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RAD-D_{12} MODULES

Recep Kılıç and Burcu Nişancı Türkmen

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Abstract Let R be a ring and M be a right R-module. M is called *(cofinitely) Rad-*D_{12} if, for every (cofinite) submodule N of M, there exist a direct summand K of M and an epimorphism $\psi : K \to \frac{M}{N}$ with $ker(\alpha) \subseteq Rad(K)$. In this paper, we provide various properties of Rad-D_{12} modules and cofinitely Rad-D_{12} modules. In particular, we characterize semiperfect rings, perfect rings and artinian serial rings using (cofinitely) Rad-D_{12} modules. Moreover, we prove that every quasi-projective Rad-D_{12} module is Rad-$\oplus$-supplemented. Finally, we show that any factor module of a (cofinitely) Rad-D_{12} module by a fully invariant submodule is (cofinitely) Rad-D_{12}.

1 Introduction

Throughout this paper, it is assumed that R is an associative ring with identity and all modules are unital right R-modules. Let M be a module. A submodule N of an R-module M will be denoted by $N \subseteq M$. A submodule $N \subseteq M$ is said to be *cofinite* if $\frac{M}{N}$ is finitely generated. Maximal submodules are cofinite. Also, every submodule of a finitely generated module is cofinite. A submodule $L \subseteq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every non-zero submodule $N \subseteq M$. M is said to be *uniform* if its submodules is essential in M, and it is said to be *extending* (or a CS-module) if every submodule of M is essential in a direct summand of M. Dually, a module M is called *lifting* (or D_1) if, every submodule N of M contains a direct summand L of M such that $M = L \oplus K$ and $N \cap K$ is small in M. Here a submodule S of M is called *small* in M if $S + K \neq M$ for every proper submodule K of M. In [5, 29.10], every right R-module is lifting if and only if R is a left and right artinian serial ring with $J^2 = 0$, where J is the Jacobson radical of R. A module M is called *hollow* (or couniform) if every submodule is small in M. Hollow and semisimple modules are lifting. If M has a largest proper submodule, i.e. a proper submodule which contains all other proper submodules, then M is called *local* [23].

As a generalization of direct summands, a submodule V of M is called *a supplement* of a submodule U in M if $M = U + V$ and $U \cap V \ll V$ [23, pp. 348]. A module M is called *(cofinitely) supplemented* if every (cofinite) submodule has a supplement in M, and it is called *amply (cofinitely) supplemented* if, whenever (cofinite submodule N) $M = N + K$, N has a supplement $V \subseteq K$ in M. Every right R-module is (cofinitely) supplemented if and only if R is right (semi)perfect [1, Theorem 2.13] and [23, 43.9]. It can be seen that M is lifting if and only if it is amply supplemented and every supplement in M is a direct summand of M. Following [18], M is said to be *cofinitely lifting* if it is amply cofinitely supplemented and supplements of a cofinite submodule of M is a direct summand of M.

Mohamed and Müller [12] call a module M *$\oplus$-supplemented* (according to [8], (D_{11})) if every submodule N of M has a supplement that is a direct summand of M. Lifting modules are $\oplus$-supplemented. Zöschinger proved in [25] that every supplemented module over a dedekind domain is $\oplus$-supplemented. Moreover, it follows from [8, Theorem 1.1] a commutative ring R is an artinian principal ideal ring if and only if every right R-module is $\oplus$-supplemented. In [7], a module M is called *$\oplus$-cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand of M. For a module M, consider the following condition:

(D_{12}) For every submodule N of M, there exist a direct summand K of M and an epimorphism $\alpha : \frac{M}{K} \to \frac{M}{N}$ such that $ker(\alpha) \ll \frac{M}{K}$.

Modules with the property (D_{12}) are extensively studied in [11]. In addition, it is proven in [11, Proposition 4.3] that every $\oplus$-supplemented module has the property (D_{12}). Wang [22] generalizes modules with (D_{12}) to cofinitely (D_{12})-modules. A module M is called *cofinitely* (D_{12}) if, for every cofinite submodule N of M, there exist a direct summand K of M and an epimorphism $\alpha : \frac{M}{K} \to \frac{M}{N}$ such that $ker(\alpha) \ll \frac{M}{K}$. He obtained various properties of these modules in the same paper.

We will denote by $Rad(M)$, namely radical of M, the sum of all small submodules of a module M. We say that a submodule V of a module M is *Rad-supplement* (in [24], *generalized supplement*) of a submodule U in M if $M = U + V$ and $U \cap V \subseteq Rad(V)$ as in [5, pp. 100]. Clearly, we have the following diagram on submodules.

direct summand $\Longrightarrow$ supplement $\Longrightarrow$ Rad-supplement

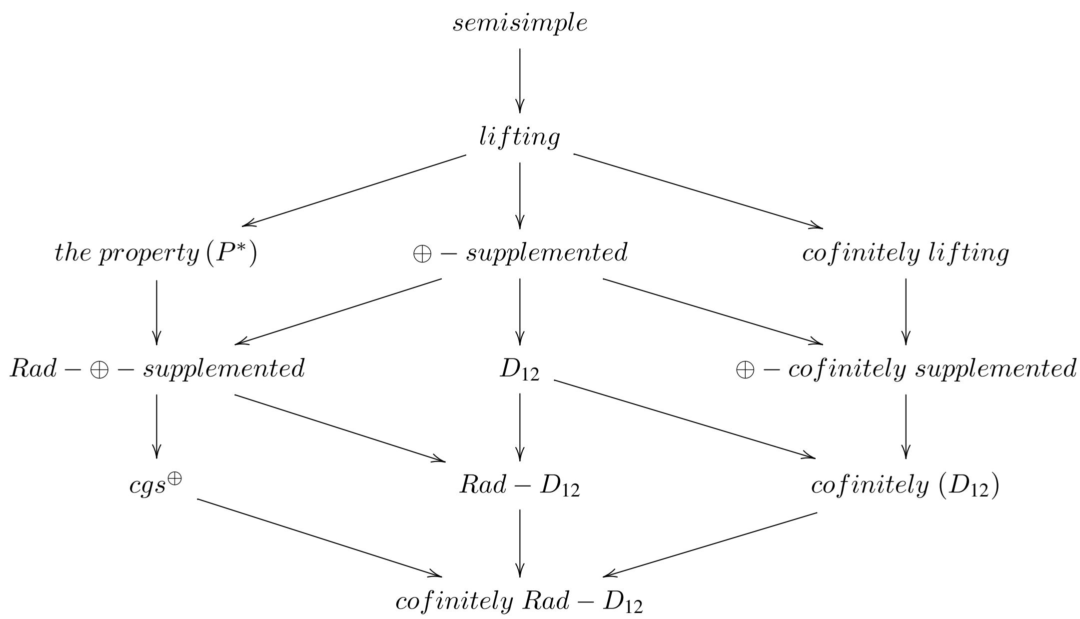
Motivated by the above definitions, we say a module M is *(cofinitely) Rad-supplemented* if every (cofinite) submodule of M has a Rad-supplement in M, and M is *(cofinitely) Rad-$\oplus$-supplemented* if every (cofinite) submodule of M has a Rad-supplement that is a direct summand in M as in these papers [4], [13] and [19]. Characterizations of those modules are studied in the same papers. From [13], we will use $cgs^{\oplus}$ instead of cofinitely Rad-$\oplus$-supplemented.

Recall from [1] that a module M has the property (P^*) if, for every submodule N of M, M has the decomposition $M = L \oplus K$ such that $L \subseteq N$ and $N \cap K \subseteq Rad(K)$. Modules with the property (P^*) is a dual notion of modules with the property (P) which is a generalization of extending modules [1]. Clearly lifting modules have the property (P^*). Also, by [19, Proposition 2.9], a projective module with the property (P^*) is lifting.

Talebi et.al. call a module M *Rad-D_{12}* if, for every submodule N of M, there exist a direct summand K of M and an epimorphism $\psi : K \to \frac{M}{N}$ with $ker(\alpha) \subseteq Rad(K)$. Some properties of Rad-D_{12} modules are given in [17]. It is shown in [17, Proposition 2.1] that every Rad-$\oplus$-supplemented module is Rad-D_{12}. It is of obvious interest to study characterizations of Rad-D_{12} modules by rings. In Theorem 2.9, we will prove that a ring R is right perfect if and only if every projective right R-module is Rad-D_{12}. In particular, we shall show in Theorem 2.16 that every right R-module is Rad-D_{12} if and only if a commutative ring R is an artinian serial ring.

Let M be a module. We call a module M *cofinitely Rad-D_{12}* if for every cofinite submodule N of M, there exist a direct summand K of M and an epimorphism $\alpha : K \to \frac{M}{N}$ such that $ker(\alpha) \subseteq Rad(K)$. We will investigate various properties of cofinitely Rad-D_{12} modules in section 2.

Under given definitions, we clearly have the following implication on modules:



In this paper, we give a new characterization of semiperfect rings via cofinitely Rad-D_{12} modules. Every non-radical indecomposable cofinitely Rad-D_{12}-module is ω-local. We show that if every right R-module is cofinitely Rad-D_{12}, then R is a noetherian serial ring.

2 (Cofinitely) Rad-D_{12} Modules

In this section, we will give characterizations of (semi)perfect rings in terms of (cofinitely) Rad-D_{12}. In particular, we will determine commutative rings whose modules are Rad-D_{12}.

Recall from [23] that an epimorphism $f : P \longrightarrow M$ is called a *cover* if $Ker(f) << P$, and a cover f is called a *projective cover* if P is a projective module. In the spirit of [23], a module M is said to be *semiperfect* if every factor module of M has a projective cover. Every semiperfect module is supplemented. A ring R is called *semiperfect*, if every finitely generated right (or left) R-module has a projective cover, and a ring R is called *right perfect* if every right R-module has a projective cover.

The proof of the next result is taken from [17, Proposition 2.1], but is given for the sake of completeness.

Proposition 2.1. *Every* $cgs^{\oplus}$*-module is cofinitely Rad-*D_{12}.

Proof. Let N be a cofinite submodule of M. Since M is $cgs^{\oplus}$, then there exist direct summands K and $K^{'}$ of M such that $M = N + K = K \oplus K^{'}$ and $N \cap K \subseteq Rad(K)$. Now we have the epimorphism g from K to $\frac{M}{N}$ which is defined by $k \to k + N$ with $ker(g) = N \cap K \subseteq Rad(K)$. Hence M is a cofinitely Rad-D_{12} module. □

The following example shows that a cofinitely Rad-D_{12} module need not $cgs^{\oplus}$.

Example 2.2. (See [11, Examples 4.5§4.6]) Let R be a local artinian ring with radical J such that $J^2 = 0$, $Q = \frac{R}{J}$ is commutative, $dim(_QJ) = 2$ and $dim(J_Q) = 1$. Consider the indecomposable injective right R-module $U = [\frac{(R \oplus R)}{D}]$ with $J = Ru + Rv$ and $D = \{(ur, -vr) \mid r \in R\}$. Now let $S = \frac{R}{J}$, the simple R-module, and $M = U \oplus S$. By [11, Example 4.6], M is cofinitely Rad-D_{12}, but not $cgs^{\oplus}$.

Recall from [23] that an R-module M is called *quasi-projective* if, for every R-module K, every R-epimorphism $\xi : M \to K$, and every R-homomorphism $f : M \to K$, there is an $\gamma \in End_R(M)$ such that $\xi \circ \gamma = f$. Now we prove that every quasi-projective (cofinitely) Rad-D_{12} module is Rad-$\oplus$-supplemented ($cgs^{\oplus}$).

Theorem 2.3. *Let M be a quasi-projective module.*

(i) If M is Rad-D_{12}, then M is a Rad-$\oplus$-supplemented module.

(ii) If M is cofinitely Rad-D_{12}, then M is a $cgs^{\oplus}$-module.

Proof. (1) Let N be a submodule of M. Then there exist a direct summand K of M and an epimorphism $\alpha : K \to \frac{M}{N}$ with $ker(\alpha) \subseteq Rad(K)$. Let $\pi : M \to \frac{M}{N}$ be the natural epimorphism. Since M is a quasi-projective, we have the homomorphism $h : M \to K$ with $\pi = \alpha \circ h$. Since K is M-projective, h splits. Hence there is a direct summand $K^{'}$ of M with $h|_{K^{'}} : K^{'} \cong K$. So $\pi|_{K^{'}}$ is an epimorphism. Therefore $M = K^{'} + N$ and $N \cap K^{'} = ker(\pi|_{K^{'}}) \subseteq Rad(M)$. Since $K^{'}$ is a direct summand of M, $N \cap K^{'} \subseteq Rad(K^{'})$. Thus M is Rad-$\oplus$-supplemented.

(2) The proof can be made similar to (1). □

Clearly, every cofinitely (D_{12})-module is cofinitely Rad-D_{12}. But the converse is not always true the following example shows. Recall from [4] that a module M is called ω*-local* if it has a unique maximal submodule. It is clear that a module is ω-local if and only if its radical is maximal.

Example 2.4. (See [16, Theorem 4.3 and Remark 4.4]) Let M be a biuniform module and let $S = End(M)$. Assume that P is a projective S-module with $dim(P) = (1, 0)$. Then P is an indecomposable ω-local module. Since $dim(P) = (1, 0)$, we conclude that P is not finitely generated. Hence, P is a $cgs^{\oplus}$-module but not $\oplus$-cofinitely supplemented. Thus, P is cofinitely Rad-D_{12} but not cofinitely (D_{12}).

Example 2.5. (See [9, 11.3]) Let R denote the ring $K[[x]]$ of all power series $\sum_{i=0}^{\infty} k_i x^i$ in an indeterminate x and with coefficients from a field K which is a local ring. Note that R is a semiperfect ring that is not perfect. Then by [7, Theorem 2.9] and [10, Corollary 2.11] , the free (projective) R-module $R^{(N)}$ is $\oplus$-cofinitely supplemented but not $\oplus$-supplemented. It follows that $R^{(N)}$ is $cgs^{\oplus}$. By [19, Theorem 2.2], $R^{(N)}$ is not Rad-$\oplus$-supplemented. Therefore $R^{(N)}$ is cofinitely Rad-D_{12} but not Rad-D_{12} by Theorem 2.3.

Theorem 2.6. *For a ring R, R is semiperfect if and only if every free right R-module is cofinitely Rad-*D_{12}*.*

Proof. ($\Rightarrow$) Suppose that a ring R is semiperfect. By [13, Theorem 2.4], every free right R-module is $cgs^{\oplus}$. Then by Proposition 2.1, every free right R-module is cofinitely Rad-D_{12}.

($\Leftarrow$) Since every free right R-module is cofinitely Rad-D_{12}, it is $cgs^{\oplus}$ by Theorem 2.3. It follows from [13, Theorem 2.4] that R is semiperfect. □

Proposition 2.7. *Let M be a cofinitely Rad-D_{12} module. If $Rad(M) \ll M$, then M is a cofinitely (D_{12})-module.*

Proof. Let N be a cofinite submodule of M. Since M is cofinitely Rad-D_{12}, there exist a direct summand K of M and an epimorphism $\alpha : K \to \frac{M}{N}$ such that $ker(\alpha) \ll M$. Since K is a direct summand of M, $ker(\alpha) \ll K$. Hence M is a cofinitely (D_{12})-module. □

A module M is called *coatomic* if every proper submodule is contained in a maximal submodule of M. Every coatomic module has a small radical. Using the above proposition, we obtain the following corollary.

Corollary 2.8. *Every coatomic cofinitely Rad-D_{12} module is cofinitely (D_{12}).*

Theorem 2.9. *For a ring R, R is right perfect if and only if every projective right R-module is Rad-D_{12}.*

Proof. The proof follows from Theorem 2.3(1) and [19, Corollary 2.3]. □

A module M is called *radical* if $Rad(M) = M$.

Proposition 2.10. *Let M be a non-radical indecomposable module. Suppose that M is a cofinitely Rad-D_{12} module. Then M is ω-local.*

Proof. Suppose that $Rad(M) \neq M$. Then M contains a maximal submodule N. By the hypothesis, there exist a direct summand K of M and an epimorphism $\alpha : K \to \frac{M}{N}$ with $ker(\alpha) \subseteq Rad(K)$. Note that $K \neq 0$. Since M is indecomposable, $K = M$. Therefore $\alpha : M \to \frac{M}{N}$ is an epimorphism with $ker(\alpha) \subseteq Rad(M)$. It follows that $\frac{M}{ker(\alpha)} \cong \frac{M}{N}$. Since N is a maximal submodule of M, $ker(\alpha)$ is a maximal submodule of M. But $ker(\alpha) \subseteq Rad(M)$. Thus $Rad(M)$ is a maximal submodule of M. Hence M is ω-local. □

Corollary 2.11. *Every finitely generated indecomposable, (cofinitely) Rad-D_{12} module is local.*

In [15, 1.4] a module M is called *uniserial* if its lattice of submodules is a chain. By [5, 2.17], a module M is uniserial if and only if every submodule of M is hollow. A module M is said to be *serial* if M is a direct sum of uniserial modules. A commutative ring R is called *uniserial* if the module $_RR$ (or R_R) is uniserial, and the ring is called *serial* if the module $_RR$ (or R_R) is serial.

Recall from [5, 1.5] that a module M is uniform if and only if every non-zero submodule of M is indecomposable.

Proposition 2.12. *Let M be a uniform module over a local commutative ring R. Then the following statements are equivalent.*

(i) M is uniserial.

(ii) Every submodule of M is cofinitely Rad-D_{12}.

Proof. $(i) \Rightarrow (ii)$ Clear.

$(ii) \Rightarrow (i)$ Let N be a finitely generated submodule of M. By (2), N is Rad-D_{12}. Since N is indecomposable, appliying Corollary 2.11 we obtain that N is local. It follows from [5, 2.17] that M is uniserial. □

By $E(M)$ we denote the injective hull of a module M. Note that the injective hull of a simple module is uniform.

Corollary 2.13. *Let R be a local commutative ring. Suppose that M is the module $E(\frac{R}{Rad(R)})$, and every submodule of M is cofinitely Rad-D_{12}. Then, R is uniserial.*

Proof. Since M is uniform, the hypothesis implies that M is uniserial by Proposition 2.12. It follows from [20, 6.2] that R is uniserial. □

Lemma 2.14. *(See [8, Theorem 1.1], [19, Corollary 2.15]) Let R be a commutative ring. Then the following statements are equivalent.*

(i) R is an artinian serial ring.

(ii) Every R-module is $\oplus$-supplemented.

(iii) Every R-module is Rad-$\oplus$-supplemented.

By Lemma 2.14, every module over an artinian serial ring is Rad-D_{12}. Now we show that the converse of this fact is true in the following Theorem. Firstly, we have:

Proposition 2.15. *Let R be a commutative ring. If every right R-module is cofinitely Rad-D_{12}, then R is a serial ring.*

Proof. Let M be a free R-module. By the hypothesis, M is cofinitely Rad-D_{12}. It follows from Theorem 2.6 that R is semiperfect. Note that $R = R_1 \oplus R_2 \oplus ... \oplus R_n$ such that the ring R_i is local for all $1 \leq i \leq n$ with $n \in \mathbb{N}$ ([23, 42.6]). For all $1 \leq i \leq n$, R_i is commutative and every R_i-module is cofinitely Rad-D_{12} by assumption. Using Corollary 2.13, we get R_i is uniserial. Thus R is a serial ring. □

Theorem 2.16. *The following statements are equivalent for a commutative ring R.*

(i) R is an artinian serial ring.

(ii) Every R-module is Rad-D_{12}.

Proof. $(i) \Rightarrow (ii)$ Clear.

$(ii) \Rightarrow (i)$ Applying Theorem 2.9, we obtain that R is perfect. It follows from ([23, 42.6]) that we can write $R = R_1 \oplus R_2 \oplus ... \oplus R_n$, where each R_i is a local perfect ring for all $1 \leq i \leq n$. By Corollary 2.13 and the hypothesis, it can be seen easily that each R_i is noetherian. Therefore, R is a serial noetherian ring as a finite direct sum of uniserial noetherian rings R_i. Applying [9, 11.6.4(c)], we deduce that R is an artinian serial ring. □

Let M be a module. $U \subseteq M$ is called *QSL* in M if $\frac{(A+U)}{U}$ is a direct summand of $\frac{M}{U}$, then there exists a direct summand P of M such that $P \subseteq A$ and $A + U = P + U$ [3]. M is said to be *cofinitely weak Rad-supplemented* if every cofinite submodule U of M has a weak Rad-supplement in M, i.e. there exists a submodule V of M such that $M = U + V$ and $U \cap V \subseteq Rad(M)$ [6].

Proposition 2.17. *Let M be a cofinitely weak Rad-supplemented module with $Rad(M)$ QSL in M. Then M is cofinitely Rad-D_{12}.*

Proof. Let N be a cofinite submodule of M. Then $\frac{M}{N}$ is finitely generated, and so

$$\frac{\frac{M}{Rad(M)}}{\frac{N+Rad(M)}{Rad(M)}} \cong \frac{\frac{M}{N}}{\frac{N+Rad(M)}{N}}$$

is finitely generated. Thus $\frac{N+Rad(M)}{Rad(M)}$ is a cofinite submodule of $\frac{M}{Rad(M)}$. By [6, Corollary 2.5], $\frac{N+Rad(M)}{Rad(M)}$ is a direct summand of $\frac{M}{Rad(M)}$. Since $Rad(M)$ is QSL in M, there exists a decomposition $M = K \oplus L$ such that $K \subseteq N$ and $N + Rad(M) = K + Rad(M)$. Now consider the epimorphism $\alpha : L \to \frac{M}{N}$ defined by $\alpha(l) = l + N$ $(l \in L)$. Since $M = K \oplus L$, then $Rad(M) = Rad(K) \oplus Rad(L)$. It follows that $N + Rad(L) = K + Rad(L)$ and, so $L \cap N + Rad(L) = L \cap K + Rad(L) = Rad(L)$. Note that $Ker(\alpha) = L \cap N \subseteq Rad(L)$. Hence M is cofinitely Rad-D_{12}. □

A module M is called *refinable* if for any submodules U, V of M with $M = U + V$, there exists a direct summand U' of M with $U' \subseteq U$ and $M = U' + V$ [5, 11.26]. It is easy to see that M is *refinable* if and only if every submodule of M is QSL.

Corollary 2.18. *Let M be a cofinitely weak Rad-supplemented refinable module. Then M is cofinitely Rad-D_{12}.*

Proof. Clear by Proposition 2.17. □

Proposition 2.19. *Let M be a cofinitely Rad-D_{12} module. If $Rad(M) \neq M$, then M has a non-zero ω-local direct summand.*

Proof. Let N be a maximal submodule of M. Then N is a cofinite submodule of M. Since M is a cofinitely Rad-D_{12} module, there exist a direct summand K of M and an epimorphism $\alpha : K \to \frac{M}{N}$ such that $ker(\alpha) \subseteq Rad(K)$. Clearly, $K \neq 0$ and $ker(\alpha)$ is a maximal submodule of K. Therefore $ker(\alpha) = Rad(K)$ and hence K is a non-zero ω-local direct summand of M. □

Recall from [23] that an R-module M has *the summand sum property (SSP)* if the sum of two direct summands of M is again a direct summand of M, and a submodule U of an R-module M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M. Let M be an R-module and let τ be a preradical for the category of R-modules. Then $Rad(M)$, $P(M)$ and $\tau(M)$ are fully invariant submodules of M. An R-module M is called a *(weak) duo module* if every (direct summand) submodule of M is fully invariant. Note that weak duo modules have SSP (See [14]).

The following Example shows a cofinitely Rad-D_{12} module that contains a direct summand which is not cofinitely Rad-D_{12}.

Example 2.20. Consider the right R-module $M = U \oplus S$ in Example 2.2. The module M is cofinitely Rad-D_{12}, but the submodule U is not cofinitely Rad-D_{12}.

Theorem 2.21. *Let $M = M_1 \oplus M_2$. Then M_2 is cofinitely Rad-D_{12} if and only if for every cofinite submodule N of M containing M_1, there exist a direct summand K of M_2 and an epimorphism $\varphi : M \to \frac{M}{N}$ such that K is a direct summand Rad-supplement of $ker(\varphi)$ in M.*

Proof. Suppose that M_2 is a cofinitely Rad-D_{12} module. Let N be a cofinite submodule of M with $M_1 \subseteq N$. Consider the submodule $N \cap M_2$ of M_2. Since $\frac{M_2}{N\cap M_2} \cong \frac{M}{N}$, $N \cap M_2$ is a cofinite submodule of M_2. Then there exist a direct summand K of M_2 and an epimorphism $\alpha : K \to \frac{M_2}{N\cap M_2}$ such that $ker(\alpha) = N \cap K \subseteq Rad(K)$. Note that $M = N + M_2$ and K is a direct summand of M. Let $M = K \oplus K'$ for some submodule K' of M. Consider the projection map $\xi : M \to K$ and the isomorphism $\beta : \frac{M_2}{N\cap M_2} \to \frac{M}{N}$ defined by $\beta(x + N \cap M_2) = x + N$. Thus $\beta \circ \alpha \circ \xi : M \to \frac{M}{N}$ is an epimorphism. Let $\varphi = \beta \circ \alpha \circ \xi$. Clearly, we have $ker(\varphi) = N + K' = ker(\alpha) \oplus K'$. Therefore $M = K + ker(\varphi)$. Moreover $K \cap ker(\varphi) = K \cap N = ker(\alpha) \subseteq Rad(K)$.

Conversely, suppose that every cofinite submodule of M containing M_1 has the stated property. Let H be a cofinite submodule of M_2. Consider the submodule $H \oplus M_1$ of M. Since $\frac{M}{H\oplus M_1} \cong \frac{M_2}{H}$ is finitely generated, $H \oplus M_1$ is a cofinite submodule of M. By the hypothesis, there exist a direct summand K of M_2 and an epimorphism $\mu : M \to \frac{M}{H\oplus M_1}$ such that $M = K + ker(\mu)$ and $K \cap ker(\mu) \subseteq Rad(K)$. Let $g : K \to \frac{M}{H\oplus M_1}$ be the restriction of μ to K. Consider the isomorphism $\eta : \frac{M}{H\oplus M_1} \to \frac{M_2}{H}$ defined by $\eta(m_1 + m_2 + (H \oplus M_1)) = m_2 + H$. Therefore $\eta \circ g : K \to \frac{M_2}{H}$ is an epimorphism. Let $\kappa = \eta \circ g$. Clearly, $ker(\kappa) \subseteq Rad(K)$. Hence M_2 is a cofinitely Rad-D_{12} module. □

Theorem 2.22. *Let $\{M_i\}_{i\in I}$ be any family of cofinitely Rad-D_{12} modules on a ring R and $M = \oplus_{i\in I} M_i$. If every cofinite submodule of M is fully invariant, then M is a cofinitely Rad-D_{12} module.*

Proof. Let N be a cofinite submodule of M. Since N is fully invariant, we have $N = \oplus_{i\in I}(N \cap M_i)$. Since $\frac{M}{N} \cong \oplus_{i\in I} \frac{M_i}{N\cap M_i}$, for every $i \in I$, $N \cap M_i$ is a cofinite submodule of M_i. Then there exist a direct summand K_i of M_i and an epimorphism $\alpha_i : K_i \to \frac{M_i}{N\cap M_i}$ with $ker(\alpha_i) \subseteq Rad(K_i)$. Now we define the homomorphism $\alpha : \oplus_{i\in I} K_i \to \oplus_{i\in I} \frac{M_i}{N\cap M_i}$ by $k_{i_1} + ... + k_{i_n} \mapsto \alpha_{i_1}(k_{i_1}) + ... + \alpha_{i_n}(k_{i_n})$ with $k_{i_j} \in K_{i_j}$ for every $j = 1, 2, ..., n$. It is not hard to check that α is an epimorphism with $ker(\alpha) \subseteq Rad(\oplus_{i\in I} K_i)$ and $\oplus_{i\in I} K_i$ is a direct summand of M. It follows that M is cofinitely Rad-D_{12}. □

Proposition 2.23. *Let M be a cofinitely Rad-D_{12} module with the property SSP. Suppose that L is a direct summand of M. Then, $\frac{M}{L}$ is a cofinitely Rad-D_{12} module.*

Proof. Let M be a cofinitely Rad-D_{12} module and $\frac{N}{L}$ be a cofinite submodule of $\frac{M}{L}$. Then N is a cofinite submodule of M. Since M is a cofinitely Rad-D_{12} module, there exist a direct summand K of M and an epimorphism $\alpha : K \to \frac{M}{N}$ with $ker(\alpha) \subseteq Rad(K)$. Since M has the property SSP, $K + L$ is a direct summand of M. Therefore there exists a submodule X of M such that $M = (K + L) \oplus X$. Note that $\frac{M}{L} = \frac{K+L}{L} \oplus \frac{X+L}{L}$. Because $\frac{K+L}{L} \cap \frac{X+L}{L} \subseteq \frac{X\cap(K+L)+L\cap(K+L+X)}{L} = \frac{L}{L}$. Since $\frac{\frac{M}{L}}{\frac{N}{L}} \cong \frac{M}{N}$, we can define the homomorphism $\alpha' : \frac{K+L}{L} \to \frac{M}{N}$ by $k + l + L = k + L \mapsto \alpha(k)$ with $k \in K,\ l \in L$. It's easy to see that α' is an epimorphism with $ker(\alpha') \subseteq Rad(\frac{K+L}{L})$ and $\frac{K+L}{L}$ is a direct summand of $\frac{M}{L}$. Hence $\frac{M}{L}$ is a cofinitely Rad-D_{12} module. □

Theorem 2.24. *Let M be a (cofinitely) Rad-D_{12} module. If L is a fully invariant submodule of M, then $\frac{M}{L}$ is a (cofinitely) Rad-D_{12} module.*

Proof. Let $\frac{N}{L}$ be a (cofinite) submodule of $\frac{M}{L}$. Then N is a (cofinite) submodule of M. Since M is a (cofinitely) Rad-D_{12} module, there exist a direct summand K of M and an epimorphism $\alpha : K \to \frac{M}{N}$ with $ker(\alpha) \subseteq Rad(K)$. It follows that there exists a submodule K' of M such that $M = K \oplus K'$. Since L is a fully invariant submodule of M, $L = (L \cap K) \oplus (L \cap K')$. It is clear that $\frac{M}{L} = \frac{K+L}{L} \oplus \frac{K'+L}{L}$. Since $\frac{\frac{M}{L}}{\frac{N}{L}} \cong \frac{M}{N}$, we can define the homomorphism $\beta : \frac{K+L}{L} \to \frac{M}{N}$ by $k + L \mapsto \beta(k + L) = \alpha(k)$ with $k \in K$. Then β is an epimorphism and $ker(\beta) \subseteq Rad(\frac{K+L}{L})$. Hence $\frac{M}{L}$ is a (cofinitely) Rad-D_{12} module. □

References

[1] I. Al-Khazzi and P.F. Smith, Modules with Chain Conditions on Superfluous Submodules, *Communications in Algebra*, **19(8)**, 2331-2351, (1991).

[2] R. Alizade, G. Bilhan and P. F. Smith, Modules whose maximal submodules have supplements, *Comm. Algebra*, **29(6)**, 2389-2405 (2001).

[3] M. Alkan, On τ-Lifting and τ-Semiperfect Modules, *Turkish J. Math.*,volume=**33**, 117-139, (2009).

[4] E. Büyükaşık and C. Lomp, On a Recent Generalization of Semiperfect Rings,*Bulletin of The Australian Mathematical Society*, **78(2)**, 317-325, (2008).

[5] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting Modules. Supplements and Projectivity in Module Theory, *Frontiers in Mathematics, Birkhäuser-Basel*, 406, (2006).

[6] S.K. Choubey, B.M. Pandeya, A.J. Gupta and H. Ranjan, Cofinitely Weak Rad-Supplemented Modules, *Research Journal of Pure Algebra*, **2(9)**, 259-262, (2012).

[7] H. Çalışıcı and A. Pancar, $\oplus$-Cofinitely Supplemented Modules, *Chech. Math. J.*, **54(129)**, 1083-1088, (2004).

[8] A. Idelhadj and R. Tribak, Modules for Which Every Submodule Has a Supplement That Is a Direct Summand, *The Arabian Journal for Sciences and Engineering*, **25(2C)**, 179-189, (2000).

[9] F. Kasch, Modules and Rings, publisher=*Academic Press Inc.*, (1982).

[10] D. Keskin, P.F. Smith and W. Xue, Rings Whose Modules Are $\oplus$-Supplemented, *Journal of Algebra*, **218**, 470-487, (1999).

[11] D. Keskin and W. Xue, Generalizations of Lifting Modules, *Acta Math. Hungar.*, **91(3)**, 253-261, (2001).

[12] S. H. Mohamed and B. J. Müller, Continuous and Discrete modules, series=*London Math. Soc. LNS 147 Cambridge University-Cambridge*, 190, (1990).

[13] B. Nişancı Türkmen and A. Pancar, On Generalization of $\oplus$-Cofinitely Supplemented Modules, *Ukrainian Mathematical Journal*, **62(2)**, 2003-209, (2010).

[14] A. Ç. Özcan, A. Harmacı and P.F. Smith, Duo Modules, *Glasgow Math. J.*, **48**, 533-545, (2006).

[15] G. Puninski, Serial Rings, *Kluwer Academic Publishers, Dordrecht, Boston, London, ISBN 0-7923-7187-9*, (2001).

[16] G. Puninski, Projective Modules over the Endomorphism Ring of a Biuniform Module, *J. Pure Appl. Algebra*, **188**, 227-246, (2004).

[17] Y. Talebi, A.R.M. Hamzekolaee and D. Keskin Tütüncü, On Rad-D_{12} Modules, *An. Şt. Univ. Ovidius Constanta*, **21(1)**, 201-208, (2013).

[18] R. Tribak, On Cofinitely Lifting and Cofinitely Weak Lifting Modules, *Communications in Algebra*, **36(12)**, 4448-4460, (2008).

[19] E. Türkmen, Rad-$\oplus$-Supplemented Modules, *An. Şt. Univ. Ovidius Constanta*, **21(1)**, 225-238, (2013).

[20] D. W. Sharpe and P. Vamos, Injective Modules, *Lectures in Pure Mathematics University of Sheffield-The Great Britain*, (1972).

[21] Y. Wang and N. Ding, Generalized Supplemented Modules, *Taiwanese J. Math.*, **6**, 1589-1601, (2006).

[22] Y. Wang, Cofinitely (D_{12})-Modules, *JP Journal of Algebra, Number Theory and Applications*, **27(2)**, 143-149, (2012).

[23] R. Wisbauer, Foundations of modules and rings, *Gordon and Breach*, (1991).

[24] W. Xue, Characterizations of Semiperfect and Perfect Modules, *Publicacions Matematiques*, **40(1)**, 115-125, (1996).

[25] H. Zöschinger, Komplementierte moduln über Dedekindringen, *Jounal of Algebra*, **29**, 42-56 (1974).

Author information

Recep Kılıç and Burcu Nişancı Türkmen, Department of Mathematics, Amasya University, Amasya, 05100, Turkey.
E-mail: burcunisancie@hotmail.com

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