(\varepsilon)}(k) + \frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}|x_k-l|\chi_{K^c(\varepsilon)}(k) \\ \leq & M\frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}\chi_{K(\varepsilon)}(k) + \varepsilon\frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}\chi_{K^c(\varepsilon)}(k)\end{aligned}$$

holds. After taking limit when $n \to \infty$ in the above inequality the proof is obtained. □

Corollary 2.7. *If the sequence $x = (x_n)$ is bounded and $d_p(x, y) = 0$ where $y_n = l$ for all $n \in \mathbb{N}$ then $x = (x_n)$ is N(p) convergent to l.*

Theorem 2.8. *st-N(p) convergence implies st-N(q) convergence if and only if there exists a positive constant M such that for every n,*

$$|k_1|P_n + |k_2|P_{n-1} + ... + |k_n|P_1 \leq MQ_n \tag{2.1}$$

and

$$\lim_{n\to\infty} \frac{k_n}{Q_n} = 0 \tag{2.2}$$

are hold.

Proof. Let (t_n^p) and (t_n^q) be st-N(p) and st-N(q) transformations of the sequence (x_n) respectively, i.e.,

$$t_n^p = \frac{1}{P_n}\sum_{k=1}^{n} p_{n-k+1}\chi_{K(\varepsilon)}(k) \text{ and } t_n^q = \frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k+1}\chi_{K(\varepsilon)}(k)$$

for an arbitrary $\varepsilon > 0$. From this, we have

$$Q_n t_n^q = q_n\chi_{K(\varepsilon)}(1) + q_{n-1}\chi_{K(\varepsilon)}(2) + ... + q_1\chi_{K(\varepsilon)}(n)$$

and by using (1.3) we obtain

$$Q_n t_n^q = k_n P_1 t_1^p + k_{n-1}P_2 t_2^p + ... + k_1 P_n t_n^p.$$

Therefore, the sequence (t_n^q) can be represent as follows:

$$t_n^q = \sum_{m=1}^{\infty} b_{n,m} t_m^p$$

where the matrix $(b_{n,m})$ as

$$b_{nm} := \begin{cases} \frac{k_{n-m+1}P_m}{Q_n}, & m \leq n, \\ 0, & m > n. \end{cases}$$

Under the sufficiency hypothesis of the theorem, $\lim_{n\to\infty} b_{nm} = 0$ for every m, since

$$\lim_{n\to\infty} b_{nm} = \lim_{n\to\infty} \frac{k_{n-m+1}P_m}{Q_n} = \lim_{n\to\infty} \frac{k_{n-m+1}P_m}{Q_{n-m+1}} = P_m \lim_{n\to\infty} \frac{k_n}{Q_n} = 0.$$

Moreover, we have

$$\sum_{m=1}^{\infty} |b_{nm}| = \frac{|k_1|P_n + ... + |k_n|P_1}{Q_n} \leq M$$

and finally

$$\sum_{m=1}^{\infty} b_{nm} = \frac{k_1 P_n + ... + k_n P_1}{Q_n} = \frac{Q_n}{Q_n} = 1$$

for every n.

Hence, $B = (b_{nm})$ is a regular matrix; it is also easy to see that the conditions of the theorem are necessary, and this completes the proof. □

Theorem 2.9. *st-N(q) convergence implies st-N(p) convergence if and only if there exists a positive constant M such that*

$$|h_1|Q_n + |h_2|Q_{n-1} + ... + |h_n|Q_1 \leq MP_n \tag{2.3}$$

and

$$\lim_{n\to\infty} \frac{h_n}{P_n} = 0 \tag{2.4}$$

are hold.

The proof can be obtained easily from the proof of Theorem 2.8. So it is omitted here.

Theorem 2.10. *st-N(q) (or st-N(p)) convergence implies st-N(p) (or st-N(q)) convergence if and only if the associated series $\sum_{n=1}^{\infty}|k_n|$ (or $\sum_{n=1}^{\infty}|h_n|$) is convergent.*

Proof. "$\Rightarrow$" Since st-N(q) convergence implies st-N(p) convergence, then from (2.1) there exists $M>0$ such that

$$0<k_1P_n<MQ_n$$

hold for all $n\in\mathbb{N}$. It is clear from the last inequality that the sequence $\left(\frac{P_n}{Q_n}\right)_{n\in\mathbb{N}}$ is bounded. So, from (2.1), we have

$$|k_1|+|k_2|\frac{P_{n-1}}{P_n}+...+|k_m|\frac{P_{n-m+1}}{P_n}\le M\frac{Q_n}{P_n}.$$

for every $m<n$. Therefore,

$$|k_1|+|k_2|+...+|k_m|\le M\frac{Q_n}{P_n}$$

because of

$$\limsup_{n\to\infty}\frac{P_{n-k}}{P_n}=1,$$

for all $k\in\{1,2,...,m\}$. The last inequality gives that $\sum_{m=1}^{\infty}|k_m|$ is convergent.

"$\Leftarrow$" It is enough to show that (2.1) and (2.2) are hold for st-N(q) convergence implies st-N(p) convergence.

From the assumption $\lim_{n\to\infty}|k_n|=0$ and this gives

$$\lim_{n\to\infty}\frac{k_n}{Q_n}=0.$$

Also, from (1.4),

$$P_n=Q_1h_n+...+Q_nh_1\le Q_n\sum_{n==1}^{\infty}|h_n|$$

holds for all $n\in\mathbb{N}$. So that

$$P_n|k_1|+P_{n-1}|k_2|+...+P_1|k_n|\le Q_n\sum_{n=1}^{\infty}|h_n|\sum_{n=1}^{\infty}|k_n|$$

holds. This gives the expected result.

The other case can be proved by doing suitable changes above. □

Corollary 2.11. *If (p_n) is increasing, then st-N(p) is stronger than statistical convergence.*

If we consider monotone increasing (p_n) and $q_n=1$, for all $n\in\mathbb{N}$ in Theorem 2.9, it is easy to see that (2.3) and (2.4) are hold. So, the proof is omitted here.

Theorem 2.12. *If the sequence (x_n) is st-N(p) and st-N(q) convergent to l, then there exists a sequence (r_n) of positive integers such that (x_n) is st-N(r) convergent to l.*

Proof. Let us denote st-N(q) transformation of the sequence (x_n) by (t_n^q). Take into consider the sequence (r_n) as follows:

$$r_n:=p_nq_1+p_{n-1}q_2+...+p_1q_n, \tag{2.5}$$

for all $n=1,2,....$ The st-N(r) transformation of (x_n) is

$$t_n^r:=\frac{1}{R_n}\sum_{k=1}^{n}r_{n-k+1}\chi_{K(\varepsilon)}(k)$$

where $R_n=r_1+r_2+...+r_n$. From (2.5), we have

$$\begin{aligned}
t_n^r &= \frac{r_n\chi_{K(\varepsilon)}(1)+r_{n-1}\chi_{K(\varepsilon)}(2)+...+r_1\chi_{K(\varepsilon)}(n)}{p_1q_1+(p_1q_2+p_2q_1)+...+(p_1q_n+...+p_nq_1)}\\
&= \frac{p_1q_1\chi_{K(\varepsilon)}(n)+(p_1q_2+p_2q_1)\chi_{K(\varepsilon)}(n-1)+...+(p_nq_1+...+p_1q_n)\chi_{K(\varepsilon)}(1)}{p_1q_1+(p_1q_2+p_2q_1)+...+(p_1q_n+...+p_nq_1)}\\
&= \frac{p_1(q_1\chi_{K(\varepsilon)}(n)+q_2\chi_{K(\varepsilon)}(n-1)+...+q_n\chi_{K(\varepsilon)}(1))+....+p_nq_1\chi_{K(\varepsilon)}(1)}{p_1(q_1+...+q_n)+...+p_nq_1}\\
&= \frac{p_1Q_nt_n^q+...+p_nQ_1t_1^q}{p_1Q_n+p_2Q_{n-1}+...+p_nQ_1}.
\end{aligned}$$

Therefore, if we consider the matrix $T = (t_{nk})$ as

$$t_{nk} := \begin{cases} \frac{p_{n-k+1}Q_k}{\sum_{i=1}^{n} p_{n-i+1}Q_i}, & k \leq n, \\ 0, & k > n. \end{cases}$$

then st-N(r) transformation of (x_n) is the T-transformation of the sequence (t_n^q).

Now, it is enough for to completion of the proof the matrix $T = (t_{nk})$ is regular. It is evident that

$$\sum_{k=1}^{n} t_{nk} = \sum_{k=1}^{n} |t_{nk}| = 1, \ n = 1, 2, ...$$

hold. Since, $\sum_{i=1}^{n} p_{n-i+1}Q_i > KP_n$ for $K \geq q_1 > 0$ and for all fixed $k \in \mathbb{N}$,

$$0 \leq \lim_{n\to\infty} \frac{p_{n-k+1}Q_n}{\sum_{i=1}^{n} p_{n-i+1}Q_i} < \lim_{n\to\infty} \frac{p_{n-k+1}Q_n}{KP_n} = 0.$$

Therefore, the sequence (x_n) is also st-N(r) convergent to l.

The st-N(r) convergency of (x_n) when it is st-N(p) convergence can be obtain easily by follows the proof. $\square$

Definition 2.13. It is said to be a properties $u(n)$ is hold statistically almost all n (st-a.a.n) (or statistically Nörlund almost all n (st-N(p) a.a.n)), if the set $A = \{n \in \mathbb{N} \mid u(n) \quad \text{is not satisfied}\}$ has natural density (or st-N(p) density) zero.

In the following theorems st-N(p) and Statistical convergence is compared under the same restriction on $p = (p_n)$.

Theorem 2.14. *Assume that $p_n \geq 1$ (st-a.a.n). Then, $x_n \to l$ st-N(p) implies $x_n \to l$ (st) if and only if*

$$\limsup_{n\to\infty} \frac{P_n}{n} < \infty. \tag{2.6}$$

Proof. From the assumption

$$\delta(A) = 0$$

where $A = \{n \in \mathbb{N} \mid p_n < 1\}$. Therefore, following inequality

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} \chi_{K(\varepsilon)}(k) &= \frac{1}{n}\left(\sum_{k\in K(\varepsilon)\cap A} \chi_{K(\varepsilon)}(k) + \sum_{k\in K(\varepsilon)\cap A^c} \chi_{K(\varepsilon)}(k)\right) \\ &\leq \frac{1}{n}\sum_{k\in K(\varepsilon)\cap A} \chi_{K(\varepsilon)}(k) + \frac{1}{n}\sum_{k\in K(\varepsilon)\cap A^c} \chi_{K(\varepsilon)}(k) \\ &\leq \frac{1}{n}\sum_{k\in K(\varepsilon)\cap A} \chi_{K(\varepsilon)}(k) + \left(\frac{P_n}{n}\right)\frac{1}{P_n}\sum_{k\in K(\varepsilon)\cap A^c} p_{n-k+1}\chi_{K(\varepsilon)}(k) \quad (2.7) \end{aligned}$$

hold. Since $K(\varepsilon) \cap A \subset A$ and $K(\varepsilon) \cap A^c \subset K(\varepsilon)$ then

$$\frac{1}{n}\sum_{k\in K(\varepsilon)\cap A} \chi_{K(\varepsilon)}(k) \to 0 \quad \text{and} \quad \frac{1}{P_n}\sum_{k\in K(\varepsilon)\cap A^c} p_{n-k+1}\chi_{K(\varepsilon)}(k) \to 0$$

when $n \to \infty$. So, if we take limit each side of (2.7), we obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k\in K(\varepsilon)} \chi_{K(\varepsilon)}(k) = 0$$

if and only if (2.6) holds. $\square$

Corollary 2.15. *If $p_n \geq 1$ for every $n \in \mathbb{N}$, then, $x_n \to l$(st-N(p)) implies $x_n \to l$(st) if and only if* (2.6) *holds.*

Theorem 2.16. *Assume that $p_n < 1$ (st-N(p) a.a.n). Then, $x_n \to l$ (st) implies $x_n \to l$ (st-N(p)) if and only if*

$$\limsup_{n\to\infty} \frac{n}{P_n} < \infty. \tag{2.8}$$

Proof. Let us consider the set

$$A = \{n \in \mathbb{N} \mid p_n \geq 1\}.$$

Then, from the assumption $\delta_p(A) = 0$.

Therefore, the following inequality

$$\begin{aligned}\frac{1}{P_n} \quad & \sum_{k=1}^{n} p_{n-k+1}\chi_{K(\varepsilon)}(k) = \\ = \quad & \frac{1}{P_n}\left(\sum_{k=1}^{n} p_{n-k+1}\chi_{K(\varepsilon)\cap A}(k) + \sum_{k=1}^{n} p_{n-k+1}\chi_{K(\varepsilon)\cap A^c}(k)\right) \\ \leq \quad & \frac{1}{P_n}\sum_{k=1}^{n} p_{n-k+1}\chi_{K(\varepsilon)\cap A}(k) + \frac{1}{P_n}\sum_{k=1}^{n} p_{n-k+1}\chi_{K(\varepsilon)\cap A^c}(k) \\ \leq \quad & \frac{1}{P_n}\sum_{k=1}^{n} p_{n-k+1}\chi_{K(\varepsilon)\cap A}(k) + \frac{n}{P_n}\frac{1}{n}\sum_{k=1}^{n} p_{n-k+1}\chi_{K(\varepsilon)}(k)\end{aligned}$$

hold for all $n \in \mathbb{N}$.

So, the st-N(p) convergence of x_n to l is obtained by st-convergence if and only if

$$\limsup_{n\to\infty} \frac{n}{P_n} < \infty.$$

□

Corollary 2.17. *If $p_n < 1$ for all $n \in \mathbb{N}$, then, $x_n \to$ l(st) implies $x_n \to$ l(st-N(p)) if and only if* (2.8) *holds.*

Definition 2.18. The real valued sequences $x = (x_n)$ and $y = (y_n)$ are called st-N(p) equivalent if the set $\{n \mid x_k \neq y_k\}$ has zero st-N(p) density. It is denoted by $x \asymp y$(st-N(p)).

Theorem 2.19. *If $x \asymp$ y(st-N(p)), then $x_n \to$ l(st-N(p)) if and only if $y_n \to$ l(st-N(p)).*

Proof. Assume that $x \asymp y$(st-N(p)) and $x_n \to l$(st-N(p)). If we consider the set $A = \{n \mid x_n = y_n\}$, it is clear that $\delta_p(A^c) = 0$. Also, let us denote the set for every $\varepsilon > 0$,

$$K_x = \{k \mid k \leq n, |x_k - l| \geq \varepsilon\}$$

and

$$K_y = \{k \mid k \leq n, |y_k - l| \geq \varepsilon\}$$

such that from the assumption we have

$$\delta_p(K_x) = 0.$$

It is clear that

$$K_y = (K_y \cap A) \cup (K_y \cap A^c) \subseteq K_x \cup A^c$$

and the inequality

$$\delta_p(K_y) \leq \delta_p(K_x) + \delta_p(A^c)$$

hold. If we take limit from each side above we obtain the proof of our assertion.

Similarly, it is proved that $y_n \to l$(st-N(p)) when $x_n \to l$(st-N(p)). □

Remark 2.20. The converse of Theorem 2.19 is not true in general.

Let us consider the sequence $x = (x_n)$ and $y = (y_n)$ as follows:

$$x_n = 1 + \frac{1}{n}, \qquad y_n = 1 - \frac{1}{n}$$

for all $n \in \mathbb{N}$. Since $x_n \to 1$, $y_n \to 1$ $(n \to \infty)$, then $x_n \to 1$ (st-N(p)) and $y_n \to 1$ (st-N(p)) for any regular N(p) method. But the set

$$A = \{n \mid x_n \neq y_n\} = \mathbb{N}$$

and

$$\delta_p(A) = 1.$$

Therefore, (x_n) and (y_n) are not st-N(p) equivalent.

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Author information

Maya Altınok, Zehra Kurtdişi and Mehmet Küçükaslan, Mersin University Faculty of Science, 33343 Mersin TURKEY, Turkey.
E-mail: mayaaltinok@mersin.edu.tr, zehrakurtdisi@hotmail.com and mkucukaslan@mersin.edu.tr or mkkaslan@gmail.com

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