RAD-$D_{12}$ MODULES

Recep Kılıç and Burcu Nişancı Türkmen

MSC 2010 Classifications: Primary 16D10; Secondary 16P70.

Keywords and phrases: (cofinitely) Rad-$D_{12}$ module, (semi-)perfect ring, artinian serial ring.

This work is supported by Scientific Research Project in Amasya University (FMB-BAP-13-051).

Abstract Let $R$ be a ring and $M$ be a right $R$-module. $M$ is called (cofinitely) Rad-$D_{12}$ if, for every (cofinite) submodule $N$ of $M$, there exist a direct summand $K$ of $M$ and an epimorphism $\psi : K \to \frac{M}{N}$ with $\ker(\alpha) \subseteq \text{Rad}(K)$. In this paper, we provide various properties of Rad-$D_{12}$ modules and cofinitely Rad-$D_{12}$ modules. In particular, we characterize semiperfect rings, perfect rings and artinian serial rings using (cofinitely) Rad-$D_{12}$ modules. Moreover, we prove that every quasi-projective Rad-$D_{12}$ module is Rad-$\oplus$-supplemented. Finally, we show that any factor module of a (cofinitely) Rad-$D_{12}$ module by a fully invariant submodule is (cofinitely) Rad-$D_{12}$.

1 Introduction

Throughout this paper, it is assumed that $R$ is an associative ring with identity and all modules are unital right $R$-modules. Let $M$ be a module. A submodule $N$ of an $R$-module $M$ will be denoted by $N \subseteq M$. A submodule $N \subseteq M$ is said to be cofinite if $\frac{M}{N}$ is finitely generated. Maximal submodules are cofinite. Also, every submodule of a finitely generated module is cofinite. A submodule $L \subseteq M$ is said to be essential in $M$, if every submodule $L \subseteq M$, $L \cap N \neq 0$ for every nonzero submodule $N \subseteq M$. $M$ is said to be uniform if its submodules is essential in $M$, and it is said to be extending (or a CS-module) if every submodule of $M$ is essential in a direct summand of $M$. Dually, a module $M$ is called lifting (or $D_1$) if, every submodule $N$ of $M$ contains a direct summand $L$ of $M$ such that $M = L \oplus K$ and $N \cap K$ is small in $M$. Here a submodule $S$ of $M$ is called small in $M$ if $S + K \neq M$ for every proper submodule $K$ of $M$. In [5, 29.10], every right $R$-module is lifting if and only if $R$ is a left and right artinian serial ring with $J^2 = 0$, where $J$ is the Jacobson radical of $R$. A module $M$ is called hollow (or couniform) if every submodule is small in $M$. Hollow and semisimple modules are lifting. If $M$ has a largest proper submodule, i.e., a proper submodule which contains all other proper submodules, then $M$ is called local [23].

As a generalization of direct summands, a submodule $V$ of $M$ is called a supplement of a submodule $U$ in $M$ if $M = U + V$ and $U \cap V \ll V$ [23, pp. 348]. A module $M$ is called (cofinitely) supplemented if every (cofinite) submodule has a supplement in $M$, and it is called amply (cofinitely) supplemented if, whenever (cofinite submodule $N$) $M = N + K$, $N$ has a supplement $V \subseteq K$ in $M$. Every right $R$-module is (cofinitely) supplemented if and only if $R$ is right (semi)perfect [1, Theorem 2.13] and [23, 43.9]. It can be seen that $M$ is lifting if and only if it is amply supplemented and every supplement in $M$ is a direct summand of $M$. Following [18], $M$ is said to be cofinitely lifting if it is amply cofinitely supplemented and supplements of a cofinite submodule $M$ is a direct summand of $M$.

Mohamed and Müller [12] call a module $M$ $\oplus$-supplemented (according to [8], $(D_{11})$) if every submodule $N$ of $M$ has a supplement that is a direct summand of $M$. Lifting modules are $\oplus$-supplemented. Zöschinger proved in [25] that every supplemented module over a dedekind domain is $\oplus$-supplemented. Moreover, it follows from [8, Theorem 1.1] a commutative ring $R$ is an artinian principal ideal ring if and only if every right $R$-module is $\oplus$-supplemented. In [7], a module $M$ is called $\oplus$-cofinitely supplemented if every cofinite submodule of $M$ has a supplement that is a direct summand of $M$. For a module $M$, consider the following condition:

$$(D_{12})$$

For every submodule $N$ of $M$, there exist a direct summand $K$ of $M$ and an epimorphism $\alpha : \frac{M}{K} \to \frac{M}{N}$ such that $\ker(\alpha) \ll \frac{M}{K}$.

Modules with the property $(D_{12})$ are extensively studied in [11]. In addition, it is proven in [11, Proposition 4.3] that every $\oplus$-supplemented module has the property $(D_{12})$. Wang [22]
generalizes modules with \((D_{12})\) to cofinitely \((D_{12})\)-modules. A module \(M\) is called cofinitely \((D_{12})\) if, for every cofinite submodule \(N\) of \(M\), there exist a direct summand \(K\) of \(M\) and an epimorphism \(\alpha : \frac{M}{N} \to \frac{M}{N}\) such that \(\ker(\alpha) \subseteq \frac{M}{N}\). He obtained various properties of these modules in the same paper.

We will denote by \(\text{Rad}(M)\), namely radical of \(M\), the sum of all small submodules of a module \(M\). We say that a submodule \(V\) of a module \(M\) is Rad-supplement \((\text{in} [24], \text{generalized supplement})\) of a submodule \(U\) in \(M\) if \(M = U + V\) and \(U \cap V \subseteq \text{Rad}(V)\) as in [5, pp. 100]. Clearly, we have the following diagram on submodules.

| direct summand | \(\implies\) | supplement | \(\implies\) | Rad-supplement |

Motivated by the above definitions, we say a module \(M\) is (cofinitely) Rad-supplemented if every (cofinite) submodule of \(M\) has a Rad-supplement in \(M\), and \(M\) is (cofinitely) Rad-\(\oplus\)-supplemented if every (cofinite) submodule of \(M\) has a Rad-supplement that is a direct summand in \(M\) as in these papers [4], [13] and [19]. Characterizations of those modules are studied in the same papers. From [13], we will use \(cgs\) instead of cofinitely Rad-\(\oplus\)-supplemented.

Recall from [1] that a module \(M\) has the property \((P^*)\) if, for every submodule \(N\) of \(M\), \(M\) has the decomposition \(M = L \oplus K\) such that \(L \subseteq N\) and \(N \cap K \subseteq \text{Rad}(K)\). Modules with the property \((P^*)\) is a dual notion of modules with the property \((P)\) which is a generalization of extending modules [1]. Clearly lifting modules have the property \((P^*)\). Also, by [19, Proposition 2.9], a projective module with the property \((P^*)\) is lifting.

Talebi et.al. call a module \(M\) Rad-\(D_{12}\) if, for every submodule \(N\) of \(M\), there exist a direct summand \(K\) of \(M\) and an epimorphism \(\psi : K \to \frac{M}{N}\) with \(\ker(\psi) \subseteq \text{Rad}(K)\). Some properties of Rad-\(D_{12}\) modules are given in [17]. It is shown in [17, Proposition 2.1] that every Rad-\(\oplus\)-supplemented module is Rad-\(D_{12}\). It is of obvious interest to study characterizations of Rad-\(D_{12}\) modules by rings. In Theorem 2.9, we will prove that a ring \(R\) is right perfect if and only if every projective right \(R\)-module is Rad-\(D_{12}\). In particular, we shall show in Theorem 2.16 that every right \(R\)-module is Rad-\(D_{12}\) if and only if a commutative ring \(R\) is an artinian serial ring.

Let \(M\) be a module. We call a module \(M\) cofinitely Rad-\(D_{12}\) if for every cofinite submodule \(N\) of \(M\), there exist a direct summand \(K\) of \(M\) and an epimorphism \(\alpha : K \to \frac{M}{N}\) such that \(\ker(\alpha) \subseteq \text{Rad}(K)\). We will investigate various properties of cofinitely Rad-\(D_{12}\) modules in section 2.

Under given definitions, we clearly have the following implication on modules:

\[
\begin{align*}
\text{semisimple} & \quad \downarrow \quad \text{lifting} \\
\text{the property \((P^*)\)} & \quad \oplus - \text{supplemented} \\
\text{Rad} - \oplus - \text{supplemented} & \quad \downarrow \quad \text{cofinitely lifting} \\
D_{12} & \quad \downarrow \quad \oplus - \text{cofinitely supplemented} \\
cgs \quad \downarrow \quad \text{cofinitely} Rad - D_{12} \\
\text{cofinitely} \text{Rad} - D_{12} & \quad \downarrow \quad \text{cofinitely} (D_{12}) \\
\end{align*}
\]

In this paper, we give a new characterization of semiperfect rings via cofinitely Rad-\(D_{12}\) modules. Every non-radical indecomposable cofinitely Rad-\(D_{12}\)-module is \(\omega\)-local. We show that if every right \(R\)-module is cofinitely Rad-\(D_{12}\), then \(R\) is a noetherian serial ring.

## 2 (Cofinitely) Rad-\(D_{12}\) Modules

In this section, we will give characterizations of (semi)perfect rings in terms of (cofinitely) Rad-\(D_{12}\). In particular, we will determine commutative rings whose modules are Rad-\(D_{12}\).
Recall from [23] that an epimorphism \( f : P \rightarrow M \) is called a cover if \( \text{Ker}(f) \ll P \), and a cover \( f \) is called a projective cover if \( P \) is a projective module. In the spirit of [23], a module \( M \) is said to be semiperfect if every factor module of \( M \) has a projective cover. Every semiperfect module is supplemented. A ring \( R \) is called semiperfect, if every finitely generated right (or left) \( R \)-module has a projective cover, and a ring \( R \) is called right perfect if every right \( R \)-module has a projective cover.

The proof of the next result is taken from [17, Proposition 2.1], but is given for the sake of completeness.

**Proposition 2.1.** Every \( \text{cgs}^{\oplus} \)-module is cofinitely \( \text{Rad-}D_{12} \).

**Proof.** Let \( N \) be a cofinite submodule of \( M \). Since \( M \) is \( \text{cgs}^{\oplus} \), then there exist direct summands \( K \) and \( K' \) of \( M \) such that \( M = N + K = K \oplus K' \) and \( N \cap K \subseteq \text{Rad}(K) \). Now we have the epimorphism \( g \) from \( K \) to \( \frac{M}{K} \) which is defined by \( k \rightarrow k + N \) with \( \text{Ker}(g) = N \cap K \subseteq \text{Rad}(K) \).

Hence \( M \) is a cofinitely \( \text{Rad-}D_{12} \)-module.

The following example shows that a cofinitely \( \text{Rad-}D_{12} \) module need not \( \text{cgs}^{\oplus} \).

**Example 2.2.** (See [11, Examples 4.5\&4.6]) Let \( R \) be a local artinian ring with radical \( J \) such that \( J^2 = 0, Q = \frac{J}{J^2} \) is commutative, \( \text{dim}(QJ) = 2 \) and \( \text{dim}(JQ) = 1 \). Consider the indecomposable injective right \( R \)-module \( U = \frac{(R/B)D}{D} \) with \( J = Ru + Rv \) and \( D = \{(ar − vr) | r \in R \} \). Now let \( S = \frac{Q}{J} \), the simple \( R \)-module, and \( M = U \oplus S \). By [11, Example 4.6], \( M \) is cofinitely \( \text{Rad-}D_{12} \), but not \( \text{cgs}^{\oplus} \).

Recall from [23] that an \( R \)-module \( M \) is called quasi-projective if, for every \( R \)-module \( K \), every \( R \)-epimorphism \( \xi : M \rightarrow K \), and every \( R \)-homomorphism \( f : M \rightarrow K \), there is an \( \gamma \in \text{End}_R(M) \) such that \( \xi \circ \gamma = f \). Now we prove that every quasi-projective (cofinitely) \( \text{Rad-}D_{12} \) module is \( \oplus \)-supplemented (\( \text{cgs}^{\oplus} \)).

**Theorem 2.3.** Let \( M \) be a quasi-projective module.

(i) If \( M \) is \( \text{Rad-}D_{12} \), then \( M \) is a \( \oplus \)-supplemented module.

(ii) If \( M \) is cofinitely \( \text{Rad-}D_{12} \), then \( M \) is a \( \text{cgs}^{\oplus} \)-module.

**Proof.** (1) Let \( N \) be a submodule of \( M \). Then there exist a direct summand \( K \) of \( M \) and an epimorphism \( \alpha : K \rightarrow \frac{M}{N} \) with \( \text{Ker}(\alpha) \subseteq \text{Rad}(K) \). Let \( \pi : M \rightarrow \frac{M}{N} \) be the natural epimorphism. Since \( M \) is a quasi-projective, we have the homomorphism \( h : M \rightarrow K \) with \( \pi = \alpha \circ h \). Since \( K \) is \( M \)-projective, \( h \) splits. Hence there is a direct summand \( K' \) of \( M \) with \( h|_{K'} : K' \cong K \). So \( \pi|_{K'} \) is an epimorphism. Therefore \( M = K' + N \) and \( N \cap K' = \text{Ker}(\pi|_{K'}) \subseteq \text{Rad}(M) \).

Since \( K' \) is a direct summand of \( M \), \( N \cap K' \subseteq \text{Rad}(K') \). Thus \( M \) is \( \oplus \)-supplemented.

(2) The proof can be made similar to (1).

Clearly, every cofinitely \( D_{12} \)-module is cofinitely \( \text{Rad-}D_{12} \). But the converse is not always true the following example shows. Recall from [4] that a module \( M \) is called \( \omega \)-local if it has a unique maximal submodule. It is clear that a module is \( \omega \)-local if and only if its radical is maximal.

**Example 2.4.** (See [16, Theorem 4.3 and Remark 4.4]) Let \( M \) be a biuniform module and let \( S = \text{End}_R(M) \). Assume that \( P \) is a projective \( S \)-module with \( \text{dim}(P) = (1, 0) \). Then \( P \) is an indecomposable \( \omega \)-local module. Since \( \text{dim}(P) = (1, 0) \), we conclude that \( P \) is not finitely generated. Hence, \( P \) is a \( \text{cgs}^{\oplus} \)-module but not \( \oplus \)-cofinitely supplemented. Thus, \( P \) is cofinitely \( \text{Rad-}D_{12} \) but not cofinitely \( \text{cgs}^{\oplus} \).

**Example 2.5.** (See [9, 11.3]) Let \( R \) denote the ring \( K[[x]] \) of all power series \( \sum_{i=0}^{\infty} k_i x^i \) in an indeterminate \( x \) and with coefficients from a field \( K \) which is a local ring. Note that \( R \) is a semiperfect ring that is not perfect. Then by [7, Theorem 2.9] and [10, Corollary 2.11], the free (projective) \( R \)-module \( R^X \) is \( \oplus \)-cofinitely supplemented but not \( \oplus \)-supplemented. It follows that \( R^X \) is \( \text{cgs}^{\oplus} \). By [19, Theorem 2.2], \( R^X \) is not \( \text{Rad-} \)-supplemented. Therefore \( R^X \) is cofinitely \( \text{Rad-}D_{12} \) but not \( \text{Rad-}D_{12} \) by Theorem 2.3.

**Theorem 2.6.** For a ring \( R \), \( R \) is semiperfect if and only if every free right \( R \)-module is cofinitely \( \text{Rad-}D_{12} \).

**Proof.** (\( \Rightarrow \)) Suppose that a ring \( R \) is semiperfect. By [13, Theorem 2.4], every free right \( R \)-module is \( \text{cgs}^{\oplus} \). Then by Proposition 2.1, every free right \( R \)-module is cofinitely \( \text{Rad-}D_{12} \).

(\( \Leftarrow \)) Since every free right \( R \)-module is cofinitely \( \text{Rad-}D_{12} \), it is \( \text{cgs}^{\oplus} \) by Theorem 2.3. It follows from [13, Theorem 2.4] that \( R \) is semiperfect. \( \Box \)
Proposition 2.7. Let $M$ be a cofinitely Rad-$D_{12}$ module. If $\text{Rad}(M) \ll M$, then $M$ is a cofinitely $(D_{12})$-module.

Proof. Let $N$ be a cofinite submodule of $M$. Since $M$ is cofinitely Rad-$D_{12}$, there exist a direct summand $K$ of $M$ and an epimorphism $\alpha : K \to M/N$ such that $\ker(\alpha) \ll M$. Since $K$ is a direct summand of $M$, $\ker(\alpha) \ll K$. Hence $M$ is a cofinitely $(D_{12})$-module.

A module $M$ is called coatomic if every proper submodule is contained in a maximal submodule of $M$. Every coatomic module has a small radical. Using the above proposition, we obtain the following corollary.

Corollary 2.8. Every coatomic cofinitely Rad-$D_{12}$ module is cofinitely $(D_{12})$.

Theorem 2.9. For a ring $R$, $R$ is right perfect if and only if every projective right $R$-module is Rad-$D_{12}$.

Proof. The proof follows from Theorem 2.3(1) and [19, Corollary 2.3].

A module $M$ is called radical if $\text{Rad}(M) = M$.

Proposition 2.10. Let $M$ be a non-radical indecomposable module. Suppose that $M$ is a cofinitely Rad-$D_{12}$ module. Then $M$ is $\omega$-local.

Proof. Suppose that $\text{Rad}(M) \neq M$. Then $M$ contains a maximal submodule $N$. By the hypothesis, there exist a direct summand $K$ of $M$ and an epimorphism $\alpha : K \to M/N$ with $\ker(\alpha) \subseteq \text{Rad}(K)$. Note that $K \neq 0$. Since $M$ is indecomposable, $K = M$. Therefore $\alpha : M \to M/N$ is an epimorphism with $\ker(\alpha) \subseteq \text{Rad}(M)$. It follows that $M/\ker(\alpha) \cong M/N$. Since $N$ is a maximal submodule of $M$, $\ker(\alpha)$ is a maximal submodule of $M$. But $\ker(\alpha) \subseteq \text{Rad}(M)$. Thus $\text{Rad}(M)$ is a maximal submodule of $M$. Hence $M$ is $\omega$-local.

Corollary 2.11. Every finitely generated indecomposable, (cofinitely) Rad-$D_{12}$ module is local.

In [15, 1.4] a module $M$ is called uniserial if its lattice of submodules is a chain. By [5, 2.17], a module $M$ is uniserial if and only if every submodule of $M$ is hollow. A module $M$ is said to be serial if $M$ is a direct sum of uniserial modules. A commutative ring $R$ is called uniserial if the module $R$ is uniserial, and the ring is called serial if the module $R$ is serial.

Recall from [5, 1.5] that a module $M$ is uniform if and only if every non-zero submodule of $M$ is indecomposable.

Proposition 2.12. Let $M$ be a uniform module over a local commutative ring $R$. Then the following statements are equivalent.

(i) $M$ is uniserial.

(ii) Every submodule of $M$ is cofinitely Rad-$D_{12}$.

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (i) Let $N$ be a finitely generated submodule of $M$. By (2), $N$ is Rad-$D_{12}$. Since $N$ is indecomposable, applying Corollary 2.11 we obtain that $N$ is local. It follows from [5, 2.17] that $M$ is uniserial.

By $E(M)$ we denote the injective hull of a module $M$. Note that the injective hull of a simple module is uniform.

Corollary 2.13. Let $R$ be a local commutative ring. Suppose that $M$ is the module $E(M/R_{\text{Rad}(R)})$, and every submodule of $M$ is cofinitely Rad-$D_{12}$. Then, $R$ is uniserial.

Proof. Since $M$ is uniform, the hypothesis implies that $M$ is uniserial by Proposition 2.12. It follows from [20, 6.2] that $R$ is uniserial.

Lemma 2.14. (See [8, Theorem 1.1], [19, Corollary 2.15]) Let $R$ be a commutative ring. Then the following statements are equivalent.

(i) $R$ is an artinian serial ring.

(ii) Every $R$-module is $\oplus$-supplemented.

(iii) Every $R$-module is Rad-$\oplus$-supplemented.
By Lemma 2.14, every module over an artinian serial ring is Rad-$D_{12}$. Now we show that the converse of this fact is true in the following Theorem. Firstly, we have:

**Proposition 2.15.** Let $R$ be a commutative ring. If every right $R$-module is cofinitely Rad-$D_{12}$, then $R$ is a serial ring.

**Proof.** Let $M$ be a free $R$-module. By the hypothesis, $M$ is cofinitely Rad-$D_{12}$. It follows from Theorem 2.6 that $R$ is semiperfect. Note that $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$ such that the ring $R_i$ is local for all $1 \leq i \leq n$ with $n \in \mathbb{N}$ (23, 42.6). For all $1 \leq i \leq n$, $R_i$ is commutative and every $R_i$-module is cofinitely Rad-$D_{12}$ by assumption. Using Corollary 2.13, we get $R_i$ is uniserial. Thus $R$ is a serial ring. \hfill \Box

**Theorem 2.16.** The following statements are equivalent for a commutative ring $R$.

(i) $R$ is an artinian serial ring.

(ii) Every $R$-module is Rad-$D_{12}$.

**Proof.** (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (i) Applying Theorem 2.9, we obtain that $R$ is perfect. It follows from ([23, 42.6]) that we can write $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$, where each $R_i$ is a local perfect ring for all $1 \leq i \leq n$. By Corollary 2.13 and the hypothesis, it can be seen easily that each $R_i$ is noetherian. Therefore, $R$ is a serial noetherian ring as a finite direct sum of uniserial noetherian rings $R_i$. Applying [9, 11.6.4(c)], we deduce that $R$ is an artinian serial ring. \hfill \Box

Let $M$ be a module. $U \subseteq M$ is called QSL in $M$ if $(A+U)/A$ is a direct summand of $M/A$, then there exists a direct summand $P$ of $M$ such that $P \subseteq A$ and $A+U = P+U$ [3]. $M$ is said to be cofinitely weak Rad-supplemented if every cofinite submodule $U$ of $M$ has a weak Rad-supplement in $M$, i.e. there exists a submodule $V$ of $M$ such that $M = U+V$ and $U \cap V \subseteq Rad(M)$ [6].

**Proposition 2.17.** Let $M$ be a cofinitely weak Rad-supplemented module with $Rad(M)$ QSL in $M$. Then $M$ is cofinitely Rad-$D_{12}$.

**Proof.** Let $N$ be a cofinite submodule of $M$. Then $M/N$ is cofinitely generated, and so $M/N$ is a module. Thus $M/N = M/\text{Rad}(M)$ is a submodule of $M/\text{Rad}(M)$. By [6, Corollary 2.5], $N/\text{Rad}(M)$ is a direct summand of $M/\text{Rad}(M)$. Since $Rad(M)$ is QSL in $M$, there exists a decomposition $M = K \oplus L$ such that $K \subseteq N$ and $N + \text{Rad}(M) = K + \text{Rad}(M)$. Now consider the epimorphism $\alpha : L \rightarrow M$ defined by $\alpha(l) = l + N (l \in L)$. Since $M = K \oplus L$, then $\text{Rad}(M) = \text{Rad}(K) \oplus \text{Rad}(L)$. It follows that $N + \text{Rad}(L) = K + \text{Rad}(L)$ and, so $L \cap N + \text{Rad}(L) = L \cap K + \text{Rad}(L) = \text{Rad}(L)$. Note that $\text{Ker}(\alpha) = L \cap N \subseteq \text{Rad}(L)$. Hence $M$ is cofinitely Rad-$D_{12}$. \hfill \Box

A module $M$ is called refinable if for any submodules $U, V$ of $M$ with $M = U + V$, there exists a direct summand $U'$ of $M$ with $U' \subseteq U$ and $M = U' + V$ [5, 11.26]. It is easy to see that $M$ is refinable if and only if every submodule of $M$ is QSL.

**Corollary 2.18.** Let $M$ be a cofinitely weak Rad-supplemented refinable module. Then $M$ is cofinitely Rad-$D_{12}$.

**Proof.** Clear by Proposition 2.17. \hfill \Box

**Proposition 2.19.** Let $M$ be a cofinitely Rad-$D_{12}$ module. If $\text{Rad}(M) \neq M$, then $M$ has a non-zero $\omega$-local direct summand.

**Proof.** Let $N$ be a maximal submodule of $M$. Then $N$ is a cofinite submodule of $M$. Since $M$ is a cofinitely Rad-$D_{12}$ module, there exist a direct summand $K$ of $M$ and an epimorphism $\alpha : K \rightarrow M$ such that $\text{ker}(\alpha) \subseteq \text{Rad}(K)$. Clearly, $K \neq 0$ and $\text{ker}(\alpha)$ is a maximal submodule of $K$. Therefore $\text{ker}(\alpha) = \text{Rad}(K)$ and hence $K$ is a non-zero $\omega$-local direct summand of $M$. \hfill \Box
Recall from [23] that an $R$-module $M$ has the summand sum property (SSP) if the sum of two direct summands of $M$ is again a direct summand of $M$, and a submodule $U$ of an $R$-module $M$ is called fully invariant if $f(U)$ is contained in $U$ for every $R$-endomorphism $f$ of $M$. Let $M$ be an $R$-module and let $\tau$ be a preradical for the category of $R$-modules. Then $\text{Rad}(M)$, $P(M)$ and $\tau(M)$ are fully invariant submodules of $M$. An $R$-module $M$ is called a (weak) duo module if every (direct summand) submodule of $M$ is fully invariant. Note that weak duo modules have SSP (See [14]).

The following Example shows a cofinitely Rad-$D_{12}$ module that contains a direct summand which is not cofinitely Rad-$D_{12}$.

**Example 2.20.** Consider the right $R$-module $M = U \oplus S$ in Example 2.2. The module $M$ is cofinitely Rad-$D_{12}$, but the submodule $U$ is not cofinitely Rad-$D_{12}$.

**Theorem 2.21.** Let $M = M_1 \oplus M_2$. Then $M_2$ is cofinitely Rad-$D_{12}$ if and only if for every cofinite submodule $N$ of $M$ containing $M_1$, there exist a direct summand $K$ of $M_2$ and an epimorphism $\varphi : M \to \frac{N}{K}$ such that $K$ is a direct summand Rad-supplement of $\ker(\varphi)$ in $M$.

**Proof.** Suppose that $M_2$ is a cofinitely Rad-$D_{12}$ module. Let $N$ be a cofinite submodule of $M$ with $M_1 \subseteq N$. Consider the submodule $N \cap M_2$ of $M_2$. Since $\frac{M_2}{N \cap M_2} \cong \frac{M_1}{N \cap M_2}$, $N \cap M_2$ is a cofinite submodule of $M_2$. Then there exist a direct summand $K$ of $M_2$ and an epimorphism $\alpha : K \to \frac{M_1}{N \cap M_2}$ such that $\ker(\alpha) = N \cap K \subseteq \text{Rad}(K)$. Note that $M = N + M_2$ and $K$ is a direct summand of $M$. Let $M = K \oplus K'$ for some submodule $K'$ of $M$. Consider the projection map $\xi : M \to K$ and the isomorphism $\beta : \frac{M_1}{N \cap M_2} \to \frac{M}{K}$ defined by $\beta(x + N \cap M_2) = x + N$. Thus $\beta \circ \alpha \circ \xi : M \to \frac{M}{K}$ is an epimorphism. Let $\varphi = \beta \circ \alpha \circ \xi$. Clearly, we have $\ker(\varphi) = N + K' \subseteq \ker(\alpha) \subseteq \text{Rad}(K)$. Moreover $K \cap \ker(\varphi) = K \cap N = \ker(\alpha) \subseteq \text{Rad}(K)$.

Conversely, suppose that every cofinite submodule of $M$ containing $M_1$ has the stated property. Let $H$ be a cofinite submodule of $M_2$. Consider the submodule $H \oplus M_1$ of $M$. Since $\frac{M_1}{H \oplus M_1} \cong \frac{M}{H}$ is finitely generated, $H \oplus M_1$ is a cofinite submodule of $M$. By the hypothesis, there exist a direct summand $K$ of $H \oplus M_1$ and an epimorphism $\mu : M \to \frac{M}{H \oplus M_1}$ such that $M = K + \ker(\mu)$ and $K \cap \ker(\mu) \subseteq \text{Rad}(K)$. Let $g : K \to \frac{M}{H \oplus M_1}$ be the restriction of $\mu$ to $K$. Consider the isomorphism $\eta : \frac{M}{H \oplus M_1} \to \frac{M}{K}$ defined by $\eta(m_1 + m_2 + (H \oplus M_1)) = m_2 + H$. Therefore $\eta \circ g : K \to \frac{M}{K}$ is an epimorphism. Let $\kappa = \eta \circ g$. Clearly, $\ker(\kappa) \subseteq \text{Rad}(K)$. Hence $M_2$ is a cofinitely Rad-$D_{12}$ module.

**Theorem 2.22.** Let $\{M_i\}_{i \in I}$ be any family of cofinitely Rad-$D_{12}$ modules on a ring $R$ and $M = \bigoplus_{i \in I} M_i$. If every cofinite submodule of $M$ is fully invariant, then $M$ is a cofinitely Rad-$D_{12}$ module.

**Proof.** Let $N$ be a cofinite submodule of $M$. Since $N$ is fully invariant, we have $N = \bigoplus_{i \in I}(N \cap M_i)$. Since $\frac{M_i}{N \cap M_i} \cong \bigoplus_{i \in I} \frac{M_i}{N \cap M_i}$ for every $i \in I$, $N \cap M_i$ is a cofinite submodule of $M_i$. Then there exist a direct summand $K_i$ of $M_i$ and an epimorphism $\alpha_i : K_i \to \frac{M_i}{N \cap M_i}$ with $\ker(\alpha_i) \subseteq \text{Rad}(K_i)$. Now we define the homomorphism $\alpha : \bigoplus_{i \in I} K_i \to \bigoplus_{i \in I} \frac{M_i}{N \cap M_i}$ by $\alpha_i(k_i) = \alpha_i(k_i) + \ldots + \alpha_i(k_i) = k_i, \ldots, k_n \mapsto \alpha_i(k_i)$ for every $i = 1, 2, \ldots, n$. It is not hard to check that $\alpha$ is an epimorphism with $\ker(\alpha) \subseteq \text{Rad}(\bigoplus_{i \in I} K_i)$ and $\bigoplus_{i \in I} K_i$ is a direct summand of $M$. It follows that $M$ is a cofinitely Rad-$D_{12}$ module.

**Proposition 2.23.** Let $M$ be a cofinitely Rad-$D_{12}$ module with the property SSP. Suppose that $L$ is a direct summand of $M$. Then, $\frac{M}{L}$ is a cofinitely Rad-$D_{12}$ module.

**Proof.** Let $M$ be a cofinitely Rad-$D_{12}$ module and $\frac{M}{L}$ be a cofinite submodule of $\frac{M}{L}$. Then $N$ is a cofinite submodule of $M$. Since $M$ is a cofinitely Rad-$D_{12}$ module, there exist a direct summand $K$ of $M$ and an epimorphism $\alpha : K \to \frac{M}{L}$ with $\ker(\alpha) \subseteq \text{Rad}(K)$. Since $M$ has the property SSP, $K + L$ is a direct summand of $M$. Therefore there exists a submodule $X$ of $M$ such that $M = (K + L) \oplus X$. Note that $\frac{M}{L} = \frac{K + L}{L} \oplus \frac{X}{L}$. Because $\frac{K + L}{L} \cap \frac{X}{L} \subseteq \frac{X \cap (K + L) + L \cap (K + L)}{L} = \frac{L}{L}$, we can define the homomorphism $\alpha' : \frac{K + L}{L} \to \frac{M}{L}$ by $k + l + L = k + L \mapsto \alpha(k)$ with $k \in K$, $l \in L$. It’s easy to see that $\alpha'$ is an epimorphism with $\ker(\alpha') \subseteq \text{Rad}(\frac{K + L}{L})$ and $\frac{K + L}{L}$ is a direct summand of $\frac{M}{L}$. Hence $\frac{M}{L}$ is a cofinitely Rad-$D_{12}$ module.

**Theorem 2.24.** Let $M$ be a (cofinitely) Rad-$D_{12}$ module. If $L$ is a fully invariant submodule of $M$, then $\frac{M}{L}$ is a (cofinitely) Rad-$D_{12}$ module.
Proof. Let $\frac{M}{T}$ be a (cofinite) submodule of $\frac{M}{\alpha(T)}$. Then $N$ is a (cofinite) submodule of $M$. Since $M$ is a (cofinitely) Rad-$D_{12}$ module, there exist a direct summand $K$ of $M$ and an epimorphism $\alpha: K \to \frac{M}{\alpha(T)}$ with $\ker(\alpha) \subseteq \text{Rad}(K)$. It follows that there exists a submodule $K'$ of $M$ such that $M = K \oplus K'$. Since $L$ is a fully invariant submodule of $M$, $L = (L \cap K) \oplus (L \cap K')$. It is clear that $\frac{M}{L} = \frac{K+L}{L} \oplus \frac{K'+L}{L}$. Since $\frac{M}{L} \cong \frac{M}{\alpha(T)}$, we can define the homomorphism $\beta: \frac{K+L}{L} \to \frac{M}{L}$ by $k + L \mapsto \beta(k + L) = \alpha(k)$ with $k \in K$. Then $\beta$ is an epimorphism and $\ker(\beta) \subseteq \text{Rad}(\frac{K+L}{L})$. Hence $\frac{M}{L}$ is a (cofinitely) Rad-$D_{12}$ module. \qed

References


Author information

Recep Kılıç and Burcu Nişancı Türkmen, Department of Mathematics, Amasya University, Amasya, 05100, Turkey.

E-mail: burcuniisancie@hotmail.com

Received: March 6, 2015.

Accepted: April 29, 2015.