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ON ANNIHILATOR GRAPHS OF NEAR-RINGS

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Abstract. Graphs from commutative rings are well studied through zero-divisor graphs, total graphs and other several graph constructions. Through these constructions, the interplay between algebraic structures and graphs are studied. Indeed, it is worthwhile to relate algebraic properties of commutative rings to the combinatorial properties of assigned graphs. Near-rings are generalized rings. In this paper, we introduce and study about annihilating ideal graphs of near-rings and in turn they generalize the results obtained for commutative rings.

1 Introduction

Graph constructions from commutative rings was by the concept of zero-divisor graph introduced and studied by Beck [8]. Subsequently several authors [1, 2, 3, 4, 5, 7, 14, 16, 17] have extensively studied various graph constructions from commutative rings. Some of the worthwhile constructions from commutative rings are zero-divisor graphs, total graphs, annihilator graphs and Cayley graphs.

Near-rings are generalized rings. In fact if we drop one of the distributive law and abelian nature of addition in the axioms of a ring, then one gets a near-ring. If the near-ring satisfies right distributive law, it is called a right near-ring. Let R be a commutative ring and $Z^*(R)$ is the set of all non-zero zero-divisors of R. Badawi [5] defined and studied the annihilator graph $AG(R)$ of a commutative ring R. Note that, for a commutative ring R, $AG(R)$ is the simple undirected graph with vertices $Z^*(R)$ and two distinct vertices $x, y \in R$ are adjacent if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. In parallel to this notation, we introduce and study the annihilator graph of a near-ring. Actually we construct three types of annihilator graphs of near-rings and study about their fundamental properties.

Through out this paper N denotes a commutative right near-ring with non-zero identity and $Z(N)$ be its set of all zero-divisors. For $x \in Z(N)$, $ann_N(x) = \{y \in N/yx = 0\}$. For convenience we denote $ann_N(x)$ by $ann(x)$ for $x \in N$. For basic properties regarding near-rings, one may refer Pilz [12] and basic properties on graph theory we refer [9].

2 Annihilator graph-I of near-rings

In this section, first we introduce and study about a class of annihilator graph corresponding to near-rings. After introducing this definition, we study about the new graph and its inter link with the zero-divisor graphs of near-rings.

Let N be a commutative near-ring. The *annihilator graph-I* of a near-ring N is the simple undirected graph with vertex set N and two distinct vertices x and y in N are adjacent if $ann(x) \cap ann(y) \neq \{0\}$. This graph is denoted by $\Gamma_1(N)$.

Example 2.1. Consider the near-ring N defined on the Kelin's 4-group $\{0, a, b, c\}$ with multiplication corresponding to the scheme 15:(0,13,0,13), p. 340 Pilz [12]. One can see that for this near-ring N, $\Gamma_1(N) = K_4$, the complete graph on vertices. Note that two non-isomorphic near-rings may have the same annihilator graph-I. For consider the near-rings considered in Example 2.1. For example the zero near-ring on the Kelin's 4-group $\{0, a, b, c\}$ and the near-ring corresponding to the scheme 15:(0,13,0,13), p. 340 Pilz [12] are one and the same where as the near-rings are not isomorphic.

First let us see a relation between the annihilator ideal graph-I $\Gamma_1(N)$ and the zero-divisor graph $\Gamma(N)$ of near-ring N.

Theorem 2.2. *The graphs $\Gamma(N) \cup \Gamma_1(N)$ and $\Gamma(N)^2$ are identical.*

Proof. Since the vertex set of $\Gamma(N)$ and $\Gamma_1(N)$ are same, it is enough to show the edge sets of $\Gamma(N) \cup \Gamma_1(N)$ and $\Gamma(N)^2$ are also same.

For, let $xy \in \Gamma(N) \cup \Gamma_1(N)$. This implies that either $xy = 0$ or $ann(x) \cap ann(y) \neq \{0\}$, which implies either $xy = 0$ or there exists a $w \in Z(N)^*$ such that $xw = 0, yw = 0$. From this we have $d(x, y) = 2$ and hence $xy \in E(\Gamma(N)^2)$.

Next, consider an edge $xy \in E(\Gamma(N)^2)$. Then $d(x, y) \leq 2$ in $\Gamma(N)$. If $d(x, y) = 1$, then $xy = 0$ and so, $xy \in E(\Gamma(N))$. If $d(x, y) = 2$, then there exists a $w \in Z(N)^*$ such that $xw = 0, yw = 0$ and so, $w \in ann(x) \cap ann(y)$. From this we get that $xy \in E(\Gamma_1(N))$ which implies that $xy \in \Gamma(N) \cup \Gamma_1(N)$. □

Theorem 2.3. *Let N be a commutative near-ring having no non-zero zero divisors of nilpotent elements of order ≥ 2. If $\Gamma(N)$ is a bipartite graph, then $G_1(N)$ has exactly two components.*

Proof. Assume that $\Gamma(N)$ be a bipartite graph and is isomorphic to $K_{m,n}$(say). Let X, Y be the bipartition of $\Gamma(N)$, such that $X = \{x_i : i = 1, 2, \ldots, m\}, Y = \{y_j : j = 1, 2, \ldots, n\}$.

Claim (i). $ann(x_i) \cap ann(y_j) = \{0\}$ for all i and j.

Suppose $t \in ann(x_i) \cap ann(y_j)$ for some i, j. Then $tx_i = 0, ty_j = 0$. Since N has no non-zero nilpotent element of index ≥ 2, $t \neq x_i$ or $t \neq y_j$. Now $tx_i = 0 = ty_j$ implies $t \in X \cap Y$ which is a contradiction. Hence claim (i) is true.

Claim (ii). $\Gamma_1[X]$ and $\Gamma_1[Y]$, the induced subgraphs of $\Gamma_1(N)$ are two components of $\Gamma_1(N)$.

Let $u, v \in X$. Since the graph $\Gamma(N)$ is connected, there exists a $u - v$ path, say $ux_1y_1 \cdots y_rv$ in $\Gamma(N)$. Then $uy_1y_2 \cdots v$ will be a path in $\Gamma_1(N)$. So u and v belong to the same component. Similarly for $\Gamma_1[Y]$. Hence the claim (ii) is true. □

Theorem 2.4. *Let N be a commutative near-ring without nilpotent elements. If $\Gamma(N)$ is a cycle of length m, then $\Gamma_1(N)$ is isomorphic to $\Gamma(N)$, when m is odd. Otherwise it is isomorphic to disjoint union of two even cycles of same length.*

Proof. **Case(i)**. m is odd.

Let $\Gamma(N) = x_1x_2x_3 \cdots x_mx_1$. Since $x_i^2 \neq 0$ for all i and x_1 is adjacent to x_2 and x_m, the edge x_2x_m must be in $\Gamma_1(N)$. Also, for all $i = 2$, x_i has common neighbours x_{i-1} and x_{i+1}. Hence the edges $x_{i-1}x_{i+1}$ should be in $\Gamma_1(N)$ for all $i \geq 2$. Obviously, these edges together with x_2x_m form a cycle of length m. So in this case $\Gamma(N)$ is isomorphic to $\Gamma_1(N)$.

Case(ii). m is even and say $m = 2k$.

Let $\Gamma(N) = x_1x_2 \cdots x_{2k}x_1$. As mentioned in the above case, x_1 has common neighbours x_2 and x_{2k}. From this the edge x_2x_{2k} exists in $\Gamma_1(N)$ and x_2 has common neighbours x_1 and x_3 in $\Gamma(N)$, so the edge x_1x_3 exists. Proceeding like this, we get two sequences of edges $x_1x_3, x_3x_5, \ldots, x_{2k-1}x_1$ and $x_2x_4, x_4x_6, \ldots, x_{2k}x_2$. These two sequences of edges form a disjoint union of two cycles in $\Gamma_1(N)$. □

Lemma 2.5. *Let N be a commutative near-ring without nilpotent elements of order ≥ 2. If $\Gamma(N)$ is a path, then $\Gamma_1(N)$ is a disjoint union of two paths.*

Proof. Let $\Gamma(N) = x_1x_2x_3 \cdots x_n$ be a path. By the definition of $\Gamma_1(N)$, the edges x_1x_3, x_3x_5, $\ldots$, and $x_2x_4, x_4x_6, \ldots$, are in $\Gamma_1(N)$. It is clear that these two sequences form two distinct paths. □

Remark 2.6. $\Gamma_1(N)$ need not be connected always. For, when $N = (Z_6, +_6, \cdot_6)$, then $\Gamma_1(N)$ is the disjoint union of the complete graphs K_1 and K_2.

Next, we give a necessary condition for the graph $\Gamma_1(N)$ to be connected.

Theorem 2.7. *If N is a commutative near-ring such that $x^n = 0$ for all $x \in N$ and for some $n \geq 2$, then the graph $\Gamma_1(N)$ is connected.*

Proof. Let $x, y \in V(\Gamma_1(N))$. Since $x^n = 0, x^{n-1} \in ann(x)$ and $y^{n-1} \in ann(y)$. If $ann(x) \cap ann(y) \neq \{0\}$, the edge xy exists in $\Gamma_1(N)$. If $ann(x) \cap ann(y) = \{0\}$ consider the product $x^{n-1}y^{n-1} \in N$. If $x^{n-1}y^{n-1} = 0$, then $x^{n-1}y^{n-2} \in ann(y)$ and $x^{n-2}y^{n-1} \in ann(x)$. We have $y^{n-1} \in ann(y), x^{n-2} \in N$, by the property of ideal, $x^{n-2}y^{n-1} \in ann(y)$. That is, $x^{n-2}y^{n-1} \in ann(x) \cap ann(y)$. In this case also the edge xy exists. If $x^{n-1}y^{n-1} \neq 0$, then $x(x^{n-1}y^{n-1}) = x^n y^{n-1} = 0$ and $y(x^{n-1}y^{n-1}) = x^{n-1}y^n = 0$ which implies that $x^{n-1}y^{n-1} \in ann(x) \cap ann(y)$ and hence, the edge xy exists. Since x and y are arbitrary, $\Gamma_1(N)$ is connected. □

Theorem 2.8. *If $\Gamma_1(N)$ is connected and N has at most one nilpotent element of order ≥ 2, then $diam(G_1(N)) \leq 3$.*

Proof. Let x and y be two distinct vertices in $\Gamma_1(N)$. If $ann(x) \cap ann(y) \neq \{0\}$, then the edge xy exists, and in this situation $d(x, y) = 1$.

If $ann(x) \cap ann(y) = \{0\}$, then three cases may arise.

Case(i). $x^2 = 0, y^2 = 0$.

If $xy = 0$, then $x \in ann(y)$ together with $x^2 = 0$ implies $x \in ann(x)$ which gives $x \in ann(y) \cap ann(x)$ which is a contradiction to our assumption. Hence $xy \neq 0$. On the other hand $x(xy) = x^2 y = 0$ and $y(xy) = xy^2 = 0$, which implies that $xy \in ann(x) \cap ann(y)$, which is a contradiction to $ann(x) \cap ann(y) = \{0\}$.

Case(ii). $x^2 = 0, y^2 \neq 0$.

As proved in Case (i), $xy \neq 0$. But $(xy)(xy) = x^2y^2 = 0$ implies $xy \in ann(xy)$ and $x(xy) = x^2 y = 0$ gives $xy \in ann(x)$. Hence $xy \in ann(x) \cap ann(y)$ so the edge xxy exists. Since $ann(y) \neq 0$ and $y^2 \neq 0$, there exist $b(\neq = y) \in ann(y)$. Consider the product $bx \in N$. But $bx \neq 0$, for if $bx = 0$, then $b \in ann(x)$ which gives $b \in ann(x) \cap ann(y)$, which is not possible.

Again $(bx)y = x(by) = 0$ gives $bx \in ann(y)$ and $(bx)(by) = x^2by = 0$ gives $bx \in ann(xy)$. That is the edge xyy exists. Hence a path $xxyy$ exists in $\Gamma_1(N)$. In this case $d(x, y) = 2$.

Case(iii). $x^2 \neq 0, y^2 = 0$. One can get proof as in the case (ii). □

3 Annihilator graph-II of near-rings

In this section, we construct a new class of annihilator graphs denoted by $\Gamma_2(N)$ called as annihilator graph-II corresponding to the near-ring N. Some general properties satisfied by the graph are obtained and some comparisons with $\Gamma(N)$ are also studied.

We define the simple graph $\Gamma_2(N)$ with vertex set $Z^*(N)$ of N and two distinct vertices x and y are adjacent in $\Gamma_2(N)$ if $ann(i) \subseteq ann(j)$ or $ann(j) \subseteq ann(i)$. We call $\Gamma_2(N)$ as the *annihilator graph-II* of the near-ring N.

Theorem 3.1. *If N is a commutative near-ring, then $\Gamma_2(N)$ is a spanning subgraph of $\Gamma_1(N)$.*

Proof. Since the vertex set of both $\Gamma_1(N)$ and $\Gamma_2(N)$ are same, we need only to consider only the edge sets of $\Gamma_1(N)$ and $\Gamma_2(N)$.

Let $e = xy \in E(\Gamma_2(N))$. Then $ann(x) \subseteq ann(y)$ or $ann(y) \subseteq ann(x)$. Since $x, y \in Z^*(N)$, $ann(x), ann(y) \neq \{0\}$. On the other hand $ann(x) \subseteq ann(y)$ or $ann(y) \subseteq ann(x)$ implies $ann(x) \cap ann(y) \neq \{0\}$. It follows that $xy \in E(\Gamma_1(N))$. □

Theorem 3.2. *Let N be a commutative near-ring. If an edge $xy \in \Gamma_2(N)$, then $d(x, y) \leq 2$ in $\Gamma(N)$.*

Proof. Suppose $e = xy \in E(\Gamma_2(N))$ and so $ann(x) \subseteq ann(y)$ or $ann(y) \subseteq ann(x)$. Without loss of generality, we may assume that $ann(x) \subseteq ann(y)$. Since $x \in Z^*(N)$, we have $ann(x) \neq \{0\}$. From this we have that there exists an $n(\neq 0) \in ann(x)$ such that $nx = 0$. Since $ann(x) \subseteq ann(y)$, $n \in ann(y)$ and so $ny = 0$.

If $n = x$, then $xy = 0$. It follows $e = xy \in E(\Gamma(N))$ and so $d(x, y) = 1$ in $\Gamma(N)$.

If $n = y$, then $xy = 0$ which implies that $xy \in E(\Gamma(N))$, and hence theorem follows.

If $n \neq x$and $n \neq y$, then $nx = 0$ and $ny = 0$ which implies that $x - n - y$ is a path in $\Gamma(N)$. From this we have that $d(x, y) \leq 2$. Hence in all the cases $d(x, y) \leq 2$. □

Theorem 3.3. *The path of length two cannot be realized as $\Gamma_2(N)$for some near-ring N.*

Proof. If possible, let $Z^*(N) = \{i, j, k\}$ and assume $i - j - k$ be the path of length two viz., P_2, which is $\Gamma_2(N)$. By the definition, we have $ann(i) \subseteq ann(j)$or $ann(j) \subseteq ann(i)$ and $ann(j) \subseteq ann(k)$ or $ann(k) \subseteq ann(j)$.

Case(i) Assume that $ann(i) \subseteq ann(j)$ and $ann(j) \subseteq ann(k)$. Then $ann(i) \subseteq ann(k)$, which implies i and k are adjacent in $\Gamma_2(N)$, which is not possible.

Case(ii) $ann(i) \subseteq ann(j)$ and $ann(k) \subseteq ann(j)$. Then it follows that $ann(i) \cup ann(k) \subseteq ann(j)$. Since $ann(i) \neq \{0\}$, assume that $n(\neq 0) \in ann(i)$. Then $ni = 0$ and by assumption, $n \in ann(j)$. That is $nj = 0$, which implies $n(i + j) = 0$. i.e, $i + j$ is a zero divisor.

If $i + j = 0$ then $i = -j$ and hence $ann(i) = ann(j)$. Then $ann(i) = ann(j)$ and $ann(k) \subseteq ann(i)$ implies i is adjacent to k, which is not possible.

If $i + j \neq 0$, then $i + j$ is a non- zero zero divisor. So the only possibility is $i + j = k$. But then $nk = n(i + j) = ni + nj = 0$. That is, $r \in ann(k)$. Since $n$is arbitrary, $ann(i) \subseteq ann(k)$. That is, i and k are adjacent, which is not possible.

The remaining cases reduce to the cases similar to (i) or (ii). □

Theorem 3.4. *Every simple graph with order less than or equal to three (except P_3) can be realized as $\Gamma_2(N)$for some commutative near-ring N.*

Proof. We know that there are only five non isomorphic simple graphs of order less than or equal to three other than P_3. The existence of such near-rings are listed below:

$K_1 = \Gamma_2(\mathbb{Z}_4)$, $2K_1 = \Gamma_2(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $K_2 = \Gamma_2(\mathbb{Z}_9)$, $K_1 \cup K_2 = \Gamma_2(\mathbb{Z}_6)$and $K_3 = \Gamma_2(\frac{\mathbb{Z}_2[x]}{<x^3>})$. □

Remark 3.5. The proof of the theorem reveals that non isomorphic near-rings may have isomorphic graphs.

4 Annihilator graph-III of near-rings

In this section, we define another class of annihilator graphs of near-rings. This notion extends the annihilator graph defined by Badawi []. This graph is denoted by $\Gamma_3(N)$. The annihilator graph $\Gamma_3(N)$ for a near-ring N, is the (undirected) graph $\Gamma_3(N)$ with vertices $Z(N)^*$ and two distinct vertices x and y are adjacent if and only if $ann(xy) \neq ann(x) \cup ann(y)$.

Remark 4.1. Note that each edge (path) of $\Gamma(N)$ is an edge (path) of $\Gamma_3(N)$. For let $e = xy \in E(\Gamma(N))$. From this, we have $xy = 0$ and $ann(xy) \neq ann(x) \cup ann(y)$. Hence $xy \in \Gamma_3(N)$.

Lemma 4.2. *Let Nbe a commutative near-ring. Then the following are hold.*

(i) *Let x, y be distinct elements of $Z(N)^*$. Then $x - y$is not an edge of $\Gamma_3(N)$if and only if $ann(xy) = ann(x)$ or $ann(xy) = ann(y)$;*

(ii) *If $x - y$ is an edge of $\Gamma(N)$ for some distinct $x, y \in Z(N)^*$, then $x - y$ is an edge of $\Gamma_3(N)$. In particular if P is a path in $\Gamma(N)$, then $P$is a path in $\Gamma_3(N)$;*

(iii) *If $x - y$ is not an edge of $\Gamma_3(N)$ for some distinct $x, y \in Z(N)^*$, then $ann(x) \subseteq ann(y)$or $ann(y) \subseteq ann(x)$;*

(iv) *If $ann(x) \nsubseteq ann(y)$ and $ann(y) \nsubseteq ann(x)$ for some distinct $x, y \in Z(N)^*$, then $x - y$ is an edge of $\Gamma_3(N)$;*

(v) *If $d_{\Gamma(N)}(x, y) = 3$ for some distinct $x, y \in Z(N)^*$, then $x - y$ is an edge of $\Gamma_3(N)$;*

(vi) *If $x - y$ is not an edge of $\Gamma_3(N)$ for some distinct $x, y \in Z(N)^*$, then there exists $w \in Z(N)^* \setminus \{x, y\}$ such that $x - w - y$ is a path in $\Gamma(N)$and hence $x - w - y$ is also a path in $\Gamma_3(N)$.*

Proof. (i) Suppose that $x - y$ is not an edge of $\Gamma_3(N)$. By definition, $ann(xy) = ann(x) \cup ann(y)$. Since $ann(xy)$ is a union of two ideals, we have, $ann(xy) = ann(x)$ or $ann(xy) = ann(y)$. Conversely, suppose that $ann(xy) = ann(x)$ or $ann(xy) = ann(y)$. Then $ann(xy) = ann(x) \cup ann(y)$ and thus $x - y$ is not an edge of $\Gamma_3(N)$.

(ii) Suppose $x-y$is an edge of $\Gamma(N)$ for some distinct $x, y \in Z(N)^*$. Then $xy = 0$ and hence $ann(xy) = N$. Since $x \neq 0$and $y \neq 0$, $ann(x) \neq N$ and $ann(y) \neq N$. Thus $x - y$ is an edge of $\Gamma_3(N)$. In particular statements is clearly true from this.

(iii) Suppose $x - y$ is not an edge of $\Gamma_3(N)$ for some distinct $x, y \in Z(N)^*$. Then $ann(x) \cup ann(y) = ann(xy)$. Since $ann(xy)$is a union of two ideals, we have $ann(x) \subseteq ann(y)$ or $ann(y) \subseteq ann(x)$.

(iv) This statement is trivial consequence of (iii).

(v) Suppose that $d_{\Gamma(N)}(x, y) = 3$ for some distinct $x, y \in Z(N)^*$. Then $ann(x) \nsubseteq ann(y)$ and $ann(y) \nsubseteq ann(x)$. Hence $x - y$ is an edge of $\Gamma_3(N)$by (iv).

(vi) Suppose that $x - y$ is not an edge of $\Gamma_3(N)$ for some distinct $x, y \in Z(N)^*$. Then there is a $w \in ann(x) \cup ann(y)$ such that $w \neq 0$ by (iii). Since $xy \neq 0$, we have $w \in Z(N)^* \setminus \{x, y\}$. Hence $x - w - y$ is a path in $\Gamma(N)$ and thus $x - w - y$ is also a path in $\Gamma_3(N)$ by (iii). □

In view of Lemma 4.2, we have the following result.

Theorem 4.3. *Let Nbe a commutative near-ring with $|Z(N)^*| \geq 2$. Then $\Gamma_3(N)$ is connected and diam$(\Gamma_3(N)) \leq 2$.*

Lemma 4.4. *Let Nbe a commutative near - ring and let x, y be distinct non zero elements. Suppose that $x - y$ is an edge of $\Gamma_3(N)$ that is not an edge of $\Gamma(N)$ for some distinct $x, y \in Z(N)^*$. If there is a $w \in ann(xy) \setminus \{x, y\}$ such that $wx \neq 0$and $wy \neq 0$, then $x - w - y$ is a path in $\Gamma_3(N)$ that is not a path in $\Gamma(N)$ and hence $C : x - w - y - x$is a cycle in $\Gamma_3(N)$ of length three and each edge of C is not an edge of $\Gamma(N)$.*

Proof. Suppose that $x - y$ is an edge in $\Gamma_3(N)$ that is not an edge in $\Gamma(N)$. Then $xy \neq 0$. Assume there exists a $w \in ann(xy) \setminus \{x, y\}$ such that $wx \neq 0$ and $wy \neq 0$. Since $y \in ann(xw) \setminus (ann(x) \cup ann(w))$, we conclude that $x - w$ is an edge of $\Gamma_3(N)$. Since $x \in ann(yw) \setminus (ann(y) \cup ann(w))$, we have that $y - w$ is an edge of $\Gamma_3(N)$. Hence $x - w - y$ is a path in $\Gamma_3(N)$. Since $xw \neq 0$ and $yw \neq 0$, we have $x - w - y$ is not a path in $\Gamma(N)$. It is clear that $C : x - w - y - x$ is a cycle in $\Gamma_3(N)$ of length three and each edge of C is not an edge of $\Gamma(N)$. □

Theorem 4.5. *Let N be a commutative near - ring. Suppose that $x - y$ is an edge of $AG(N)$ that is not an edge of $\Gamma(N)$ for some distinct $x, y \in Z(N)^*$. If $xy^2 \neq 0$ and $x^2y \neq 0$, then there is a $w \in Z(N)^*$ such that $x - w - y$ is a path in $\Gamma_3(N)$ that is not a path in $\Gamma(N)$ and hence $C : x - w - y - x$ is a cycle in $\Gamma_3(N)$ of length three and each edge of C is not an edge of $\Gamma(N)$.*

Proof. Suppose that $x - y$ is an edge of $\Gamma_3(N)$ that is not an edge of $\Gamma(N)$. Then $xy \neq 0$ and there is a $w \in ann(xy) \setminus (ann(x) \cup ann(y))$. We show $w \notin \{x, y\}$. Assume $w \in \{x, y\}$. Then either $x^2y = 0$ or $y^2x = 0$, which is a contradiction. Thus $w \notin \{x, y\}$. Hence $x - w - y$ is the desired path in $\Gamma_3(N)$ by Lemma 4.4. □

Corollary 4.6. *Let N be a reduced commutative near-ring. Suppose that $x - y$ is an edge of $\Gamma_3(N)$ that is not an edge of $\Gamma(N)$ for some distinct $x, y \in Z(N)^*$. Then there is a $w \in ann(xy) \setminus \{x, y\}$ such that $x - w - y$ is a path in $\Gamma_3(N)$ that is not a path in $\Gamma(N)$ and $\Gamma_3(N)$ contains a cycle C of length 3 such that at least two edges C are not the edges of $\Gamma(N)$.*

Proof. Suppose that $x - y$ is an edge of $\Gamma_3(N)$ that is not an edge of $\Gamma(N)$ for some distinct $x, y \in Z(N)^*$. Since N is reduced, we have $(xy)^2 \neq 0, \in \Gamma$. This implies $x^2y \neq 0$ and $xy^2 \neq 0$. Thus the claim is now clear by Theorem 4.5. □

Corollary 4.7. *Let N be a reduced commutative near- ring and suppose that $\Gamma_3(N) \neq \Gamma(N)$. Then gr$(G_3(N))$ = 3. moreover, there is a cycle C of length 3 in $\Gamma_3(N)$ such that at least two edges of C are not the edges of $\Gamma(N)$.*

Proof. Since $\Gamma_3(N) \neq \Gamma(N)$, there are some distinct $x, y \in Z(N)^*$ such that $x - y$ is an edge of $\Gamma_3(N)$ that is not an edge of $\Gamma(N)$. Since N is reduced, we have $(xy)^2 \neq 0, \in \Gamma$. This implies $x^2y \neq 0$ and $xy^2 \neq 0$. Thus the claim is now clear by Theorem 4.5. □

Theorem 4.8. *Let N be a commutative near- ring and suppose that $\Gamma_3(N) \neq \Gamma(N)$ with gr$(\Gamma_3(N)$ $) \neq 3$. Then there are some distinct $x, y \in Z(N)^*$ such that $x - y$ is an edge of $\Gamma_3(N)$ that is not an edge of $\Gamma(N)$ and there is no path of length 2 from x to y in $\Gamma(N)$.*

Proof. Since $\Gamma_3(N) \neq \Gamma(N)$, there are some distinct $x, y \in Z(N)^*$ such that $x - y$ is an edge of $\Gamma_3(N)$ that is not an edge of $\Gamma(N)$. Assume that $x - w - y$ is a path of length 2 in $\Gamma(N)$. Then $x - w - y$ is a path of length 2 in $\Gamma_3(N)$ by Lemma 4.2(i). Therefore $x - w - y - x$ is a cycle of length 3 in $\Gamma_3(N)$ and hence gr($\Gamma_3(N)$)=3, a contradiction. Thus there is no path of length from x to y in $\Gamma(N)$ □

Lemma 4.9. *Let N be a reduced near- ring that is not an gamma near-integral domain and let $z \in Z(N)^*$. Then*

(i) $ann(x) = ann(z^n)$ *for each positive integer* $n \geq 2$;

(ii) *If* $c + z \in Z(N)$ *for some* $c \in ann(z) \setminus \{0\}$, *then* $ann(z + c) \subset ann(z)$. *In particular if* $Z(N)$ *is an ideal of* N *and* $c \in ann(x) \setminus \{0\}$, *then* $ann(z + c)$ *is properly contained in* $ann(z)$.

Proof. (i) Let $n \geq 2$. It is clear that $ann(z) \subseteq ann(z^n)$. Let $a \in ann(z^n)$. Since $fz^n = 0$ and N is reduced, we have $fz = 0$. Thus $ann(z^n) = ann(z)$.

(ii) Let $c \in ann(z) \setminus \{0\}$ and suppose that $c + z \in Z(N)$. Since $z^2 \neq 0$, we have $c + z \neq 0$ and hence $c + z \in Z(N)^*$. Since $c \in ann(z)$ and N is reduced, we have $c \notin ann(c + z)$. Hence $ann(c + z) \neq ann)(z)$. Since $ann(c + z) \subset ann(z(c + z)) = ann(z^2)$ and $ann(z^2) = ann(z)$, by(i), it follows that $ann(c + z) \subset ann(z)$. □

Theorem 4.10. *Let N be a reduced near- ring with $|Min(N)| \neq 3$ (possibly $Min(N)$ is infinite). Then $\Gamma_3(N) \neq \Gamma(N)$ and gr$(\Gamma_3(N)) = 3$.*

Proof. If $Z(N)$ is an ideal of N, by Theorem 4.2, $\Gamma_3(N) \neq \Gamma(N)$. Hence assume that $Z(N)$ is not an ideal of N. Since $|Min(N)| \neq 3$, we have diam($\Gamma(N)$)=3 and thus $\Gamma_3(N) \neq \Gamma(N)$ by Theorem 4.2. Since N is reduced and $\Gamma_3(N) \neq \Gamma(N)$, we have gr$(\Gamma_3(N)) = 3$. □

Theorem 4.11. *Let N be a reduced near- ring that is not an gamma near-integral domain. Then $\Gamma_3(N) = \Gamma(N)$ if and only if $|Min(N)| = 2$.*

Proof. Suppose that $\Gamma_3(N)$ =$\Gamma(N)$. Since N is a reduced near- ring that is not an gamma near-integral domain $|Min(N)| = 2$ by Theorem 4.5.

Conversely, suppose that $|Min(N)| = 2$. Let P_1, P_2 be the minimal prime ideals of N. Since N is reduced, we have $Z(N) = P_1 \cup P_2$ and $P_1 \cap P_2 = \{0\}$. Let $a, b \in Z(N)^*$. Assume that $a, b \in P_1$. Since $P_1 \cap P_2 = 0$, neither $a \notin P_2$ nor $b \notin P_2$ and thus $ab \neq 0$. Since $P_1P_2 \subseteq P_1 \cap P_2 = \{0\}$, it follows that $ann(ab) = ann(a) = ann(b) = P_2$. Thus $a - b$ is not an edge of $\Gamma_3(N)$. Similarly, if $a, b \in P_2$, then $a - b$ is not an edge of $\Gamma_3(N)$. If $a \in P_1, b \in P_2$, then $ab = 0$ and thus $a - b$ is an edge of $\Gamma_3(N)$. Hence each edge of $\Gamma_3(N)$ is an edge of $\Gamma(N)$ and therefore $\Gamma_3(N) = \Gamma(N)$. □

For the remainder of this section, we study the case when N is non reduced.

Theorem 4.12. *Let N be a non reduced near- ring with $|Nil(N)^*| \geq 2$ and let $\Gamma N_3(N)$ be the (induced) sub graph of $\Gamma_3(N)$ with vertices $Nil(N)^*$. Then $\Gamma N_3(N)$ is complete.*

Proof. Suppose there are non zero distinct elements $a, b \in Nil(N)$ such that $ab \neq 0, \in \Gamma$. Assume that $ann(ab) = ann(a) \cup ann(b)$. Hence $ann(ab) = ann(a)$ or $ann(ab) = ann(b)$. Without loss of generality, we may assume that $ann(ab) = ann(a)$. Let n be the least positive integer such that $b^n = 0$. Suppose that $ab^k \neq 0$, for each $k, 1 \leq k \leq n$. Then $b^{n-1} \in ann(ab) \setminus ann(a)$, a contradiction. Hence assume that ℓ , $1 \leq \ell \leq n$ is the least positive integer such that $ab^\ell = 0$. Since $ab^{i-1} \neq 0, 1 < i < n$, we have $b^{k-1} \in ann(ab) \setminus ann(a)$, a contradiction. Thus $a - b$ is an edge of $\Gamma N_3(N)$. □

Theorem 4.13. *Let N be a non reduced near- ring with $|Nil(N)^*| \geq 2$ and let $\Gamma N_3(N)$ be the induced sub graph of $\Gamma(N)$ with vertices $Nil(N)^*$. Then $\Gamma N_3(N)$ is complete if and only if $Nil(N)^2 = 0$.*

Proof. If $Nil(N)^2 = \{0\}$, then it is clear that $\Gamma N_3(N)$ is complete. Conversely assume that $\Gamma N_3(N)$ is complete. We need only show that $w^2 = 0$ for each $w \in Nil(N)^*$. Let $\in Nil(N)^*$ and assume that $w^2 \neq 0$. Let n be the least positive integer such that $w^n = 0$. Then $n \geq 3$. Thus $w(w^n - 1 + w) = 0$ and $w^n = 0$. From these, we have $w^2 = 0$, which is a contradiction. Thus $w^2 = 0$ for each $w \in Nil(N)$. □

Theorem 4.14. *Let N be a near- ring such that $\Gamma_3(N) \neq \Gamma(N)$. Then the following statements are equivalent:*

(i) $\Gamma(N)$ *is a star graph;*

(ii) $\Gamma(N) = K_{1,2}$ *;*

(iii) $\Gamma_3(N) = K_3$.

Proof. (i) $\Longrightarrow$ (ii). Since $\Gamma(N)$ is a star graph, gr$(\Gamma(N)) = \infty$ and $\Gamma_3(N) \neq \Gamma(N)$. From Theorem 4.10, we have N is non reduced and $|Z(N)^*| \geq 3$. Since $\Gamma(N)$ is a star graph, there are two sets A, B such that $Z(N)^* = A \cup B$ with $|A| = 1, A \cap B = \phi, AB = \{0\}$ and $b_1 b_2 \neq 0$ for every $b_1, b_2 \in B$. Since $|A| = 1$, we may assume that $A = \{w\}$ for some $w \in Z(N)^*$. Since each edge of $\Gamma(N)$ is an edge of $\Gamma_3(N)$ and $\Gamma_3(N) \neq \Gamma(N)$, there are some $x, y \in B$ such that xy is an edge of $\Gamma(N)$, but not an edge of $A_G(N)$. Since $ann(c) = w$ for each $c \in B$ and $ann(xy) \neq ann(x) \cup ann(y)$. From this we have $ann(xy) \neq w$. Thus $ann(xy) = B$ and $xy = w$. Since $A = \{xy\}$ and $AB = \{0\}$, we have $(xy)x = x^2 y = 0$ and $(xy)y = y^2 x = 0$. Now we show that $B = \{x, y\}$ and hence $|B| = 2$. Thus assume there is a $c \in B$ such that $c \neq x$ and $c \neq y$. From this $wc = xyc = 0$.

We claim that $(xc+xy) \neq x$ and $(xc+xy) \neq xy$ (note that $xy = w$). Suppose that $(xc+xy) = x$. Then $(xc + xy)y = xcy + xy^2 = 0$ and $xy = 0$, a contradiction. Hence $x \neq (xc + xy)$. Since $x, c \in B$, we have $xc \neq 0$ and thus $(xc + xy), xy$ are distinct elements of $Z(N)^*$. Since $x^2 y = 0$ and $y \in B$ either $x^2 = 0$ or $x^2 = xy$ or $x^2 = y$. Suppose that $x^2 = y$. Since $xy = w \neq 0$. We have $xy = x(x^2) = x^3 = w \neq 0$. Since $x^2 y = 0$, we have $x^4 = 0$. Since $x^4 = 0$, and $x^3 \neq 0$, we have $x^2, x^3, x^2 + x^3$ are distinct elements of $Z(N)^*$, and thus $x^2 - x^3 - x^2 + x^3 - x^2$ is a cycle of length three in $\Gamma(N)$, which is a contradiction. Hence we assume that either $x^2 = 0$ or $x^2 = xy = w$. In both cases, we have $x^2 c = 0$. Since $x, (xc + xy), xy$ are distinct elements of $Z(N)^*$ and $xy^2 = yx^2 = x^2 c = 0$. Now we have $x - (xc + xy) - xy - x$ is a cycle of length three in $\Gamma(N)$, again a contradiction. Thus $B = \{x, y\}$ and $|B| = 2$. Hence $\Gamma(N) = K_{1,2}$.

(ii) $\Longrightarrow$ (iii). Note that each edge of $\Gamma(N)$ is an edge of $\Gamma_3(N)$ and $\Gamma_3(N) \neq \Gamma(N)$. Hence from $\Gamma(N) = K_{1,2}$, we have that $\Gamma_3(N)$ must be K_3.

(iii) $\Longrightarrow$ (i). Since $|Z(N)^*| = 3$, $\Gamma(N)$ is connected and $\Gamma_3(N) \neq \Gamma(N)$ exactly one edge of $AG(N$ is not and edge of $\Gamma(N)$. Thus $\Gamma(N)$ is a star graph. □

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