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# Some modular relations for the Rogers-Ramanujan type functions of order fifteen and its applications to partitions

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**Abstract** In a manuscript of Ramanujan, published with his Lost Notebook [18] there are forty identities involving the Rogers- Ramanujan functions. In this paper, we establish modular relations involving the Rogers-Ramanujan functions, the Rogers-Ramanujan type functions of order ten and the Rogers-Ramanujan-Slater type functions of order fifteen which are analogues to Ramanujan forty identities. We also give partition theoretic interpretations of our modular relations.

## 1 Introduction

Throughout the paper, we assume $|q| < 1$ and for positive integer $n$, we use the standard notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

In [18], Ramanujan stated that

$$H(q)\{G(q)\}^{11} - q^2G(q)\{H(q)\}^{11} = 1 + 11q\{G(q)H(q)\}^6,$$

where

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}, \tag{1.1}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} \tag{1.2}$$

are the famous Rogers-Ramanujan functions. In 1975, B. J. Birch [10] published 40 identities conjectured by Ramanujan involving the functions $G(q)$ and $H(q)$, which are called Ramanujan's forty identities. In 1921, H. B. C. Darling [14] proved one of the identities in the proceedings of London Mathematical Society. In the same issue of the journal, L. J. Rogers [20] established 10 of the 40 identities including the one proved by Darling. In 1933, G. N. Watson [24] proved 8 of the 40 identities, 2 of which had been previously established by Rogers. In 1977, D. Bressoud [12] in his doctoral thesis, proved 15 more from the list of 40. In 1989, A. F. J. Biagioli [9] proved 8 of the remaining 9 identities by invoking the theory of modular forms. Recently, B. C. Berndt et al. [8] have found new proofs for 35 of the forty identities in the spirit of Ramanujan.

S. -S. Huang [16] and S. -L. Chen and Huang [13] have established several modular relations for the Göllnitz-Gordan functions by techniques which have been used by Rogers, Watson and Bressoud to prove some of Ramanujan's 40 identities. In 2008, N. D. Baruah, J. Bora and N. Saikia [6] have given alternative proofs some of them by using Schröter's formulas and some simple theta functions identities of Ramanujan. In 2003, H. Hahn [15] has established several modular relations for the septic analogues of the Rogers-Ramanujan functions. In 2007, Baruah and Bora [5] have established several modular relations for the nonic analogues of the Rogers-Ramanujan functions as well as relations that are connected with the Rogers-Ramanujan, Göllnitz-Gordan and septic analogues of Rogers-Ramanujan type functions. In 2007, Baruah and Bora [4] have established several modular relations involving two functions analogues to the Rogers-Ramanujan functions. Some of these relations are connected with Rogers-Ramanujan, Göllnitz-Gordan, septic and nonic analogues of Rogers-Ramanujan type functions.

In 2008, C. Adiga, K. R. Vasuki and B. R. Srivatsa Kumar [3] have established modular relations involving two functions of Rogers-Ramanujan type. In 2010, Vasuki, G. Sharath and K. R. Rajanna [23] have established modular relations for cubic functions and are shown to be connected to the Ramanujan cubic continued fraction. In 2012, Adiga, Vasuki and N. Bhaskar [2] have established modular relations for cubic functions. Vasuki and P. S. Guruprasad [22] have established certain modular relations for the Rogers-Ramanujan type functions of order twelve of which some of them are proved by Baruah and Bora [4] on employing different method. Recently, Adiga and N. A. S. Bulkhali [1] have established several modular relations for the Rogers-Ramanujan type functions of order ten, namely,

$$\begin{aligned} J(q) :=& \sum_{n=0}^{\infty} \frac{(-q;q)_n \, q^{n(n+1)/2}}{(q;q^2)_{n+1}(q;q)_n} \\ =& \frac{(-q;q)_\infty (q^3;q^{10})_\infty (q^7;q^{10})_\infty (q^{10};q^{10})_\infty}{(q;q)_\infty}, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} K(q) :=& \sum_{n=0}^{\infty} \frac{(-q;q)_n \, q^{n(n+3)/2}}{(q;q^2)_{n+1}(q;q)_n} \\ =& \frac{(-q;q)_\infty (q;q^{10})_\infty (q^9;q^{10})_\infty (q^{10};q^{10})_\infty}{(q;q)_\infty}. \end{aligned} \tag{1.4}$$

Almost all of these functions which have been studied so far are due to Rogers [19] and L. G. Slater [21].

In [18, p. 33], Ramanujan stated the following identity:

$$\frac{f(aq^3,\ a^{-1}q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}\ (-a^{-1}q;\ q^2)_n\ (-aq;\ q^2)_n}{(q^2;\ q^2)_{2n}}. \tag{1.5}$$

The preceding result of Ramanujan yields infinitely many identities of Rogers-Ramanujan-Slater type when $a$ is set to $\pm q^r$ for $r \in \mathbb{Q}$. In [17, p. 20] J. Mc Laughlin, A. V. Sills and P. Zimmer have listed the following Rogers-Ramanujan-Slater type $mod15$ identities:

$$\begin{aligned} A(q) :=& \frac{f(-q^7,-q^8)}{f(-q^5)} \\ =& \sum_{n=0}^{\infty} \frac{q^{5n^2}\ (q^2;q^5)_n\ (q^3;q^5)_n}{(q^5;q^5)_{2n}} \quad ((1.5)\ \text{ with } a=-q^{1/5}), \end{aligned} \tag{1.6}$$

$$\begin{aligned} B(q) :=& \frac{f(-q^4,-q^{11})}{f(-q^5)} \\ =& 1-\sum_{n=1}^{\infty} \frac{q^{5n^2-1}\ (q^4;q^5)_{n-1}\ (q;q^5)_{n+1}}{(q^5;q^5)_{2n}} \quad ((1.5) \text{ with } a=-q^{7/5}), \end{aligned} \tag{1.7}$$

$$\begin{aligned} C(q) :=& \frac{f(-q^2,-q^{13})}{f(-q^5)} \\ =& 1-\sum_{n=1}^{\infty} \frac{q^{5n^2-3}\ (q^2;q^5)_{n-1}\ (q^3;q^5)_{n+1}}{(q^5;q^5)_{2n}} \quad ((1.5) \text{ with } a=-q^{11/5}), \end{aligned} \tag{1.8}$$

$$\begin{aligned} D(q) :=& \frac{f(-q,-q^{14})}{f(-q^5)} \\ =& 1-\sum_{n=1}^{\infty} \frac{q^{5n^2-4}\ (q;q^5)_{n-1}\ (q^4;q^5)_{n+1}}{(q^5;q^5)_{2n}} \quad ((1.5) \text{ with } a=-q^{13/5}). \end{aligned} \tag{1.9}$$

The main purpose of this paper is to establish several modular relation involving $A(q)$, $B(q)$, $C(q)$ and $D(q)$, which are analogues of Ramanujan's forty identities and further we extract partition theoretic interpretations of our results.

## 2 Definitions and Preliminary results

In this section, we present some basic definitions and preliminary results on Ramanujan's theta functions. Ramanujan's general theta function is

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.1}$$

Then it is easy to verify that

$$f(a,b) = f(b,a), \tag{2.2}$$
$$f(1,a) = 2f(a,a^3), \tag{2.3}$$
$$f(-1,a) = 0. \tag{2.4}$$

The well-known Jacobi triple product identity is given by

$$f(a,b) = (-a;ab)_\infty\, (-b;ab)_\infty\, (ab;ab)_\infty \tag{2.5}$$

and, if $n$ is an integer,

$$f(a,b) = a^{n(n+1)/2}\, b^{n(n-1)/2}\, f(a(ab)^n, b(ab)^{-n}). \tag{2.6}$$

The three most interesting special cases of (2.1) are

$$\varphi(q) \;:=\; f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_\infty (q^2;q^2)_\infty, \tag{2.7}$$
$$\psi(q) \;:=\; f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.8}$$

and

$$f(-q) \;:=\; f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{2.9}$$

Also, after Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty. \tag{2.10}$$

Throughout the paper, we shall use $f_n$ to denote $f(-q^n)$. The following lemma is a consequence of (2.5) and Entry 24 of [7, p. 39].

**Lemma 2.1.** *We have*

$$\varphi(q) = \frac{f_2^5}{f_1^2\, f_4^2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \quad \varphi(-q) = \frac{f_1^2}{f_2}, \quad \psi(-q) = \frac{f_1\, f_4}{f_2},$$

$$f(q) = \frac{f_2^3}{f_1\, f_4}, \quad \chi(q) = \frac{f_2^2}{f_1\, f_4} \quad \textit{and} \quad \chi(-q) = \frac{f_1}{f_2}.$$

The following lemma is a consequence of (2.5) and the simple identity

$$(-q;q)_\infty = \frac{f_2}{f_1}.$$

**Lemma 2.2.** *We have*

$$f(q,q^2) = \frac{\varphi(-q^3)}{\chi(-q)} = \frac{f_2\, f_3^2}{f_1\, f_6}.$$

## 3 Main Results

In this section, we prove the following four modular relations involving $A(q)$, $B(q)$, $C(q)$ and $D(q)$, Rogers-Ramanujan functions $G(q)$ and $H(q)$ and Rogers-Ramanujan type functions $J(q)$ and $K(q)$ of order ten. For Simplicity, for a positive integer $k$, we set $A_k := A(q^k)$, $B_k := B(q^k)$, $C_k := C(q^k)$, $D_k := D(q^k)$, $G_k := G(q^k)$, $H_k := H(q^k)$, $J_k := J(q^k)$ and $K_k := K(q^k)$.

$$A_4A_1 + q^2B_4B_1 + q^5C_4C_1 + q^7D_4D_1 = \frac{f_2f_6f_3^2}{f_1f_5f_{12}f_{20}} - q, \tag{3.1}$$

$$A_3A_2 + q^2B_3B_2 + q^5C_3C_2 + q^7D_3D_2 = \frac{f_2f_3^2f_9^2}{f_1f_6f_{10}f_{15}f_{18}} - q, \tag{3.2}$$

$$A_2G_1 - qB_2H_1 - q^2C_2G_1 - q^3D_2H_1 = \frac{f_3^2}{f_6f_{10}}, \tag{3.3}$$

$$A_1J_1 + qB_1K_1 + qC_1J_1 - q^2D_1K_1 = \frac{2f_2^4}{f_5f_1^3} - \frac{f_2f_5^2}{f_{10}f_1^2}. \tag{3.4}$$

### 3.1 Proofs of (3.1) - (3.4):

We will prove relations (3.1) - (3.4) by using ideas similar to those of Watson [24]. In all proofs, one expresses the left sides of the identities in terms of theta functions by using (2.5). After clearing fractions, we see that the right side can be expressed as a product of two theta functions, say with summations indices $m$ and $n$. One then tries to find a change of indices of the form

$$\alpha m + \beta n = 5M + a \quad and \quad \gamma m + \delta n = 5N + b,$$

so that the product on the right side decomposes into the requisite sum of two products of theta functions on the left side.

**Theorem 3.1.** *The identity (3.1) holds.*

*Proof.* Using (1.6) - (1.9), we may rewrite (3.1) in the form

$$\begin{aligned} &f(-q^{32},-q^{28})\, f(-q^7,-q^8) + q^2 f(-q^{44},-q^{16})\, f(-q^{11},-q^4) \\ &+q^5 f(-q^8,-q^{52})\, f(-q^2,-q^{13}) + q^7 f(-q^{56},-q^4)\, f(-q^{14},-q) \\ &= f(q,q^2)\, f(-q^6,-q^6) - f(-q^5)\, f(-q^{20})\, q. \end{aligned} \tag{3.5}$$

Set

$$m + n = 5M + a \text{ and } m - 4n = 5N + b$$

where $a$ and $b$ will have values from the set $\{0, \pm1, \pm2\}$. Then

$$m = 4M + N + (4a + b)/5 \quad \text{and} \quad n = M - N + (a - b)/5.$$

It follows easily that $a = b$, and so $m = 4M + N + a$ and $n = M - N$, where $-2 \le a \le 2$. Thus there is a one-to-one correspondence between the set of all pairs of integers $(m, n)$, $-\infty < m, n < \infty$, and triple of integers $(M, N, a)$, $-\infty < M, N < \infty$, $-2 \le a \le 2$. Using (2.1) and (2.7), we obtain

$$\begin{aligned} f(q,q^2)\,\varphi(-q^6) &= \sum_{m,n=-\infty}^{\infty} (-1)^n\, q^{(3m^2+m+12n^2)/2} \\ &= \sum_{a=-2}^{2} q^{\frac{(3a^2+a)}{2}} \sum_{M=-\infty}^{\infty} (-1)^M\, q^{\frac{60M^2+(24a+4)M}{2}} \\ &\quad\times \sum_{N=-\infty}^{\infty} (-1)^N\, q^{\frac{15N^2+(6a+1)N}{2}} \\ &= \sum_{a=-2}^{2} q^{(3a^2+a)/2}\, f(-q^{32+12a}, -q^{28-12a})\, f(-q^{8+3a}, -q^{7-3a}) \end{aligned}$$

$$
\begin{aligned}
&= q^5 f(-q^8,-q^{52})\, f(-q^2,-q^{13}) + q\, f(-q^{20},-q^{40})\, f(-q^5,-q^{10})\\
&\quad + f(-q^{32},-q^{28})\, f(-q^7,-q^8) + q^2 f(-q^{44},-q^{16})\, f(-q^{11},-q^4)\\
&\quad + q^7 f(-q^{56},-q^4)\, f(-q^{14},-q)
\end{aligned}
$$

which is nothing but (3.5). This completes the proof of the theorem. □

**Theorem 3.2.** *The identity (3.2) holds.*

*Proof.* Using (1.6) - (1.9), we may rewrite (3.2) in the form

$$
\begin{aligned}
&f(-q^{24},-q^{21})\, f(-q^{16},-q^{14}) + q^2 f(-q^{33},-q^{12})\, f(-q^{22},-q^8)\\
&+q^5 f(-q^6,-q^{39})\, f(-q^4,-q^{26}) + q^7 f(-q^{42},-q^3)\, f(-q^{28},-q^2)\\
&= f(q,q^2)\, f(-q^9,-q^9) - f(-q^{10}) f(-q^{15})\, q.
\end{aligned} \tag{3.6}
$$

Set

$$m+2n = 5M+a \;\text{ and }\; m-3n = 5N+b$$

where $a$ and $b$ will have values from the set $\{0,\pm1,\pm2\}$. Then

$$m = 3M+2N+(3a+2b)/5 \quad\text{and}\quad n = M-N+(a-b)/5.$$

It follows easily that $a=b$, and so $m=3M+2N+a$ and $n=M-N$, where $-2\le a\le 2$. Thus there is a one-to-one correspondence between the set of all pairs of integers $(m,n)$, $-\infty< m,n<\infty$, and triple of integers $(M,N,a)$, $-\infty<M,N<\infty$, $-2\le a\le 2$. Using (2.1) and (2.7), we obtain

$$
\begin{aligned}
f(q,q^2)\,\varphi(-q^9) &= \sum_{m,n=-\infty}^{\infty} (-1)^n\, q^{(3m^2+m+18n^2)/2}\\
&= \sum_{a=-2}^{2} q^{\frac{(3a^2+a)}{2}} \sum_{M=-\infty}^{\infty} (-1)^M\, q^{\frac{45M^2+(18a+3)M}{2}}\\
&\quad\times \sum_{N=-\infty}^{\infty} (-1)^N\, q^{\frac{30N^2+(12a+2)N}{2}}\\
&= \sum_{a=-2}^{2} q^{(3a^2+a)/2}\, f(-q^{24+9a},-q^{21-9a})\, f(-q^{16+6a},-q^{14-6a})\\
&= q^5 f(-q^6,-q^{39})\, f(-q^4,-q^{26}) + q\, f(-q^{15},-q^{30})\, f(-q^{10},-q^{20})\\
&\quad + f(-q^{24},-q^{21})\, f(-q^{16},-q^{14}) + q^2 f(-q^{33},-q^{12})\, f(-q^{22},-q^8)\\
&\quad + q^7 f(-q^{42},-q^3)\, f(-q^{28},-q^2)
\end{aligned}
$$

which is nothing but (3.6). This completes the proof of the theorem. □

**Theorem 3.3.** *The identity (3.3) holds.*

*Proof.* Using (1.1), (1.2) and (1.6) - (1.9), we may write (3.3) in the form

$$
\begin{aligned}
&f(-q^{14},-q^{16})\, f(-q^2,-q^3) - q f(-q^8,-q^{22})\, f(-q,-q^4)\\
&-q^2 f(-q^4,-q^{26})\, f(-q^2,-q^3) - q^3 f(-q^2,-q^{28})\, f(-q,-q^4)\\
&= f(-q,-q)\, f(q,q^2).
\end{aligned} \tag{3.7}
$$

Set

$$m+n = 5M+a \;\text{ and }\; 2m-3n = 5N+b$$

where $a$ and $b$ will have values from the set $\{0,\pm1,\pm2\}$. Then

$$m = 3M+N+(3a+b)/5 \quad\text{and}\quad n = 2M-N+(2a-b)/5.$$

It follows that values of $a$ and $b$ are associated as in the following table:

| | | | |
|---|---|---|---|
| $a$ | 0 | $\pm1$ | $\pm2$ |
| $b$ | 0 | $\pm2$ | $\mp1$ |

When $a$ assumes the values -2, -1, 0, 1, 2 in succession, it is easy to see that the corresponding values of $2m^2+3n^2+n$ are, respectively,

$$\begin{aligned}30M^2-22M+5N^2+N+4,\\30M^2-10M+5N^2-5N+2,\\30M^2+2M+5N^2-N,\\30M^2+14M+5N^2-3N+2,\\30M^2+26M+5N^2-3N+6.\end{aligned}$$

Hence, by using (2.7), (2.1) and (2.4) we get

$$\begin{aligned}\varphi(-q)\,f(q,q^2)&=\sum_{m,n=-\infty}^{\infty}(-1)^m q^{\frac{2m^2+3n^2+n}{2}}\\&=-q^2f(-q^4,-q^{26})\,f(-q^2,-q^3)+f(-q^{14},-q^{16})\,f(-q^2,-q^3)\\&\quad-qf(-q^8,-q^{22})\,f(-q,-q^4)-q^3f(-q^2,-q^{28})\,f(-q,-q^4)\end{aligned}$$

which is nothing but (3.7). This completes the proof of the theorem. □

**Remark 3.4.** This result can also be proved by applying a formula proved by R. Bleckmith, J. Brillhart and I. Gerst [11, Theorem 2].

**Theorem 3.5.** *The identity (3.4) holds.*

*Proof.* Using (1.3), (1.4) and (1.6) - (1.9), we may write (3.4) in the form

$$\begin{aligned}&f(-q^7,-q^8)\,f(-q^3,-q^7)+qf(-q^4,-q^{11})\,f(-q,-q^9)\\&+qf(-q^2,-q^{13})\,f(-q^3,-q^7)-q^2f(-q^{14},-q)\,f(-q^9,-q)\\&=f(1,\,q)\,f(-q^2,-q^4)-f(-q^{10},-q^5)\,f(-q^5,-q^5).\end{aligned}\tag{3.8}$$

Set

$$m+2n=5M+a\ \text{ and }\ m-3n=5N+b$$

where $a$ and $b$ will have values from the set $\{0,\pm1,\pm2\}$. Then

$$m=3M+2N+(3a+2b)/5\quad\text{and}\quad n=M-N+(a-b)/5.$$

It follows easily that $a=b$, and so $m=3M+2N+a$ and $n=M-N$, where $-2\le a\le 2$. Thus there is a one-to-one correspondence between the set of all pairs of integers $(m,n)$, $-\infty<m,n<\infty$, and triple of integers $(M,N,a)$, $-\infty<M,N<\infty,\ -2\le a\le 2$. Using (2.1), (2.9) and (2.6), we obtain

$$\begin{aligned}f(1\,,q)\,f(-q^2,-q^4)&=\sum_{m,n=-\infty}^{\infty}(-1)^n\,q^{(m^2+m+6n^2+2n)/2}\\&=\sum_{a=-2}^{2}q^{\frac{(a^2+a)}{2}}\sum_{M=-\infty}^{\infty}(-1)^M\,q^{\frac{15M^2+(6a+15)M}{2}}\\&\quad\times\sum_{N=-\infty}^{\infty}(-1)^N\,q^{\frac{10N^2+4aN}{2}}\\&=\sum_{a=-2}^{2}q^{(a^2+a)/2}\,f(-q^{10+3a},-q^{5-3a})\,f(-q^{5+2a},-q^{5-2a})\\&=qf(-q^4,-q^{11})\,f(-q,-q^9)+f(-q^7,-q^8)\,f(-q^3,-q^7)\\&\quad+f(-q^5,-q^5)\,f(-q^{10},-q^5)+qf(-q^2,-q^{13})\,f(-q^3,-q^7)\\&\quad-q^2f(-q,-q^{14})\,f(-q^9,-q)\end{aligned}$$

which is nothing but (3.8). This completes the proof of the theorem. □

## 4 Applications to the theory of partitions

In this section, we present partition theoretic interpretations of (3.1)-(3.4). For simplicity, we adopt the standard notation

$$(a_1, a_2, \dots a_n; q)_\infty := \prod_{j=1}^{n} (a_j; q)_\infty$$

and define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty,$$

where $r$ and $s$ are positive integers and $r < s$.

We also need the notation of colored partitions. A positive integer $n$ has $k$ color if there are $k$ copies of $n$ available and all of them are viewed as distinct objects. Partitions of positive integer into parts with colors are called “colored partitions".
For example, if 1 is allowed to have two colors, say $r$ (red), and $g$ (green), then all colored partitions of 3 are 3, $2 + 1_r$, $2 + 1_g$, $1_r + 1_r + 1_r$, $1_r + 1_r + 1_g$, $1_r + 1_g + 1_g$, and $1_g + 1_g + 1_g$.

An important fact is that

$$\frac{1}{(q^u;\ q^v)_\infty^k}$$

is the generating function for the number of partitions of $n$, where all the parts are congruent to $u \ (mod\ v)$ and have $k$ colors.

**Theorem 4.1.** *Let $P_1(n)$ denote the number of partitions of $n$ into parts not congruent to* ±7, ±8, ±15, ±22, ±23, ±28 $(mod\ 60)$*, parts congruent to* ±3, ±9, ±12, ±20, ±21, ±24, ±27, 30 $(mod\ 60)$ *with two colors and parts congruent to* ±6, ±18 $(mod\ 60)$ *with three colors.*
*Let $P_2(n)$ denote the number of partitions of $n$ into parts not congruent to* ±4, ±11, ±15, ±16, ±19, ±26 $(mod\ 60)$*, parts congruent to* ±3, ±9, ±12, ±20, ±21, ±24, ±27, 30 $(mod\ 60)$ *with two colors and parts congruent to* ±6, ±18 $(mod\ 60)$ *with three colors.*
*Let $P_3(n)$ denote the number of partitions of $n$ into parts not congruent to* ±2, ±8, ±13, ±15, ±17, ±28 $(mod\ 60)$*, parts congruent to* ±3, ±9, ±12, ±20, ±21, ±24, ±27, 30 $(mod\ 60)$ *with two colors and parts congruent to* ±6, ±18 $(mod\ 60)$ *with three colors.*
*Let $P_4(n)$ denote the number of partitions of $n$ into parts not congruent to* ±1, ±4, ±14, ±15, ±16, ±29 $(mod\ 60)$*, parts congruent to* ±3, ±9, ±12, ±20, ±21, ±24, ±27, 30 $(mod\ 60)$ *with two colors and parts congruent to* ±6, ±18 $(mod\ 60)$ *with three colors.*
*Let $P_5(n)$ denote the number of partitions of $n$ into parts not congruent to* ±6, ±12, ±15, ±18, ±24, 30 $(mod\ 60)$*, parts congruent to* ±1, ±5, ±7, ±11, ±13, ±17, ±19, ±20, ±23, ±25, ±29, $(mod\ 60)$ *with two colors each.*
*Let $P_6(n)$ denote the number of partitions of $n$ into parts not congruent to* ±5, ±10, ±15, ±20, ±25, $(mod\ 60)$*, parts congruent to* ±3, ±9, ±12, ±21, ±24, ±27, 30 $(mod\ 60)$ *with two colors and parts congruent to* ±6, ±18 $(mod\ 60)$ *with three colors. Then, for any positive integer $n \geq 7$, we have*

$$P_1(n) + P_2(n-2) + P_3(n-5) + P_4(n-7) = P_5(n) - P_6(n-1).$$

*Proof.* Using (1.6)-(1.9) and (2.5) in (3.1) and simplifying we obtain

$$\frac{1}{(q^{1\pm},q^{2\pm},q^{3\pm},q^{3\pm},q^{4\pm},q^{5\pm},q^{6\pm},q^{6\pm},q^{6\pm},q^{9\pm},q^{9\pm},q^{10\pm},q^{11\pm},q^{12\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{12\pm},q^{13\pm},q^{14\pm},q^{16\pm},q^{17\pm},q^{18\pm},q^{18\pm},q^{18\pm},q^{19\pm},q^{20\pm},q^{20\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{21\pm},q^{21\pm},q^{24\pm},q^{24\pm},q^{25\pm},q^{26\pm},q^{27\pm},q^{27\pm},q^{29\pm},q^{30},q^{30};\ q^{60})_\infty}$$
$$+\frac{q^2}{(q^{1\pm},q^{2\pm},q^{3\pm},q^{3\pm},q^{5\pm},q^{6\pm},q^{6\pm},q^{6\pm},q^{7\pm},q^{8\pm},q^{9\pm},q^{9\pm},q^{10\pm},q^{12\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{12\pm},q^{13\pm},q^{14\pm},q^{17\pm},q^{18\pm},q^{18\pm},q^{18\pm},q^{20\pm},q^{20\pm},q^{21\pm},q^{21\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{22\pm},q^{23\pm},q^{24\pm},q^{24\pm},q^{25\pm},q^{27\pm},q^{27\pm},q^{28\pm},q^{29\pm},q^{30},q^{30};\ q^{60})_\infty}$$
$$+\frac{q^5}{(q^{1\pm},q^{3\pm},q^{3\pm},q^{4\pm},q^{5\pm},q^{6\pm},q^{6\pm},q^{6\pm},q^{7\pm},q^{9\pm},q^{9\pm},q^{10\pm},q^{11\pm},q^{12\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{12\pm},q^{14\pm},q^{16\pm},q^{18\pm},q^{18\pm},q^{18\pm},q^{19\pm},q^{20\pm},q^{20\pm},q^{21\pm},q^{21\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{22\pm},q^{23\pm},q^{24\pm},q^{24\pm},q^{25\pm},q^{26\pm},q^{27\pm},q^{27\pm},q^{29\pm},q^{30},q^{30};\ q^{60})_\infty}$$
$$+\frac{q^7}{(q^{2\pm},q^{3\pm},q^{3\pm},q^{5\pm},q^{6\pm},q^{6\pm},q^{6\pm},q^{7\pm},q^{8\pm},q^{9\pm},q^{9\pm},q^{10\pm},q^{11\pm},q^{12\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{12\pm},q^{13\pm},q^{17\pm},q^{18\pm},q^{18\pm},q^{18\pm},q^{19\pm},q^{20\pm},q^{20\pm},q^{21\pm},q^{21\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{22\pm},q^{23\pm},q^{24\pm},q^{24\pm},q^{25\pm},q^{26\pm},q^{27\pm},q^{27\pm},q^{28\pm},q^{30},q^{30};\ q^{60})_\infty}$$

$$=\frac{1}{(q^{1\pm},q^{1\pm},q^{2\pm},q^{3\pm},q^{4\pm},q^{5\pm},q^{5\pm},q^{7\pm},q^{7\pm},q^{8\pm},q^{9\pm},q^{10\pm},q^{11\pm},q^{11\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{13\pm},q^{13\pm},q^{14\pm},q^{16\pm},q^{17\pm},q^{17\pm},q^{19\pm},q^{19\pm},q^{20\pm},q^{20\pm},q^{21\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{22\pm},q^{23\pm},q^{23\pm},q^{25\pm},q^{25\pm},q^{26\pm},q^{27\pm},q^{28\pm},q^{29\pm},q^{29\pm};\ q^{60})_\infty}$$
$$-\frac{q}{(q^{1\pm},q^{2\pm},q^{3\pm},q^{3\pm},q^{4\pm},q^{6\pm},q^{6\pm},q^{6\pm},q^{7\pm},q^{8\pm},q^{9\pm},q^{9\pm},q^{11\pm},q^{12\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{12\pm},q^{13\pm},q^{14\pm},q^{16\pm},q^{17\pm},q^{18\pm},q^{18\pm},q^{18\pm},q^{19\pm},q^{21\pm},q^{21\pm};q^{60})_\infty}$$
$$\times\frac{1}{(q^{22\pm},q^{23\pm},q^{24\pm},q^{24\pm},q^{26\pm},q^{27\pm},q^{27\pm},q^{28\pm},q^{29\pm},q^{30},q^{30};\ q^{60})_\infty}.$$

Note that the six quotients of the above represent the generating functions for $P_1(n)$, $P_2(n)$, $P_3(n)$, $P_4(n)$, $P_5(n)$ and $P_6(n)$ respectively. Hence, it is equivalent to

$$\sum_{n=0}^{\infty}P_1(n)q^n+q^2\sum_{n=0}^{\infty}P_2(n)q^n+q^5\sum_{n=0}^{\infty}P_3(n)q^n+q^7\sum_{n=0}^{\infty}P_4(n)q^n$$
$$=\sum_{n=0}^{\infty}P_5(n)q^n-q\sum_{n=0}^{\infty}P_6(n)q^n$$

where we set $P_1(0)=P_2(0)=P_3(0)=P_4(0)=P_5(0)=P_6(0)=1$. Equating coefficients of $q^n$ $(n\geq 7)$ on both sides yields the desired result. □

**Example 4.2.** The following table illustrates the case $n = 7$ in the Theorem (4.1)

| | |
|---|---|
| $P_1(7) = 22$ | $6_r + 1,\ 6_g + 1,\ 6_w + 1,\ 5 + 2,\ 5 + 1 + 1,\ 4 + 3_r,\ 4 + 3_g,$<br>$4 + 2 + 1,\ 4 + 1 + 1 + 1,\ 3_r + 3_r + 1,\ 3_r + 3_g + 1,\ 3_g + 3_g + 1,$<br>$3_r + 2 + 2,\ 3_g + 2 + 2,\ 3_r + 2 + 1 + 1,\ 3_g + 2 + 1 + 1,$<br>$3_r + 1 + 1 + 1 + 1,\ 3_g + 1 + 1 + 1 + 1,\ 2 + 2 + 2 + 1,$<br>$2 + 2 + 1 + 1 + 1,\ 2 + 1 + 1 + 1 + 1 + 1,\ 1 + 1 + 1 + 1 + 1 + 1 + 1$ |
| $P_2(5) = 8$ | $5,\ 3_r + 2,\ 3_g + 2,\ 3_r + 1 + 1,\ 3_g + 1 + 1,$<br>$2 + 2 + 1,\ 2 + 1 + 1 + 1,\ 1 + 1 + 1 + 1 + 1$ |
| $P_3(2) = 1$ | $1 + 1$ |
| $P_4(0) = 1$ | $1$ |
| $P_5(7) = 48$ | $7_r,\ 7_g,\ 5_r + 2,\ 5_g + 2,\ 5_r + 1_r + 1_r,\ 5_r + 1_r + 1_g,$<br>$5_r + 1_g + 1_g,\ 5_g + 1_r + 1_r,\ 5_g + 1_r + 1_g,\ 5_g + 1_g + 1_g,\ 4 + 3,$<br>$4 + 2 + 1_r,\ 4 + 2 + 1_g,\ 4 + 1_r + 1_r + 1_r,\ 4 + 1_r + 1_r + 1_g,$<br>$4 + 1_r + 1_g + 1_g,\ 4 + 1_g + 1_g + 1_g,\ 3 + 3 + 1_r,\ 3 + 3 + 1_g,$<br>$3 + 2 + 1_r + 1_r,\ 3 + 2 + 1_r + 1_g,\ 3 + 2 + 1_g + 1_g,$<br>$3 + 1_r + 1_r + 1_r + 1_r,\ 3 + 1_r + 1_r + 1_r + 1_g,$<br>$3 + 1_r + 1_r + 1_g + 1_g,\ 3 + 1_r + 1_g + 1_g + 1_g,$<br>$3 + 1_g + 1_g + 1_g + 1_g,\ 3 + 2 + 2,\ 2 + 2 + 2 + 1_r,\ 2 + 2 + 2 + 1_g,$<br>$2 + 2 + 1_r + 1_r + 1_r,\ 2 + 2 + 1_r + 1_r + 1_g,\ 2 + 2 + 1_r + 1_g + 1_g,$<br>$2 + 2 + 1_g + 1_g + 1_g,\ 2 + 1_r + 1_r + 1_r + 1_r + 1_r,$<br>$2 + 1_r + 1_r + 1_r + 1_r + 1_g,\ 2 + 1_r + 1_r + 1_r + 1_g + 1_g,$<br>$2 + 1_r + 1_r + 1_g + 1_g + 1_g,\ 2 + 1_r + 1_g + 1_g + 1_g + 1_g,$<br>$2 + 1_g + 1_g + 1_g + 1_g + 1_g,\ 1_r + 1_r + 1_r + 1_r + 1_r + 1_r + 1_r,$<br>$1_r + 1_r + 1_r + 1_r + 1_r + 1_r + 1_g, 1_r + 1_r + 1_r + 1_r + 1_r + 1_g + 1_g,$<br>$1_r + 1_r + 1_r + 1_r + 1_g + 1_g + 1_g, 1_r + 1_r + 1_r + 1_g + 1_g + 1_g + 1_g,$<br>$1_r + 1_r + 1_g + 1_g + 1_g + 1_g + 1_g,\ 1_r + 1_g + 1_g + 1_g + 1_g + 1_g + 1_g,$<br>$1_g + 1_g + 1_g + 1_g + 1_g + 1_g + 1_g,$ |
| $P_6(6) = 16$ | $6_r,\ 6_g,\ 6_w,\ 4 + 2,\ 4 + 1 + 1,\ 3_r + 3_r\ ,3_r + 3_g\ ,3_g + 3_g\ ,$<br>$3_r + 2 + 1,\ 3_g + 2 + 1,\ 3_r + 1 + 1 + 1,\ 3_r + 1 + 1 + 1,\ 2 + 2 + 2$<br>$2 + 2 + 1 + 1,\ 2 + 1 + 1 + 1 + 1,\ 1 + 1 + 1 + 1 + 1 + 1$ |

**Theorem 4.3.** *Let $P_1(n)$ denote the number of partitions of $n$ into parts not congruent to* ±1, ±5, ±7, ±11, ±13, ±14, ±16, ±17, ±19, ±21, ±23, ±24, ±25, ±29, ±31, ±35, ±37, ±41, ±43, ±44, $(mod\ 90)$, *parts congruent to* ±18, ±36, 45 $(mod\ 90)$ *with two colors and parts congruent to* ±9, ±27 $(mod\ 90)$ *with three colors.*
*Let $P_2(n)$ denote the number of partitions of $n$ into parts not congruent to* ±1, ±5, ±7, ±8, ±11, ±12, ±13, ±17, ±19, ±22, ±23, ±25, ±29, ±31, ±33, ±35, ±37, ±38, ±41, ±43, $(mod\ 90)$, *parts congruent to* ±18, ±36, 45 $(mod\ 90)$ *with two colors and parts congruent to* ±9, ±27 $(mod\ 90)$ *with three colors.*
*Let $P_3(n)$ denote the number of partitions of $n$ into parts not congruent to* ±1, ±4, ±5, ±6, ±7, ±11, ±13, ±17, ±19, ±23, ±25, ±26, ±29, ±31, ±34, ±35, ±37, ±39, ±41, ±43, $(mod\ 90)$, *parts congruent to* ±18, ±36, 45 $(mod\ 90)$ *with two colors and parts congruent to* ±9, ±27 $(mod\ 90)$ *with three colors.*
*Let $P_4(n)$ denote the number of partitions of $n$ into parts not congruent to* ±1, ±2, ±3, ±5, ±7, ±11, ±13, ±17, ±19, ±23, ±25, ±28, ±29, ±31, ±32, ±35, ±37, ±41, ±42, ±43, $(mod\ 90)$, *parts congruent to* ±18, ±36, 45 $(mod\ 90)$ *with two colors and parts congruent to* ±9, ±27 $(mod\ 90)$ *with three colors.*
*Let $P_5(n)$ denote the number of partitions of $n$ into parts not congruent to* ±3, ±6, ±9, ±12,

$\pm 15$, $\pm 18$, $\pm 21$, $\pm 24$, $\pm 27$, $\pm 33$, $\pm 36$, $\pm 39$, $\pm 42$, $45$ $(mod\, 90)$.
*Let $P_6(n)$ denote the number of partitions of $n$ into parts not congruent to* $\pm 1$, $\pm 5$, $\pm 7$, $\pm 10$, $\pm 11$, $\pm 13$, $\pm 15$, $\pm 17$, $\pm 19$, $\pm 20$, $\pm 23$, $\pm 25$, $\pm 29$, $\pm 30$, $\pm 31$, $\pm 35$, $\pm 37$, $\pm 40$, $\pm 41$, $\pm 43$, $(mod\, 90)$, *parts congruent to* $\pm 18$, $\pm 36$, $45$ $(mod\, 90)$ *with two colors and parts congruent to* $\pm 9$, $\pm 27$ $(mod\, 90)$ *with three colors. Then, for any positive integer $n \geq 7$, we have*

$$P_1(n) + P_2(n-2) + P_3(n-5) + P_4(n-7) = P_5(n) - P_6(n-1).$$

*Proof.* Using (1.6)-(1.9) and (2.5) in (3.2) and simplifying we obtain

$$\begin{aligned}
&\frac{1}{(q^{2\pm}, q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{8\pm}, q^{9\pm}, q^{9\pm}, q^{9\pm}, q^{10\pm}, q^{12\pm}, q^{15\pm}, q^{18\pm}, q^{18\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{20\pm}, q^{22\pm}, q^{26\pm}, q^{27\pm}, q^{27\pm}, q^{27\pm}, q^{28\pm}, q^{30\pm}, q^{32\pm}, q^{33\pm}, q^{34\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{36\pm}, q^{36\pm}, q^{38\pm}, q^{39\pm}, q^{40\pm}, q^{42\pm}, q^{45}, q^{45};\ q^{90})_\infty} \\
&+ \frac{q^2}{(q^{2\pm}, q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{9\pm}, q^{9\pm}, q^{9\pm}, q^{10\pm}, q^{14\pm}, q^{15\pm}, q^{16\pm}, q^{18\pm}, q^{18\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{20\pm}, q^{21\pm}, q^{24\pm}, q^{26\pm}, q^{27\pm}, q^{27\pm}, q^{27\pm}, q^{28\pm}, q^{30\pm}, q^{32\pm}, q^{34\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{36\pm}, q^{36\pm}, q^{39\pm}, q^{40\pm}, q^{42\pm}, q^{44\pm}, q^{45}, q^{45};\ q^{90})_\infty} \\
&+ \frac{q^5}{(q^{2\pm}, q^{3\pm}, q^{8\pm}, q^{9\pm}, q^{9\pm}, q^{9\pm}, q^{10\pm}, q^{12\pm}, q^{14\pm}, q^{15\pm}, q^{16\pm}, q^{18\pm}, q^{18\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{20\pm}, q^{21\pm}, q^{22\pm}, q^{24\pm}, q^{27\pm}, q^{27\pm}, q^{27\pm}, q^{28\pm}, q^{30\pm}, q^{32\pm}, q^{33\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{36\pm}, q^{36\pm}, q^{38\pm}, q^{40\pm}, q^{42\pm}, q^{44\pm}, q^{45}, q^{45};\ q^{90})_\infty} \\
&+ \frac{q^7}{(q^{4\pm}, q^{6\pm}, q^{8\pm}, q^{9\pm}, q^{9\pm}, q^{9\pm}, q^{10\pm}, q^{12\pm}, q^{14\pm}, q^{15\pm}, q^{16\pm}, q^{18\pm}, q^{18\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{20\pm}, q^{21\pm}, q^{22\pm}, q^{24\pm}, q^{26\pm}, q^{27\pm}, q^{27\pm}, q^{27\pm}, q^{30\pm}, q^{33\pm}, q^{34\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{36\pm}, q^{36\pm}, q^{38\pm}, q^{39\pm}, q^{40\pm}, q^{44\pm}, q^{45}, q^{45};\ q^{90})_\infty} \\
=&\frac{1}{(q^{1\pm}, q^{2\pm}, q^{4\pm}, q^{5\pm}, q^{7\pm}, q^{8\pm}, q^{10\pm}, q^{11\pm}, q^{13\pm}, q^{14\pm}, q^{16\pm}, q^{17\pm}, q^{19\pm}, q^{20\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{22\pm}, q^{23\pm}, q^{25\pm}, q^{26\pm}, q^{28\pm}, q^{29\pm}, q^{30\pm}, q^{31\pm}, q^{32\pm}, q^{34\pm}, q^{35\pm}, q^{37\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{38\pm}, q^{40\pm}, q^{41\pm}, q^{43\pm}, q^{44\pm};\ q^{90})_\infty} \\
&- \frac{q}{(q^{2\pm}, q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{8\pm}, q^{9\pm}, q^{9\pm}, q^{9\pm}, q^{12\pm}, q^{14\pm}, q^{16\pm}, q^{18\pm}, q^{18\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{21\pm}, q^{22\pm}, q^{24\pm}, q^{26\pm}, q^{27\pm}, q^{27\pm}, q^{27\pm}, q^{28\pm}, q^{32\pm}, q^{33\pm}, q^{34\pm}; q^{90})_\infty} \\
&\times \frac{1}{(q^{36\pm}, q^{36\pm}, q^{38\pm}, q^{39\pm}, q^{42\pm}, q^{44\pm}, q^{45}, q^{45};\ q^{90})_\infty}.
\end{aligned}$$

Note that the six quotients of the above represent the generating functions for $P_1(n)$, $P_2(n)$, $P_3(n)$, $P_4(n)$, $P_5(n)$ and $P_6(n)$ respectively. Hence, it is equivalent to

$$\begin{aligned}
\sum_{n=0}^{\infty} P_1(n)q^n + q^2 \sum_{n=0}^{\infty} P_2(n)q^n + q^5 \sum_{n=0}^{\infty} P_3(n)q^n + q^7 \sum_{n=0}^{\infty} P_4(n)q^n \\
= \sum_{n=0}^{\infty} P_5(n)q^n - q \sum_{n=0}^{\infty} P_6(n)q^n
\end{aligned}$$

where we set $P_1(0) = P_2(0) = P_3(0) = P_4(0) = P_5(0) = P_6(0) = 1$. Equating coefficients of $q^n$ $(n \geq 7)$ on both sides yields the desired result. □

**Example 4.4.** The following table illustrates the case $n = 9$ in the Theorem(4.3)

| | |
|---|---|
| $P_1(9) = 7$ | $9_r,\ 9_g,\ 9_w,\ 6+3,\ 4+3+2,$ $3+3+3,\ 3+2+2+2$ |
| $P_2(7) = 2$ | $4+3,\ 3+2+2$ |
| $P_3(4) = 1$ | $2+2$ |
| $P_4(2) = 0$ | $0$ |
| $P_5(9) = 16$ | $8+1,\ 7+2,\ 7+1+1,\ 5+4,\ 5+2+2,$ $5+2+1+1,\ 5+1+1+1+1,\ 4+4+1,$ $4+2+1+1+1,\ 4+1+1+1+1+1,\ 4+2+2+1,$ $2+2+2+2+1,\ 2+2+2+1+1+1,$ $2+2+1+1+1+1+1,\ 2+1+1+1+1+1+1+1,$ $1+1+1+1+1+1+1+1+1$ |
| $P_6(8) = 6$ | $8,\ 6+2,\ 4+4,\ 4+2+2,$ $3+3+2,\ 2+2+2+2$ |

**Theorem 4.5.** *Let $P_1(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm2$, $\pm5$, $\pm7$, $\pm8$, $\pm10$, $\pm13$, $\pm14$ $(mod\ 30)$, parts congruent to $\pm3$, $\pm6$, $15$ $(mod\ 30)$ with two colors and parts congruent to $\pm9$ $(mod\ 30)$ with three colors.*
*Let $P_2(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm1$, $\pm4$, $\pm5$, $\pm8$, $\pm10$, $\pm11$, $\pm14$ $(mod\ 30)$, parts congruent to $\pm9$, $\pm12$, $15$ $(mod\ 30)$ with two colors and parts congruent to $\pm3$ $(mod\ 30)$ with three colors.*
*Let $P_3(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm2$, $\pm4$, $\pm5$, $\pm7$, $\pm8$, $\pm10$, $\pm13$ $(mod\ 30)$, parts congruent to $\pm3$, $\pm6$, $15$ $(mod\ 30)$ with two colors and parts congruent to $\pm9$ $(mod\ 30)$ with three colors.*
*Let $P_4(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm1$, $\pm2$, $\pm4$, $\pm5$, $\pm10$, $\pm11$, $\pm14$ $(mod\ 30)$, parts congruent to $\pm9$, $\pm12$, $15$ $(mod\ 30)$ with two colors and parts congruent to $\pm3$ $(mod\ 30)$ with three colors.Then, for any positive integer $n \geq 3$, we have*

$$P_1(n) - P_2(n-1) - P_3(n-2) - P_4(n-3) = 0.$$

*Proof.* Using (1.1), (1.2), (1.6)-(1.9) and (2.5) in (3.3) and simplifying we obtain

$$\begin{aligned}
&\frac{1}{(q^{1\pm}, q^{3\pm}, q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{6\pm}, q^{9\pm}, q^{9\pm}, q^{9\pm}, q^{11\pm}, q^{12\pm}, q^{15}, q^{15}; q^{30})_\infty} \\
&- \frac{q}{(q^{2\pm}, q^{3\pm}, q^{3\pm}, q^{3\pm}, q^{6\pm}, q^{7\pm}, q^{9\pm}, q^{9\pm}, q^{12\pm}, q^{12\pm}, q^{13\pm}, q^{15}, q^{15}; q^{30})_\infty} \\
&- \frac{q^2}{(q^{1\pm}, q^{3\pm}, q^{3\pm}, q^{6\pm}, q^{6\pm}, q^{9\pm}, q^{9\pm}, q^{9\pm}, q^{11\pm}, q^{12\pm}, q^{14\pm}, q^{15}, q^{15}; q^{30})_\infty} \\
&- \frac{q^3}{(q^{3\pm}, q^{3\pm}, q^{3\pm}, q^{6\pm}, q^{7\pm}, q^{8\pm}, q^{9\pm}, q^{9\pm}, q^{12\pm}, q^{12\pm}, q^{13\pm}, q^{15}, q^{15}; q^{30})_\infty} \\
=&1.
\end{aligned}$$

Note that the four quotients of the above represent the generating functions for $P_1(n)$, $P_2(n)$, $P_3(n)$ and $P_4(n)$ respectively. Hence, it is equivalent to

$$\sum_{n=0}^{\infty} P_1(n)q^n - q\sum_{n=0}^{\infty} P_2(n)q^n - q^2\sum_{n=0}^{\infty} P_3(n)q^n - q^3\sum_{n=0}^{\infty} P_4(n)q^n = 1$$

where we set $P_1(0) = P_2(0) = P_3(0) = P_4(0) = 1$. Equating coefficients of $q^n$ $(n \geq 3)$ on both sides yields the desired result. □

**Example 4.6.** The following table illustrates the case $n = 8$ in the Theorem (4.5)

| | |
|---|---|
| $P_1(8) = 12$ | $6_r + 1 + 1,\ 6_g + 1 + 1,\ 4 + 4,\ 4 + 3_r + 1,$ $4 + 3_g + 1,\ 4 + 1 + 1 + 1 + 1,\ 3_r + 3_r + 1 + 1,$ $3_r + 3_g + 1 + 1,\ 3_g + 3_g + 1 + 1,\ 3_r + 1 + 1 + 1 + 1 + 1,$ $3_g + 1 + 1 + 1 + 1 + 1,\ 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1,$ |
| $P_2(7) = 4$ | $7,\ 3_r + 2 + 2,\ 3_g + 2 + 2,\ 3_w + 2 + 2,$ |
| $P_3(6) = 8$ | $6_r,\ 6_g,\ 3_r + 3_r,\ 3_r + 3_g,\ 3_g + 3_g,$ $3_r + 1 + 1 + 1,\ 3_g + 1 + 1 + 1,\ 1 + 1 + 1 + 1 + 1 + 1,$ |
| $P_4(5) = 0$ | 0 |

**Theorem 4.7.** *Let $P_1(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm 3$, $\pm 7$ $(mod\ 30)$, parts congruent to $\pm 1$, $\pm 2$, $\pm 4$, $\pm 6$, $\pm 11$, $\pm 12$, $\pm 14$, 15 $(mod\ 30)$ with two colors and parts congruent to $\pm 5$ $(mod\ 30)$ with three colors.*
*Let $P_2(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm 9$, $\pm 11$ $(mod\ 30)$, parts congruent to $\pm 2$, $\pm 6$, $\pm 7$, $\pm 8$, $\pm 12$, $\pm 13$, $\pm 14$, 15 $(mod\ 30)$ with two colors and parts congruent to $\pm 5$ $(mod\ 30)$ with three colors.*
*Let $P_3(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm 3$, $\pm 13$ $(mod\ 30)$, parts congruent to $\pm 1$, $\pm 4$, $\pm 6$, $\pm 8$, $\pm 11$, $\pm 12$, $\pm 14$, 15 $(mod\ 30)$ with two colors and parts congruent to $\pm 5$ $(mod\ 30)$ with three colors.*
*Let $P_4(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm 1$, $\pm 9$ $(mod\ 30)$, parts congruent to $\pm 2$, $\pm 4$, $\pm 6$, $\pm 7$, $\pm 8$, $\pm 11$, $\pm 13$, 15 $(mod\ 30)$ with two colors and parts congruent to $\pm 5$ $(mod\ 30)$ with three colors.*
*Let $P_5(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm 2$, $\pm 4$, $\pm 6$, $\pm 8$, $\pm 10$, $\pm 12$, $\pm 14$ $(mod\ 30)$, parts congruent to $\pm 3$, $\pm 9$ $(mod\ 30)$ with two colors, parts congruent to $\pm 1$, $\pm 7$, $\pm 11$, $\pm 13$ $(mod\ 30)$ with three colors and parts congruent to $\pm 5$, 15 $(mod\ 30)$ with four colors.*
*Let $P_6(n)$ denote the number of partitions of $n$ into parts not congruent to $\pm 5$, $\pm 10$, 15 $(mod\ 30)$, parts congruent to $\pm 1$, $\pm 2$, $\pm 4$, $\pm 6$, $\pm 7$, $\pm 8$, $\pm 11$, $\pm 12$, $\pm 13$, $\pm 14$ $(mod\ 30)$ with two colors .*
*Then, for any positive integer $n \geq 2$, we have*

$$P_1(n) + P_2(n-1) + P_3(n-1) - P_4(n-2) = 2P_5(n) - P_6(n).$$

*Proof.* Using (1.3), (1.4) (1.6), (1.9) and (2.5) in (3.4) and simplifying we obtain

$$\frac{1}{(q^{1\pm}, q^{1\pm}, q^{2\pm}, q^{2\pm}, q^{4\pm}, q^{4\pm}, q^{5\pm}, q^{5\pm}, q^{5\pm}, q^{6\pm}, q^{6\pm}, q^{8\pm}, q^{9\pm}; q^{30})_\infty}$$
$$\times \frac{1}{(q^{10\pm}, q^{11\pm}, q^{11\pm}, q^{12\pm}, q^{12\pm}, q^{13\pm}, q^{14\pm}, q^{14\pm}, q^{15}, q^{15}; q^{30})_\infty}$$
$$+ \frac{q}{(q^{1\pm}, q^{2\pm}, q^{2\pm}, q^{3\pm}, q^{4\pm}, q^{5\pm}, q^{5\pm}, q^{5\pm}, q^{6\pm}, q^{6\pm}, q^{7\pm}, q^{7\pm}, q^{8\pm}, q^{8\pm}; q^{30})_\infty}$$
$$\times \frac{1}{(q^{10\pm}, q^{12\pm}, q^{12\pm}, q^{13\pm}, q^{13\pm}, q^{14\pm}, q^{14\pm}, q^{15}, q^{15}; q^{30})_\infty}$$
$$+ \frac{q}{(q^{1\pm}, q^{1\pm}, q^{2\pm}, q^{4\pm}, q^{4\pm}, q^{5\pm}, q^{5\pm}, q^{5\pm}, q^{6\pm}, q^{6\pm}, q^{7\pm}, q^{8\pm}, q^{8\pm}, q^{9\pm}; q^{30})_\infty}$$
$$\times \frac{1}{(q^{10\pm}, q^{11\pm}, q^{11\pm}, q^{12\pm}, q^{12\pm}, q^{14\pm}, q^{14\pm}, q^{15}, q^{15}; q^{30})_\infty}$$
$$- \frac{q^2}{(q^{2\pm}, q^{2\pm}, q^{3\pm}, q^{4\pm}, q^{4\pm}, q^{5\pm}, q^{5\pm}, q^{5\pm}, q^{6\pm}, q^{6\pm}, q^{7\pm}, q^{7\pm}, q^{8\pm}, q^{8\pm}; q^{30})_\infty}$$
$$\times \frac{1}{(q^{10\pm}, q^{11\pm}, q^{11\pm}, q^{12\pm}, q^{13\pm}, q^{13\pm}, q^{14\pm}, q^{15}, q^{15}; q^{30})_\infty}$$

$$=2\frac{1}{(q^{1\pm},q^{1\pm},q^{1\pm},q^{3\pm},q^{3\pm},q^{5\pm},q^{5\pm},q^{5\pm},q^{5\pm},q^{7\pm},q^{7\pm},q^{7\pm},q^{9\pm},q^{9\pm};q^{30})_\infty}$$
$$\times\frac{1}{(q^{11\pm},q^{11\pm},q^{11\pm},q^{13\pm},q^{13\pm},q^{13\pm},q^{15},q^{15},q^{15},q^{15};q^{30})_\infty}$$
$$-\frac{1}{(q^{1\pm},q^{1\pm},q^{2\pm},q^{2\pm},q^{3\pm},q^{4\pm},q^{4\pm},q^{6\pm},q^{6\pm},q^{7\pm},q^{7\pm},q^{8\pm},q^{8\pm},q^{9\pm};q^{30})_\infty}$$
$$\times\frac{1}{(q^{11\pm},q^{11\pm},q^{12\pm},q^{12\pm},q^{13\pm},q^{13\pm},q^{14\pm},q^{14\pm};q^{30})_\infty}.$$

Note that the six quotients of the above represent the generating functions for $P_1(n)$, $P_2(n)$, $P_3(n)$, $P_4(n)$, $P_5(n)$ and $P_6(n)$ respectively. Hence, it is equivalent to

$$\sum_{n=0}^{\infty}P_1(n)q^n+q\sum_{n=0}^{\infty}P_2(n)q^n+q\sum_{n=0}^{\infty}P_3(n)q^n-q^2\sum_{n=0}^{\infty}P_4(n)q^n$$
$$=2\sum_{n=0}^{\infty}P_5(n)q^n-\sum_{n=0}^{\infty}P_6(n)q^n$$

where we set $P_1(0)=P_2(0)=P_3(0)=P_4(0)=P_5(0)=P_6(0)=1$. Equating coefficients of $q^n$ $(n\geq 2)$ on both sides yields the desired result. □

**Example 4.8.** The following table illustrates the case $n=4$ in the Theorem (4.7)

| | |
|---|---|
| $P_1(4)=16$ | $4_r,\ 4_g,\ 2_r+2_r,\ 2_r+2_g,\ 2_g+2_g,\ 2_r+1_r+1_r,$ $2_r+1_r+1_g,\ 2_r+1_g+1_g,\ 2_g+1_r+1_r,$ $2_g+1_r+1_g,\ 2_g+1_g+1_g,\ 1_r+1_r+1_r+1_r,$ $1_r+1_r+1_r+1_g,\ 1_r+1_r+1_g+1_g,$ $1_r+1_g+1_g+1_g,\ 1_g+1_g+1_g+1_g$ |
| $P_2(3)=4$ | $3,\ 2_r+1,\ 2_g+1,\ 1+1+1,$ |
| $P_3(3)=6$ | $2+1_r,\ 2+1_g,\ 1_r+1_r+1_r$ $1_r+1_r+1_g,\ 1_r+1_g+1_g,\ 1_g+1_g+1_g$ |
| $P_4(2)=2$ | $2_r,\ 2_g$ |
| $P_5(4)=21$ | $3_r+1_r,\ 3_r+1_g,\ 3_r+1_w,\ 3_g+1_r,\ 3_g+1_g,\ 3_g+1_w,$ $1_r+1_r+1_r+1_r,\ 1_r+1_r+1_r+1_g,\ 1_r+1_r+1_r+1_w,$ $1_r+1_r+1_g+1_g,\ 1_r+1_r+1_g+1_w,\ 1_r+1_r+1_w+1_w,$ $1_r+1_g+1_g+1_g,\ 1_r+1_g+1_g+1_w,\ 1_r+1_g+1_w+1_w,$ $1_r+1_w+1_w+1_w,\ 1_g+1_g+1_g+1_g,\ 1_g+1_g+1_g+1_w,$ $1_g+1_g+1_w+1_w,\ 1_g+1_w+1_w+1_w,\ 1_w+1_w+1_w+1_w$ |
| $P_6(4)=18$ | $4_r,\ 4_g,\ 3+1_r,\ 3+1_g,\ 2_r+2_r,\ 2_r+2_g,\ 2_g+2_g,$ $2_r+1_r+1_r,\ 2_r+1_r+1_g,\ 2_r+1_g+1_g,\ 2_g+1_r+1_r,$ $2_g+1_r+1_g,\ 2_g+1_r+1_g,\ 1_r+1_r+1_r+1_r,$ $1_r+1_r+1_r+1_g,\ 1_r+1_r+1_g+1_g,\ 1_r+1_g+1_g+1_g,$ $1_g+1_g+1_g+1_g$ |

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