Arithmetical property in amalgamated algebras along an ideal
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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

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Abstract. Let \( f : A \longrightarrow B \) be a ring homomorphism and let \( J \) be an ideal of \( B \). In this paper, we investigate the transfer of the notion of valuation ring and arithmetical ring to the amalgamation \( A \triangleleft_f J \). If \( A \) and \( B \) are integral domains, then we provide necessary and sufficient conditions for \( A \triangleleft_f J \) to be an arithmetical ring and Prüfer domain.

1 Introduction
Throughout this paper all rings considered are assumed to be commutative, and have identity element and all modules are unitary.

Following Kaplansky [12], a ring \( R \) is said to be a valuation ring if for any two elements in \( R \), one divides the other. By an arithmetical ring is understood a ring \( R \) for which the ideals form a distributive lattice [11], i.e. for which

\[(a + b) \cap c = (a \cap c) + (b \cap c) \text{ for all ideals of } R.\]

In [11], it is shown that \( R \) is an arithmetical ring if and only if each localization \( R_m \) at a maximal ideal \( m \) is a valuation ring. Note that an arithmetical domain is a Prüfer domain. See for instance [1, 2, 9, 10].

Let \( A \) and \( B \) be rings, \( J \) an ideal of \( B \) and let \( f : A \longrightarrow B \) be a ring homomorphism. The following subring of \( A \times B \):

\[A \triangleleft_f J = \{(a, f(a) + j) ; a \in A, j \in J\}\]

is said to be amalgamation of \( A \) with \( B \) along \( J \) with respect to \( f \) introduced and studied by D’Anna, Finocchiaro and Fontana in [6] and in [7]. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D’Anna and Fontana in [3, 4, 5]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata’s idealizations). See for instance [3, 4, 5, 6, 7].

In this paper, we investigate the transfer of the notion of valuation ring and arithmetical ring to the amalgamation \( A \triangleleft_f J \). If \( A \) and \( B \) are integral domains, then we provide necessary and sufficient conditions for \( A \triangleleft_f J \) to be an arithmetical ring and Prüfer domain.

2 Main Results
We first develop a result on the transfer of the valuation property to amalgamation rings.

**Theorem 2.1.** Let \( A \) and \( B \) be a pair of rings, \( J \) an ideal of \( B \) and let \( f : A \longrightarrow B \) be a ring homomorphism. Then:

(1) If \( f \) is not injective, then \( A \triangleleft_f J \) is a valuation ring if and only if \( A \) is a valuation ring and \( J = (0) \).
(2) If $f$ is injective, then $A \bowtie^f J$ is a valuation ring if and only if $f(A) + J$ is a valuation ring and $f(A) \cap J = (0)$.

**Proof.** (1) Assume that $A \bowtie^f J$ is a valuation ring. Since $f$ is not injective, there is some $0 \neq a \in \ker f$. We claim that $J = (0)$.

Let $x \in J$. Then $(a,0) = (a,f(a)) \in A \bowtie^f J$ and $(0,x) \in A \bowtie^f J$. Hence $(0,x) \in (A \bowtie^f J)(a,0)$ (since $a \neq 0$) and so $(0,x) = (a,0)(b,f(b)+j)$ for some $(b,f(b)+j) \in A \bowtie^f J$. Hence $x = 0$, and so $J = (0)$.

It remains to show that $A$ is a valuation ring. Let $(\alpha, \beta) \in A^2$. Since $A \bowtie^f J$ is a valuation ring then $(\alpha, f(\alpha)) \in (A \bowtie^f J)(\beta, f(\beta))$ or $(\beta, f(\beta)) \in (A \bowtie^f J)(\alpha, f(\alpha))$. We conclude that $\alpha \in A \beta$ or $\beta \in A \alpha$, as desired.

Conversely, assume that $J = (0)$ and $A$ is a valuation ring. Then $A \bowtie^f J$ is isomorphic to $A$ and so $A \bowtie^f J$ is a valuation ring.

(2) Let $\varphi : A \bowtie^f J \longrightarrow f(A) + J$ be the ring homomorphism defined by $\varphi(a,f(a)+j) = f(a)+j$.

We have $\frac{A \bowtie^f J}{f^{-1}(J) \times (0)} \cong f(A) + J$, since $\varphi$ is surjective and $\ker \varphi = f^{-1}(J) \times (0)$. Assume that $f$ is injective. If $f(A) \cap J = (0)$ and $f(A) + J$ is a valuation ring, then $f^{-1}(J) = (0)$ and $A \bowtie^f J \cong f(A) + J$. It follows that $A \bowtie^f J$ is a valuation ring. Conversely, assume that $A \bowtie^f J$ is a valuation ring. Since $\varphi$ is a surjective ring homomorphism, then $f(A) + J$ is a valuation ring. Now suppose that $f(A) \cap J \neq (0)$, and choose an element $f(a) \neq 0$ in $J$, where $a \in A$. We have $(a,0) \in A \bowtie^f J$, and so $(a,0) \in (A \bowtie^f J)(0,f(a))$ or $(0,f(a)) \in (A \bowtie^f J)(a,0)$, a contradiction. This completes the proof of Theorem 2.1. □

**Remark 2.2.** Let $f : A \longrightarrow B$ be an injective ring homomorphism and let $J$ be an ideal of $B$. If $A \bowtie^f J$ is a valuation ring and $J \neq (0)$, then $A$ is a valuation domain.

**Proof.** Suppose that the statement is false, and choose an element $(a,b) \in A^2$ such that $a \neq 0$, $b \neq 0$ and $ab = 0$. For each $x \in J$ there is $(c,f(c)+y) \in A \bowtie^f J$ such that $(b,f(b))(c,f(c)+y) = (0,x)$. Then $bc = 0$ and $f(b)y = x$, therefore $f(a)x = 0$ and $f(a) \in (0 : J)$. For each $x \in J$, we can write $(a,f(a))(d,f(d)+z) = (0,x)$, where $(d,f(d)+z)$ is an element of $A \bowtie^f J$. Hence $x = f(a)z = 0$ which contradicts $J \neq (0)$.

□

**Corollary 2.3.** Let $A$ be a ring and let $I$ be an ideal of $A$. Then $A \bowtie^f I$ is a valuation ring if and only if $I$ is a valuation ring and $I = (0)$.

Now, we are able to give our main result about the transfer of arithmetical property to amalgamation of rings.

**Theorem 2.4.** Let $A$ and $B$ be a pair of integral domains, $f : A \longrightarrow B$ a ring homomorphism and let $J$ be a proper ideal of $B$. Then:

1. If $A \bowtie^f J$ is an arithmetical ring then $A$ is an arithmetical ring.
2. If $f$ is injective, then $A \bowtie^f J$ is an arithmetical ring if and only if $f(A)+J$ is an arithmetical ring and $f(A) \cap J = (0)$.
3. If $f$ is not injective, then $A \bowtie^f J$ is not an arithmetical ring.

The proof of this theorem draws on the following results.

**Lemma 2.5.** Let $f : A \longrightarrow B$ be a ring homomorphism, $J$ an ideal of $B$ and let $m$ be a maximal ideal of $A$. Set $S = f(A \setminus m) + J$. Then $S$ is a closed subset of $B$ and the correspondence $F : A_m \longrightarrow S^{-1}B$, defined by $F \left( \frac{a}{s} \right) = \frac{f(a)}{f(s)}$ for all $\frac{a}{s} \in A_m$ is a ring homomorphism.
Proof. Let \( s, t \in A \setminus m \) and \( x, y \in J \), we have the equality
\[
(f(s) + x)(f(t) + y) = f(st) + (f(s)y + f(t)x + xy).
\]

Then \( S \) is a closed subset of \( B \). Let \( a, b \in A \) and \( s, t \in A \setminus m \), such that \( \frac{a}{s} = \frac{b}{t} \). Then there exists \( u \in A \setminus m \) such that \( u(a) = ub \). Let \( f(u) f(a) = f(a) f(b) \). Hence, \( f(a) f(s) = f(b) f(t) \) and so \( f \) is a mapping. Let \( \frac{a}{s}, \frac{b}{t} \in A_m \). It is easy to get successively that
\[
F\left( \frac{a}{s} + \frac{b}{t} \right) = F\left( \frac{a}{s} \right) + F\left( \frac{b}{t} \right), \quad F\left( \frac{ab}{st} \right) = F\left( \frac{a}{s} \right) F\left( \frac{b}{t} \right)
\]
and \( F(1) = 1 \). We deduce that \( F \) is a ring homomorphism. \hfill \Box

Lemma 2.6. With the notations of the above lemma, set
\[
M = m \bowtie J = \{(a, f(a) + j) : a \in m, j \in J\}.
\]

Then the correspondence between the ring \( (A \bowtie J)_M \) and \( A_m \bowtie S^{-1}J \), \( \varphi : (A \bowtie J)_M \rightarrow A_m \bowtie S^{-1}J \) where
\[
\varphi\left( \frac{a}{s}, f(a) + j \right) = \left( \frac{a}{s}, f(a) + x \right)
\]
is a ring isomorphism.

Proof. We begin by showing that \( \varphi \) is a mapping. Then \( M := m \bowtie J \), is a maximal ideal of \( A \bowtie J \) by [7, Proposition 2.6]. For each \( \left( \frac{a}{s}, f(a) + x \right) \in (A \bowtie J)_M \), we have the following equalities:
\[
F\left( \frac{a}{s} \right) + \frac{f(s)x - f(a)y}{f(s)f(s) + y} = \frac{f(a)(f(s) + y) + f(s)x - f(a)y}{f(s)f(s) + y} = \frac{f(a) + x}{f(s) + y}.
\]

Therefore, \( \left( \frac{a}{s}, \frac{f(a) + x}{f(s) + y} \right) \in A_m \bowtie S^{-1}J \). Let \( a, a' \in A \), \( s, s' \in A \setminus m \), and \( x, y, x', y' \in J \), such that \( \left( \frac{a}{s}, f(a) + x \right) = \left( \frac{a'}{s'}, f(a') + x' \right) \). Then there exists \( (t, f(t) + z) \in S \) such that
\[
(t, f(t) + z)(s', f(s') + y')(a, f(a) + x) = (t, f(t) + z)(s, f(s) + y)(a', f(a') + x')
\]
and so
\[
\left\{
\begin{align*}
ts'a &= tsa' \\
(f(t) + z)(s', f(s') + y')(a, f(a) + x) &= (f(t) + z)(s, f(s) + y)(a', f(a') + y).
\end{align*}
\right.
\]
We deduce that \( \frac{a}{s} = \frac{a'}{s'} \) and \( \frac{f(a) + x}{f(s) + y} = \frac{f(a') + x'}{f(s') + y'} \). It follows that \( \varphi \) is map of the ring \( (A \bowtie J)_M \) into the ring \( A_m \bowtie S^{-1}J \). From the definition of \( \varphi \), we have \( \varphi(1) = 1 \). Let \( X = \left( \frac{a}{s}, f(a) + x \right), \quad Y = \left( \frac{b}{s}, f(b) + y \right) \) be elements of \( (A \bowtie J)_M \), we have clearly the equalities
\[
\varphi(X + Y) = \varphi(X) + \varphi(Y) \quad \text{and} \quad \varphi(XY) = \varphi(X) \varphi(Y).
\]
It follows that \( \varphi \) is a ring homomorphism.

We need only show that \( \varphi \) is bijective. Let \( X = \left( \frac{a}{s}, f(a) + x \right), \quad Y = \left( \frac{b}{s}, f(b) + y \right) \) be elements of \( (A \bowtie J)_M \), we have clearly the equalities
\[
\varphi(X + Y) = \varphi(X) + \varphi(Y) \quad \text{and} \quad \varphi(XY) = \varphi(X) \varphi(Y).
\]
It follows that \( \varphi \) is a ring homomorphism. Let \( X = \left( \frac{a}{s}, f(a) + x \right), \quad Y = \left( \frac{b}{s}, f(b) + y \right) \) be elements of \( (A \bowtie J)_M \), we have clearly the equalities
\[
\varphi(X + Y) = \varphi(X) + \varphi(Y) \quad \text{and} \quad \varphi(XY) = \varphi(X) \varphi(Y).
\]
It follows that \( \varphi \) is a ring homomorphism. Let \( a, b, c, d \in A \setminus m \) such that \( a + b + c + d = 0 \). Multiplying the above equality by \( f(t) \) we get
\[
(f(t) + x)(a, f(a) + x) = 0.
\]
We need only show that \( \varphi \) is bijective. Let \( X = \left( \frac{a}{s}, f(a) + x \right), \quad Y = \left( \frac{b}{s}, f(b) + y \right) \) be elements of \( (A \bowtie J)_M \), we have clearly the equalities
\[
\varphi(X + Y) = \varphi(X) + \varphi(Y) \quad \text{and} \quad \varphi(XY) = \varphi(X) \varphi(Y).
\]
It follows that \( \varphi \) is a ring homomorphism. Let \( a, b, c, d \in A \setminus m \) such that \( a + b + c + d = 0 \). Multiplying the above equality by \( f(t) \) we get
\[
(f(t) + x)(a, f(a) + x) = 0.
\]
We put \( b = at, u = st, z = f(a)y + f(s)x \) and \( j = f(s)y \). From the previous equalities we deduce that
\[
\left( \frac{a}{s}, F \left( \frac{a}{s} \right) + \frac{x}{f(t) + y} \right) = \varphi \left( \frac{b, f(b) + z}{(u, f(u) + j)} \right).
\]
Consequently, \( \varphi \) is surjective. We conclude that \( \varphi \) is a ring isomorphism. This completes the proof of Lemma 2.6.

\textbf{Proof.} of Theorem 2.4.

(1) straightforward.

(2) Assume that \( A \bowtie J \) is an arithmetical ring. Since \( f^{-1}(J) \subsetneq A \) there exists a maximal ideal \( m \) of \( A \) containing \( f^{-1}(J) \). Let \( S \) be as in Lemma 2.5. By [7, Proposition 2.6], \( M = m \bowtie J \) is a maximal ideal of \( A \bowtie J \). Thus \( (A \bowtie J)_M \) is a valuation ring. We can now applyLemma 2.6 to obtain that \( A \bowtie J \) is a valuation ring, where \( F: A_m \rightarrow S^{-1}B \) is the ring homomorphism defined by \( F \left( \frac{a}{s} \right) = \frac{f(a)}{f(s)} \). Let \( \frac{a}{s} \in \ker F \), there is some \( (t, j) \in (A \setminus m) \times J \) such that \( (f(t) + j)f(a) = 0 \). If \( f(t) + j = 0 \) then \( t \in f^{-1}(J) \) which contradicts the containment \( f^{-1}(J) \subseteq m \). Hence, \( f(a) = 0 \) since \( B \) is an integral domain. It follows that \( a = 0 \) and so \( F \) is injective. By applying statement (2) of Theorem 2.1, we get that \( F(A_m) \cap S^{-1}J = (0) \) and \( F(A_m) + S^{-1}J \) is a valuation ring. Now, we wish to show that \( f(A) \cap J = (0) \). Let \( a \) be an element \( A \) such that \( f(a) \in J \). We have clearly \( F \left( \frac{a}{1} \right) = \frac{f(a)}{1} \in F(A_m) \cap S^{-1}J = (0) \) and \( f(a) = 0 \). From the previous part of the proof, we deduce that \( a = 0 \) and so \( f(A) \cap J = (0) \). On the other hand, the natural projection of \( A \bowtie J \) into \( B \), \( \varphi \) is injective (since so is \( f \)). Hence \( A \bowtie J \approx f(A) + J \). Consequently, \( f(A) + J \) is an arithmetical ring and the necessary condition follows.

From the previous part of the proof, we get the sufficient condition.

(3) Suppose that \( A \bowtie J \) is an arithmetical ring, and choose \( 0 \neq j \in J \). Let \( m \) be a maximal ideal of \( A, S = f(A \setminus m) + J \) and let \( F: A_m \rightarrow S^{-1}B \) be the ring homomorphism defined by \( F \left( \frac{a}{s} \right) = \frac{f(a)}{f(s)} \) (by Lemma 2.5). It is easy to see that \( \frac{a}{1} \notin \ker F \), if \( 0 \neq a \in \ker f \). Hence \( F \) is not injective. By applying Lemma 2.6 and condition (1) of Theorem 2.1, we get successively that \( A_m \) is a valuation ring and \( S^{-1}J = (0) \). Hence there exists \( f(t_m) + j_m \in S \) such that \( (f(t_m) + j_m)j = 0 \). From the assumption, we can write \( f(t_m) + j_m = 0 \). Let \( f \) be the ideal of \( A \) generated by all \( t_m \). For every maximal ideal \( m \) of \( A \), we have \( I \subseteq m \) since \( t_m \in I \setminus m \), therefore \( I = A \). We can write \( 1 = t_1x_1 + \cdots + t_mx_n \), where \( x_i \in A, t_i \in A \setminus m \), for some maximal ideal \( m_i \) of \( A \). It follows that
\[
1 = f(t_1)f(x_1) + \cdots + f(t_n)f(x_n).
\]
We conclude that \( J = B \), since \( f(t_i) \in J \). We have the desired contradiction. This completes the proof of Theorem 2.4.

\textbf{Remark 2.7.} Let \( f: A \rightarrow B \) be a ring homomorphism and let \( J \) be an ideal of \( B \).

- If \( J = (0) \) then \( A \bowtie J \) is an arithmetical ring if and only if \( A \) is an arithmetical ring.
- If \( J = B \) then \( A \bowtie J \) is an arithmetical ring if and only if \( A \) and \( B \) are arithmetical rings.

\textbf{Proof.} Since the product \( A \times B \) is an arithmetical ring if and only if \( A \) and \( B \) are arithmetical rings, the conclusion is straightforward.

\textbf{Corollary 2.8.} Let \( A \) be an integral domain and let \( I \) be a proper ideal of \( A \). Then \( A \bowtie I \) is never an arithmetical ring.

Now, we are able to give the transfer of Prüfer domain to amalgamation of rings.
Corollary 2.9. Let $A$ and $B$ be a pair of integral domains, $f : A \rightarrow B$ a ring homomorphism and let $J$ be a proper ideal of $B$. Then $A \bowtie J$ is a Prüfer domain if and only if $f(A) + J$ is a Prüfer domain and $f(A) \cap J = (0)$.

Proof. By Theorem 2.4 and [6, Proposition 5.2].

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