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Arithmetical property in amalgamated algebras along an ideal

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Abstract. Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. In this paper, we investigate the transfer of the notion of valuation ring and arithmetical ring to the amalgamation $A \bowtie^f J$. If A and B are integral domains, then we provide necessary and sufficient conditions for $A \bowtie^f J$ to be an arithmetical ring and Prüfer domain.

1 Introduction

Throughout this paper all rings considered are assumed to be commutative, and have identity element and all modules are unitary.

Following Kaplansky [12], a ring R is said to be a valuation ring if for any two elements in R, one divides the other. By an arithmetical ring is understood a ring R for which the ideals form a distributive lattice [11], i.e. for which

$$(\mathfrak{a} + \mathfrak{b}) \cap \mathfrak{c} = (\mathfrak{a} \cap \mathfrak{c}) + (\mathfrak{b} \cap \mathfrak{c}) \text{ for all ideals of } R.$$

In [11], it is shown that R is an arithmetical ring if and only if each localization $R_\mathfrak{m}$ at a maximal ideal $\mathfrak{m}$ is a valuation ring. Note that an arithmetical domain is a Prüfer domain. See for instance [1, 2, 9, 10].

Let A and B be rings, J an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. The following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \; ; \; a \in A, j \in J\}$$

is said to be amalgamation of A with B along J with respect to f introduced and studied by D'Anna, Finocchiaro and Fontana in [6] and in [7]. In particular, they have studied amalgmations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [3, 4, 5]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations). See for instance [3, 4, 5, 6, 7].

In this paper, we investigate the transfer of the notion of valuation ring and arithmetical ring to the amalgamation $A \bowtie^f J$. If A and B are integral domains, then we provide necessary and sufficient conditions for $A \bowtie^f J$ to be an arithmetical ring and Prüfer domain.

2 Main Results

We first develop a result on the transfer of the valuation property to amalgamation rings.

Theorem 2.1. Let A and B be a pair of rings, J an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. Then:

(1) If f is not injective, then $A \bowtie^f J$ is a valuation ring if and only if A is a valuation ring and $J = (0)$.

(2) If f is injective, then $A \bowtie^f J$ is a valuation ring if and only if $f(A) + J$ is a valuation ring and $f(A) \cap J = (0)$.

Proof. (1) Assume that $A \bowtie^f J$ is a valuation ring. Since f is not injective, there is some $0 \neq a \in \ker f$. We claim that $J = (0)$.
Let $x \in J$. Then $(a, 0) = (a, f(a)) \in A \bowtie^f J$ and $(0, x) \in A \bowtie^f J$. Hence $(0, x) \in \left(A \bowtie^f J\right)(a, 0)$ (since $a \neq 0$) and so $(0, x) = (a, 0)(b, f(b)+j)$ for some $(b, f(b)+j) \in A \bowtie^f J$. Hence $x = 0$, and so $J = (0)$.
It remains to show that A is a valuation ring. Let $(\alpha, \beta) \in A^2$. Since $A \bowtie^f J$ is a valuation ring then $(\alpha, f(\alpha)) \in \left(A \bowtie^f J\right)(\beta, f(\beta))$ or $(\beta, f(\beta)) \in \left(A \bowtie^f J\right)(\alpha, f(\alpha))$. We conclude that $\alpha \in A\beta$ or $\beta \in A\alpha$, as desired.
Conversely, assume that $J = (0)$ and A is a valuation ring. Then $A \bowtie^f J$ is isomorphic to A and so $A \bowtie^f J$ is a valuation ring.

(2) Let $\varphi : A \bowtie^f J \longrightarrow f(A) + J$ be the ring homomorphism defined by

$$\varphi(a, f(a) + j) = f(a) + j.$$

We have $\dfrac{A \bowtie^f J}{f^{-1}(J) \times (0)} \simeq f(A) + J$, since φ is surjective and $\ker \varphi = f^{-1}(J) \times (0)$. Assume that f is injective. If $f(A) \cap J = (0)$ and $f(A) + J$ is a valuation ring, then $f^{-1}(J) = (0)$ and $A \bowtie^f J \simeq f(A) + J$. It follows that $A \bowtie^f J$ is a valuation ring. Conversely, assume that $A \bowtie^f J$ is a valuation ring. Since φ is a surjective ring homomorphism, then $f(A) + J$ is a valuation ring. Now suppose that $f(A) \cap J \neq (0)$, and choose an element $f(a) \neq 0$ in J, where $a \in A$. We have $(a, 0) \in A \bowtie^f J$, and so $(a, 0) \in \left(A \bowtie^f J\right)(0, f(a))$ or $(0, f(a)) \in \left(A \bowtie^f J\right)(a, 0)$, a contradiction. This completes the proof of Theorem 2.1. $\square$

Remark 2.2. Let $f : A \longrightarrow B$ be an injective ring homomorphism and let J be an ideal of B. If $A \bowtie^f J$ is a valuation ring and $J \neq (0)$, then A is a valuation domain.

Proof. Suppose that the statement is false, and choose an element $(a, b) \in A^2$ such that $a \neq 0, b \neq 0$ and $ab = 0$. For each $x \in J$ there is $(c, f(c) + y) \in A \bowtie^f J$ such that $(b, f(b))(c, f(c) + y) = (0, x)$. Then $bc = 0$ and $f(b)y = x$, therefore $f(a)x = 0$ and $f(a) \in (0 : J)$. For each $x \in J$, we can write $(a, f(a))(d, f(d) + z) = (0, x)$, where $(d, f(d) + z)$ is an element of $A \bowtie^f J$. Hence $x = f(a)z = 0$ which contradicts $J \neq (0)$. $\square$

Corollary 2.3. Let A be a ring and let I be an ideal of A. Then $A \bowtie I$ is a valuation ring if and only if A is a valuation ring and $I = (0)$.

Now, we are able to give our main result about the transfer of arithmetical property to amalgamation of rings.

Theorem 2.4. Let A and B be a pair of integral domains, $f : A \longrightarrow B$ a ring homomorphism and let J be a proper ideal of B. Then:

(1) If $A \bowtie^f J$ is an arithmetical ring then A is an arithmetical ring.

(2) If f is injective, then $A \bowtie^f J$ is an arithmetical ring if and only if $f(A) + J$ is an arithmetical ring and $f(A) \cap J = (0)$.

(3) If f is not injective, then $A \bowtie^f J$ is not an arithmetical ring.

The proof of this theorem draws on the following results.

Lemma 2.5. Let $f : A \longrightarrow B$ be a ring homomorphism, J an ideal of B and let $\mathfrak{m}$ be a maximal ideal of A. Set $S = f(A \setminus \mathfrak{m}) + J$. Then S is a closed subset of B and the correspondence $F : A_{\mathfrak{m}} \longrightarrow S^{-1}B$, defined by $F\left(\dfrac{a}{s}\right) = \dfrac{f(a)}{f(s)}$ for all $\dfrac{a}{s} \in A_{\mathfrak{m}}$ is a ring homomorphism.

Proof. Let $s, t \in A \setminus \mathfrak{m}$ and $x, y \in J$, we have the equality

$$(f(s)+x)(f(t)+y) = f(st) + (f(s)y + f(t)x + xy).$$

Then S is a closed subset of B. Let $a, b \in A$ and $s, t \in A \setminus \mathfrak{m}$, such that $\frac{a}{s} = \frac{b}{t}$. Then there exists $u \in A \setminus \mathfrak{m}$ such that $uta = usb$ and so $f(u)f(t)f(a) = f(u)f(s)f(b)$. Hence, $\frac{f(a)}{f(s)} = \frac{f(b)}{f(t)}$ and so F is a mapping. Let $\frac{a}{s}, \frac{b}{t} \in A_{\mathfrak{m}}$. It is easy to get successively that

$$F\left(\frac{a}{s} + \frac{b}{t}\right) = F\left(\frac{a}{s}\right) + F\left(\frac{b}{t}\right),\ F\left(\frac{a}{s}\frac{b}{t}\right) = F\left(\frac{a}{s}\right) F\left(\frac{b}{t}\right)$$

and $F(1) = 1$. We deduce that F is a ring homomorphism. □

Lemma 2.6. With the notations of the above lemma, set

$$M = \mathfrak{m} \bowtie^f J = \{(a, f(a)+j)\ ;\ a \in \mathfrak{m}, j \in J\}.$$

Then the correspondence between the ring $\left(A \bowtie^f J\right)_M$ and $A_{\mathfrak{m}} \bowtie^F S^{-1}J$, $\varphi : \left(A \bowtie^f J\right)_M \longrightarrow A_{\mathfrak{m}} \bowtie^F S^{-1}J$ where

$$\varphi\left(\frac{(a, f(a)+x)}{(s, f(s)+y)}\right) = \left(\frac{a}{s}, \frac{f(a)+x}{f(s)+y}\right)$$

is a ring isomorphism.

Proof. We begin by showing that φ is a mapping. Then $M := \mathfrak{m} \bowtie^f J$, is a maximal ideal of $A \bowtie^f J$ by [7, Proposition 2.6]. For each $\frac{(a, f(a)+x)}{(s, f(s)+y)} \in \left(A \bowtie^f J\right)_M$, we have the following equalities:

$$F\left(\frac{a}{s}\right) + \frac{f(s)x - f(a)y}{f(s)(f(s)+y)} = \frac{f(a)(f(s)+y) + f(s)x - f(a)y}{f(s)(f(s)+y)} = \frac{f(a)+x}{f(s)+y}.$$

Therefore, $\left(\frac{a}{s}, \frac{f(a)+x}{f(s)+y}\right) \in A_{\mathfrak{m}} \bowtie^F S^{-1}J$. Let $a, a' \in A,\ s, s' \in A \setminus \mathfrak{m}$, and $x, y, x', y' \in J$, such that $\frac{(a, f(a)+x)}{(s, f(s)+y)} = \frac{(a', f(a')+x')}{(s', f(s')+y')}$. Then there exists $(t, f(t)+z) \in S$ such that

$$(t, f(t)+z)(s', f(s')+y')(a, f(a)+x) = (t, f(t)+z)(s, f(s)+y)(a', f(a')+x')$$

and so

$$\begin{cases} ts'a = tsa' \\ (f(t)+z)(f(s')+y')(f(a)+x) = (f(t)+z)(f(s)+y)(f(a')+y). \end{cases}$$

We deduce that $\frac{a}{s} = \frac{a'}{s'}$ and $\frac{f(a)+x}{f(s)+y} = \frac{f(a')+x'}{f(s')+y'}$. It follows that φ is map of the ring $\left(A \bowtie^f J\right)_M$ into the ring $A_{\mathfrak{m}} \bowtie^F S^{-1}J$. From the definition of φ, we have $\varphi(1) = 1$. Let $X = \frac{(a, f(a)+x)}{(s, f(s)+j)}$, $Y = \frac{(b, f(b)+y)}{(s, f(s)+j)}$ be elements of $\left(A \bowtie^f J\right)_M$, we have clearly the equalities $\varphi(X+Y) = \varphi(X)+\varphi(Y)$ and $\varphi(XY) = \varphi(X)\varphi(Y)$. It follows that φ is a ring homomorphism. We need only show that φ is bijective. Let $X = \frac{(a, f(a)+x)}{(s, f(s)+y)} \in \ker\varphi$ then $\frac{a}{s} = 0$ and $\frac{f(a)+x}{f(s)+y} = 0$. There is some $(t, f(u)+j) \in A_{\mathfrak{m}} \times S$ such that $ta = 0$ and $(f(u)+j)((f(a)+x) = 0$. Multiplying the above equality by $f(t)$ we get that $(tu, f(tu) + f(t)j)(a, f(a)+x) = 0$. It follows that $X = 0$ and $\ker\varphi = (0)$, so φ is injective. Let $a \in A,\ s, t \in A \setminus \mathfrak{m}$ and let $x, y \in J$. Then we have the following equality:

$$\left(\frac{a}{s}, F\left(\frac{a}{s}\right) + \frac{x}{f(t)+y}\right) = \left(\frac{at}{st}, \frac{f(at) + f(a)y + f(s)x}{f(st) + f(s)y}\right).$$

We put $b = at, u = st, z = f(a)y + f(s)x$ and $j = f(s)y$. From the previous equalities we deduce that

$$\left(\frac{a}{s}, F\left(\frac{a}{s}\right) + \frac{x}{f(t)+y}\right) = \varphi\left(\frac{(b, f(b)+z)}{(u, f(u)+j)}\right).$$

Consequently, φ is surjective. We conclude that φ is a ring isomorphism. This completes the proof of Lemma 2.6. □

Proof. of Theorem 2.4.

(1) straightforward.

(2) Assume that $A \bowtie^f J$ is an arithmetical ring. Since $f^{-1}(J) \subsetneq A$ there exists a maximal ideal $\mathfrak{m}$ of A containing $f^{-1}(J)$. Let S be as in Lemma 2.5. By [7, Proposition 2.6], $M = \mathfrak{m} \bowtie^f J$ is a maximal ideal of $A \bowtie^f J$. Thus $\left(A \bowtie^f J\right)_M$ is a valuation ring. We can now apply Lemma 2.6 to obtain that $A_{\mathfrak{m}} \bowtie^F S^{-1}J$ is a valuation ring, where $F : A_{\mathfrak{m}} \longrightarrow S^{-1}B$ is the ring homomorphism defined by $F\left(\frac{a}{s}\right) = \frac{f(a)}{f(s)}$. Let $\frac{a}{s} \in \ker F$, there is some $(t, j) \in (A \backslash \mathfrak{m}) \times J$ such that $(f(t)+j)f(a) = 0$. If $f(t)+j = 0$ then $t \in f^{-1}(J)$ which contradicts the containment $f^{-1}(J) \subseteq \mathfrak{m}$. Hence, $f(a) = 0$ since B is an integral domain. It follows that $a = 0$ and so F is injective. By applying statement (2) of Theorem 2.1, we get that $F(A_{\mathfrak{m}}) \cap S^{-1}J = (0)$ and $F(A_{\mathfrak{m}}) + S^{-1}J$ is a valuation ring. Now, we wish to show that $f(A) \cap J = (0)$. Let a be an element A such that $f(a) \in J$. We have clearly $F\left(\frac{a}{1}\right) = \frac{f(a)}{1} \in F(A_{\mathfrak{m}}) \cap S^{-1}J = (0)$ and so $\frac{f(a)}{1} = 0$. From the previous part of the proof, we deduce that $a = 0$ and so $f(A) \cap J = (0)$. On the other hand, the natural projection of $A \bowtie^f J \subseteq A \times B$ into B, φ is injective (since so is f). Hence $A \bowtie^f J \simeq f(A) + J$. Consequently, $f(A) + J$ is an arithmetical ring and the necessary condition follows.
From the previous part of the proof, we get the sufficient condition.

(3) Suppose that $A \bowtie^f J$ is an arithmetical ring, and choose $0 \neq j \in J$. Let $\mathfrak{m}$ be a maximal ideal of A, $S = f(A \setminus \mathfrak{m}) + J$ and let $F : A_{\mathfrak{m}} \longrightarrow S^{-1}B$ be the ring homomorphism defined by $F\left(\frac{a}{s}\right) = \frac{f(a)}{f(s)}$ (by Lemma 2.5). It is easy to see that $0 \neq \frac{a}{1} \in \ker F$, if $0 \neq a \in \ker f$. Hence F is not injective. By applying Lemma 2.6 and condition (1) of Theorem 2.1, we get successively that $A_{\mathfrak{m}}$ is a valuation ring and $S^{-1}J = (0)$. Hence there exists $f(t_{\mathfrak{m}}) + j_{\mathfrak{m}} \in S$ such that $(f(t_{\mathfrak{m}}) + j_{\mathfrak{m}})\, j = 0$. From the assumption, we can write $f(t_{\mathfrak{m}}) + j_{\mathfrak{m}} = 0$. Let I be the ideal of A generated by all $t_{\mathfrak{m}}$. For every maximal ideal $\mathfrak{m}$ of A, we have $I \nsubseteq \mathfrak{m}$ since $t_{\mathfrak{m}} \in I \setminus \mathfrak{m}$, therefore $I = A$. We can write $1 = t_1x_1 + \cdots + t_nx_n$, where $x_i \in A,\ t_i \in A \setminus \mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$ of A. It follows that

$$1 = f(t_1)f(x_1) + \cdots + f(t_n)f(x_n).$$

We conclude that $J = B$, since $f(t_i) \in J$. We have the desired contradiction. This completes the proof of Theorem 2.4. □

Remark 2.7. Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B.

- If $J = (0)$ then $A \bowtie^f J$ is an arithmetical ring if and only if A is an arithmetical ring.
- If $J = B$ then $A \bowtie^f J$ is an arithmetical ring if and only if A and B are arithmetical rings.

Proof. Since the product $A \times B$ is an arithmetical ring if and only if A and B are arithmetical rings, the conclusion is straightforward. □

Corollary 2.8. Let A be an integral domain and let I be a proper ideal of A. Then $A \bowtie I$ is never an arithmetical ring.

Now, we are able to give the transfer of Prüfer domain to amalgamation of rings.

Corollary 2.9. Let A and B be a pair of integral domains, $f : A \longrightarrow B$ a ring homomorphism and let J be a proper ideal of B. Then $A \bowtie^f J$ is a Prüfer domain if and only if $f(A) + J$ is a Prüfer domain and $f(A) \cap J = (0)$.

Proof. By Theorem 2.4 and [6, Proposition 5.2]. □

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