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On some common fixed point theorems in G - metric spaces using $G-(E.A)$ - property

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Abstract In this paper a general fixed point theorem which extends the main results from [31] to G - metric spaces and generalizes Theorem 2.3 [20] and Theorem 3.1 [7] for mappings with $G-(E.A)$ - property under an implicit relation.

1 Introduction

Let (X,d) be a metric space and let S, T be two self mappings of X. In [10], Jungck defined S and T to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

The concept was frequently used to prove the existence theorems in common fixed point theory. The study of common fixed points of noncompatible mappings is also interesting. Work along these lines has been initiated by Pant in [21], [22], [23].

Recently, Aamri and El - Moutawakil [1] introduced a generalization of the concept of noncompatible mappings.

Definition 1.1 ([1]). Let S and T be two self mappings of a metric space (X,d). We say that S and T satisfy $(E.A)$ - property if there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Remark 1.2. It is clear that two self mappings S and T of a metric space (X,d) are noncompatible if there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is nonzero or non existent.

Therefore, two noncompatible self mappings of a metric space (X,d) satisfy $(E.A)-$ property.

Definition 1.3 ([11]). Two self mappings S and T of a metric space (X,d) is said to be weakly compatible if $Su = Tu$ implies $STu = TSu$.

Remark 1.4. It is known that the notions of weakly compatible mappings and mappings satisfying $(E.A)$ - property are independent.

Definition 1.5. Let S and T be two self mappings of a metric space (X,d). A point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$ and the point $w = Sx = Tx$ is said to be a point of coincidence of S and T.

The set of coincidence points of S and T is denoted by $C(S,T)$.

Lemma 1.6 ([2]). *Let f and g be weakly compatible self mappings on a nonempty set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

The following theorem is proved in [1].

Theorem 1.7. *Let S and T be weakly compatible mappings of a metric space (X,d) such that:*
(i) T *and* S *satisfy* $(E.A)$ *- property,*
(ii)

$$d(Tx,Ty) < \max\{d(Sx,Sy), \frac{1}{2}[d(Sx,Tx)+d(Sy,Ty)], \frac{1}{2}[d(Sx,Ty)+d(Sy,Tx)]\},$$

for all $x \neq y \in X$,
(iii) $T(X) \subset S(X)$.
If $S(X)$ *or* $T(X)$ *is a complete subspace of* X*, then* T *and* S *have a unique common fixed point.*

In [8], [9], Dhage introduced a new class of generalized metric space, named D - metric space. Mustafa and Sims [13], [14], proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G - metric space. In fact, Mustafa and Sims and other authors studied many fixed point results for self mappings in G - metric spaces under certain conditions [14], [15], [16], [17], [18], [19], [20], [34], and other papers.

In [24] and [25], the study of fixed points for mappings satisfying an implicit relation was introduced.

Actually, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, Tychonoff metric spaces, compact metric spaces, paracompact metric spaces, reflexive metric spaces, probabilistic metric spaces, convex metric spaces, in two or three metric spaces, for single valued functions, hybrid pairs of functions and set valued functions. Quite recently, the method is used in the study of fixed points for mappings satisfying an implicit relation of integral type, and fuzzy metric spaces. The method unified different types of contractive and extensive conditions. With this method, the proofs of some fixed point theorems are more simple. Also, this method allows the study of local and global properties of fixed point structures.

2 Preliminaries

Definition 2.1 ([14]). Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:
(G_1) : $G(x,y,z) = 0$ if $x = y = z$,
(G_2) : $0 < G(x,x,y)$ for all $x \neq y \in X$,
(G_3) : $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,
(G_4) : $G(x,y,z) = G(y,z,x) = G(z,x,y) = ...$ (symmetry in all three variables),
(G_5) : $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).
The function G is called a G - metric and the pair (X,G) is said to be a G - metric space.

Note that if $G(x,y,z) = 0$, then $x = y = z$.

Definition 2.2 ([14]). Let (X,G) be a G - metric space. A sequence (x_n) in X is said to be
a) G - convergent, if for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ and $x \in X$ such that for all $m,n \in \mathbb{N}$, $m,n \geq k$, $G(x_n,x_m,x) < \varepsilon$.
b) G - Cauchy if for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m,n,p \in \mathbb{N}$, $n,m,p \geq k$, $G(x_n,x_m,x_p) < \varepsilon$, that is $G(x_n,x_m,x_p) \to 0$ as $n,m,p \to \infty$.
A G - metric space is said to be G - complete if every G - Cauchy sequence is G - convergent.

Lemma 2.3 ([14]). *Let (X,G) be a G - metric space. Then, the following properties are equivalent:*
1) (x_n) *is* G *- convergent to* x*;*
2) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$*;*
3) $G(x_n,x,x) \to 0$ *as* $n \to \infty$*;*
4) $G(x_n,x_m,x) \to 0$ *as* $n,m \to \infty$.

Definition 2.4 ([3]). Let (X,G) be a G - metric space. The pair of functions $f,g : (X,G) \to (X,G)$ is called:
a) G - compatible if $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0$ whenever (x_n) is a sequence in X such that (fx_n) and (gx_n) are G - convergent to some $t \in X$;
b) G - noncompatible if there exists at least one sequence (x_n) in X such that (fx_n) and (gx_n) are G - convergent to some $t \in X$, but $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n)$ is either nonzero or does not exist.

Definition 2.5 ([1],[7],[20]). Let (X,G) be a G - metric space. Two self mappings f and g of X are said to be satisfying condition $G-(E.A)$ - property if there exists a sequence (x_n) in X such that (fx_n) and (gx_n) are G - convergent to some $t \in X$.

Some fixed point theorems for self mappings in G - metric spaces with $G-(E.A)$ - property are proved in [1], [7], [20].

Let Φ be the set of all functions $\phi : [0,\infty) \to [0,\infty)$ nondecreasing with $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in [0,\infty)$. If $\phi \in \Phi$, then ϕ is called a Φ - map. If ϕ is a Φ - map, then is easy to show that:

(1) $\phi(t) < t$ for all $t \in (0,\infty)$;
(2) $\phi(0) = 0$.

The following theorem is proved in [20].

Theorem 2.6 (Theorem 2.3, [20]). *Let (X,G) be a G – metric space. Suppose that the mappings $f,g : X \to X$ are weakly compatible satisfying the following conditions:*

(1) f and g satisfy the $G-(E.A)$ property,
(2) $g(X)$ is a closed subspace of X,

$$G(fx,fy,fz) \leq \phi(\max\{G(gx,gy,gy), G(gx,fx,gz), G(gz,fz,gz), G(gy,fy,gz)\}), \quad (2.1)$$

for all $x,y,z \in X$.
Then f and g have a unique common fixed point.

The following theorem is proved in [7].

Theorem 2.7 (Theorem 3.1, [7]). *Let f and g be two self mappings of a G - metric space (X,G) satisfying the inequality:*

$$G(fx,fy,fz) \leq \alpha G(fx,gy,gz) + \beta G(gx,fy,gz) + \gamma G(gx,gy,fz) \quad (2.2)$$

for every $x,y,z \in X$ and $\alpha,\beta,\gamma \geq 0$ with $\alpha + 3\beta + 3\gamma < 1$,
(2) f and g satisfy G - $(E.A)$ property,
(3) $f(X)$ is a closed subspace of X.
Then f and g have a unique common fixed point in X .

3 Implicit relations

Definition 3.1. Let $\mathfrak{F}_G$ be the set of all real continuous functions $F(t_1,...,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

$(D_1) : F(t,0,0,t,t,0) > 0, \forall t > 0,$
$(D_2) : F(t,t,0,0,t,t) \geq 0, \forall t > 0.$

In [26] a generalization of Theorem 1.7 for mappings satisfying implicit relations is proved.

Theorem 3.2 ([26]). *Let T and S be two weakly compatible self mappings of a metric space (X,d) such that*

(i) T and S satisfy $(E.A)$ - property,
(ii)

$$F(d(Tx,Ty), d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)) < 0$$

for all $x \neq y \in X$ and $F \in \mathfrak{F}_G$,
(iii) $T(X) \subset S(X)$.
If $S(X)$ or $T(X)$ is a complete subspace of X, then T and S have a unique common fixed point.

Definition 3.3. An altering distance is a function $\psi : [0,\infty) \to [0,\infty)$ satisfying:

(ψ_1) : ψ is increasing and continuous;
(ψ_2) : $\psi(t) = 0$ if and only if $t = 0$.

Fixed point problems involving an altering distance have been studied in [12], [28], [32], [33] and in other papers.

The following result is obtained in [31].

Theorem 3.4. *Let T and S be two weakly compatible self mappings of a metric space (X, d) such that*

(1) S and T satisfy $(E.A)$ - property,

(2) S and T satisfy the inequality

$$F(\psi(d(Tx,Ty)),\psi(d(Sx,Sy)),\psi(d(Sx,Tx)),\psi(d(Sy,Ty)),\psi(d(Sx,Ty)),\psi(d(Sy,Tx))) < 0 \quad (3.1)$$

for all $x \neq y \in X$, where ψ is an altering distance and $F \in \mathfrak{F}_G$,

(iii) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a complete subspace of X, then T and S have a unique common fixed point.

Definition 3.5. Let $\mathfrak{F}_G$ be the set of all continuous functions $F(t_1,...,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : $F(t,t,0,t,t,t') \leq 0$ implies $t < t'$, $\forall t, t' > 0$,

(F_2) : $F(t,0,0,t,0,t) > 0, \forall t > 0$.

Example 3.6. $F(t_1,...,t_6) = t_1 - \phi(\max\{t_2,t_3,t_4,t_5,t_6\})$, where ϕ is as in Theorem 3.2.

(F_1) : Let $t, t' > 0$ be such that $F(t,t,0,t,t,t') = t - \phi(\max\{t,t'\}) \leq 0$. If $t > t'$ then $t - \phi(t) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq \phi(t') < t'$.

(F_2) : $F(t,0,0,t,0,t) = t - \phi(t) > 0, \forall t > 0$.

Example 3.7. $F(t_1,...,t_6) = t_1 - \alpha t_4 - (\beta+\gamma)t_5 - at_2 - bt_3 - ct_6$, where $\alpha,\beta,\gamma,a,b,c \geq 0$ and $\alpha+\beta+\gamma+a+c < 1$.

(F_1) : Let $t,t' > 0$ be such that $F(t,t,0,t,t,t') = t - \alpha t - (\beta+\gamma)t - at - ct' \leq 0$ which implies $t \leq \dfrac{c}{1-(\alpha+\beta+\gamma)-a}t' < t'$.

(F_2) : $F(t,0,0,t,0,t) = t - \alpha t - ct = t[1-(\alpha+c)] > 0, \forall t > 0$.

Example 3.8. $F(t_1,...,t_6) = t_1 - h\max\{t_2,t_3,t_4,t_5,t_6\}$, where $h \in (0,1)$.

(F_1) : Let $t,t' > 0$ be such that $F(t,t,0,t,t,t') = t - h\max\{t,t'\} \leq 0$. If $t > t'$ then $t(1-h) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq ht' < t'$.

(F_2) : $F(t,0,0,t,0,t) = t(1-h) > 0, \forall t > 0$.

Example 3.9. $F(t_1,...,t_6) = t_1 - h\max\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2}\}$, where $h \in (0,1)$.

(F_1) : Let $t,t' > 0$ be such that $F(t,t,0,t,t,t') = t - h\max\{t,\dfrac{t+t'}{2}\} \leq 0$. If $t > t'$ then $t(1-h) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq ht' < t'$.

(F_2) : $F(t,0,0,t,0,t) = t(1-h) > 0, \forall t > 0$.

Example 3.10. $F(t_1,...,t_6) = t_1 - k\max\{t_2,\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\}$, where $k \in (0,1)$.

(F_1) : Let $t,t' > 0$ be such that $F(t,t,0,t,t,t') = t - k\max\{t,\dfrac{t+t'}{2}\} \leq 0$ and the proof is as in Example 3.9.

(F_2) : $F(t,0,0,t,0,t) = \dfrac{t}{2} > 0, \forall t > 0$.

Example 3.11. $F(t_1,...,t_6) = t_1 - at_2 - b\max\{t_3,t_4\} - c\max\{t_2,t_5,t_6\}$, where $a,b,c \geq 0$ and $a+b+c < 1$.

(F_1) : Let $t,t' > 0$ be such that $F(t,t,0,t,t,t') = t - at - bt - c\max\{t,t'\} \leq 0$. If $t > t'$ then $t(1-(a+b+c)) \leq 0$, a contradiction. Hence $t \leq t'$ which implies $t \leq \dfrac{c}{1-(a+b)}t' < t'$.

(F_2) : $F(t,0,0,t,0,t) = t - bt - ct = t(1-(b+c)) > 0, \forall t > 0$.

Example 3.12. $F(t_1,...,t_6) = t_1 - at_2 - b\sqrt{t_3t_4} - c\sqrt{t_5t_6}$, where $a,b,c \geq 0$ and $a+c < 1$.

(F_1) : Let $t,t' > 0$ be such that $F(t,t,0,t,t,t') = t - at - ct' \leq 0$ which implies $t \leq \dfrac{c}{1-a}t' < t'$.

(F_2) : $F(t,0,0,t,0,t) = t > 0, \forall t > 0$.

Example 3.13. $F(t_1,...,t_6) = t_1 - h\max\{t_2,t_3,t_4\} - (1-h)(at_5+bt_6)$, where $h \in (0,1)$, $a,b,c \geq 0$ and $a+b < 1$.

(F_1) : Let $t,t' > 0$ be such that $F(t,t,0,t,t,t') = t - ht - (1-h)(at+t') = (1-h)(t-at-bt') \leq 0$ which implies $t - at - bt' \leq 0$, i.e. $t \leq \dfrac{b}{1-a}t' < t'$.

(F_2) : $F(t,0,0,t,0,t) = t - ht - (1-h)bt = (1-h)(1-b)t > 0, \forall t > 0$.

The purpose of this paper is to prove a general fixed point theorem which extend Theorem 3.4 for G - metric space, generalizing Theorem 2.6 and 2.7 and to obtain other particular results. As applications, in the last part of this paper, a general fixed point theorem in G - metric spaces for mappings satisfying implicit contractive conditions of integral type is proved.

4 Main results

Theorem 4.1. *Let (X, G) be a G - metric space and let $f, g : X \to X$ be two functions satisfying the following inequality*

$$\begin{aligned} &F(\psi(G(fx, fy, fy)), \psi(G(gx, gy, gy)), \psi(G(gy, fy, fy)), \\ &\psi(G(fx, gy, gy)), \psi(G(gx, fy, gy)), \psi(G(fx, gx, gy))) \leq 0 \end{aligned} \tag{4.1}$$

for all $x, y \in X$, where F satisfies property (F_1) and ψ is an altering distance. Then, f and g have at most one point of coincidence.

Proof. Suppose that f and g have two distinct points of coincidence $u = fa = ga$ and $v = fb = gb$. By (4.1) we have successively

$$\begin{aligned} &F(\psi(G(fa, fb, fb)), \psi(G(ga, gb, gb)), \psi(G(gb, fb, fb)), \\ &\psi(G(fa, gb, gb)), \psi(G(ga, fb, gb)), \psi(G(fa, ga, gb))) \leq 0, \end{aligned}$$

$$F(\psi(G(u, v, v)), \psi(G(u, v, v)), 0, \psi(G(u, v, v)), \psi(G(u, v, v)), \psi(G(u, u, v))) \leq 0.$$

By (F_1) it follows that $\psi(G(u, v, v)) < \psi(G(u, u, v))$.
Similarly, $\psi(G(v, u, u)) < \psi(G(u, v, v))$.
Hence,

$$\psi(G(u, v, v)) < \psi(G(v, u, u)) < \psi(G(u, v, v)),$$

a contradiction. □

Theorem 4.2. *Let (X, G) be a G - metric space and let $f, g : X \to X$ be two functions satisfying the inequality (4.1), for all $x, y \in X$, $F \in \mathfrak{F}_G$ and ψ is an altering distance. If:*
1) f and g satisfy $G - (E.A)$ - property,
2) $g(X)$ is a closed subspace of X,
then, f and g have a point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Proof. Since f and g satisfy $G - (E.A)$ - property, there exists a sequence (x_n) in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Since $g(X)$ is a closed subspace of X, there exists $p \in X$ such that $gp = t$. Also, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gp$. We will prove that $fp = gp$. Suppose that $fp \neq gp$. By (4.1) we have

$$\begin{aligned} &F(\psi(G(fp, fx_n, fx_n)), \psi(G(gp, gx_n, gx_n)), \psi(G(gx_n, fx_n, fx_n)), \\ &\psi(G(fp, gx_n, gx_n)), \psi(G(gp, fx_n, gx_n)), \psi(G(fp, gp, gx_n))) \leq 0. \end{aligned}$$

Letting n tend to infinity we obtain

$$F(\psi(G(fp, gp, gp)), 0, 0, \psi(G(fp, gp, gp)), 0, \psi(G(fp, gp, gp))) \leq 0,$$

a contradiction of (F_2). Hence $fp = gp$ and $u = fp = gp$ is a point of coincidence of f and g. By Theorem 4.1, u is the unique point of coincidence. By Lemma 1.6 u is the unique common fixed point of f and g. □

If $\psi(t) = t$, by Theorem 4.2 we obtain

Theorem 4.3. *Let (X, G) be a G - metric space and let $f, g : X \to X$ be two functions satisfying the inequality*

$$\begin{aligned} &F(G(fx, fy, fy), G(gx, gy, gy), G(gy, fy, fy), \\ &G(fx, gy, gy), G(fy, gx, gy), G(fx, gx, gy)) \leq 0 \end{aligned}$$

for all $x, y \in X$ and $F \in \mathfrak{F}_G$. If:
1) f and g satisfy $G - (E.A)$ - property,
2) $g(X)$ is a closed subspace of X, then, f and g have a point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Corollary 4.4 (Theorem 2.1). *By (2.1) for $y = z$ we obtain*

$$\begin{aligned} G(fx, fy, fy) &\le \phi(\max\{G(gx, gy, gy), G(gy, fy, gy), G(fx, gx, gy)\}) \\ &\le \phi(\max\{G(gx, gy, gy), G(gy, fy, fy), \\ &G(fx, gx, gy), G(fy, gx, gy), G(fx, gx, gy)\}). \end{aligned}$$

The proof follows from Theorem 4.3 and Example 3.6.

Corollary 4.5 (Theorem 2.2). *By (2.2) for $y = z$ we obtain*

$$\begin{aligned} G(fx, fy, fy) &\le \alpha G(fx, gy, gy) + (\beta + \gamma) G(fy, gx, gy) \\ &\le \alpha G(fx, gy, gy) + (\beta + \gamma) G(fy, gx, gy) + \\ &+ a G(gx, gy, gy) + b G(fx, gy, gy) + c G(fx, gx, gz). \end{aligned}$$

The proof follows from Theorem 4.3 and Example 3.7.

Remark 4.6. By Example 3.2 it follows that Corollary 4.5 is true in condition $\alpha + \beta + \gamma < 1$ instead of $\alpha + 3\beta + 3\gamma < 1$ because $a = b = c = 0$.

Example 4.7. Let $X = [0, \infty)$ be and $G : X^3 \to \mathbb{R}$ be the metric $G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}$ for all $x, y, z \in X$. Then (X, G) is a G - metric space. We define f and g to be the self mappings

$$f(x) = \begin{cases} \frac{1}{2}, \ x \in \left[0, \frac{1}{2}\right] \\ \frac{3}{5}, \ x \in \left(\frac{1}{2}, \infty\right) \end{cases}, \quad g(x) = \begin{cases} \frac{1}{3} + x, \ x \in \left[0, \frac{1}{2}\right) \\ \frac{1}{2}, \ x = \frac{1}{2} \\ \frac{2}{5}, \ x \in \left(\frac{1}{2}, \infty\right) \end{cases}$$

Then $g(X) = \left[\frac{1}{2}, \frac{5}{6}\right]$ is a closed set. Let $x_n \in \left[0, \frac{1}{2}\right]$ be such that $x_n \to \frac{1}{6}$, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \frac{1}{2}$. Hence, f and g satisfy $G - (E.A)$ - property.

Since $C(f, g) = \left\{\frac{1}{6}, \frac{1}{2}\right\}$, then $f\left(g\left(\frac{1}{6}\right)\right) = f\left(\frac{1}{2}\right) = \frac{1}{2}$, $g\left(f\left(\frac{1}{6}\right)\right) = g\left(\frac{1}{2}\right) = \frac{1}{2}$ and $f\left(g\left(\frac{1}{2}\right)\right) = f\left(\frac{1}{2}\right) = \frac{1}{2}$, $g\left(f\left(\frac{1}{2}\right)\right) = g\left(\frac{1}{2}\right) = \frac{1}{2}$. Therefore, f and g are weakly compatible.

On the other hand,

$$g(fx, fy, fy) = \begin{cases} 0, x \in \left[0, \frac{1}{2}\right], y \in \left[0, \frac{1}{2}\right] \\ \frac{1}{6}, x \in \left[0, \frac{1}{2}\right], y \in \left(\frac{1}{2}, \infty\right) \\ \frac{1}{6}, x \in \left(\frac{1}{2}, \infty\right), y \in \left[0, \frac{1}{2}\right] \\ 0, x \in \left(\frac{1}{2}, \infty\right), y \in \left(\frac{1}{2}, \infty\right) \end{cases}, \quad G(gy, fy, fy) = \begin{cases} \left|y - \frac{1}{6}\right|, y \in \left[0, \frac{1}{2}\right] \\ \frac{1}{5}, y \in \left(\frac{1}{2}, \infty\right) \end{cases}$$

Then, there exists $h \in \left(\frac{5}{6}, 1\right)$ such that

$$G(fx, fy, fy) \le h G(gx, gy, gy),$$

which implies

$$G(fx, fy, fy) \le h \max\{G(gx, gy, gy), G(gy, fy, fy), G(fx, gy, gy), G(gx, fy, fy), G(fx, gx, gy)\}.$$

By Theorem 4.3 and Example 3.8 f and g have a unique common fixed point $x = \frac{1}{2}$.

5 Consequences

In [6], Branciari established the following theorem which opened the way to the study of fixed points for mappings satisfying a contractive condition of integral type.

Theorem 5.1 ([6]). *Let (X, G) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ a mapping such that for each $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt \tag{5.1}$$

where $h : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, \infty)$, such that for $\varepsilon > 0$, $\int_0^{\varepsilon} h(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Remark 5.2. Theorem 5.1 has been generalized in several papers. In [5], Aydi initiated the study of fixed points in G - metric spaces for mappings satisfying contractive conditions of integral type.

Lemma 5.3 ([28]). *Let $h(t)$ as in Theorem 5.1. Then, $\psi(t) = \int_0^t h(x)dx$ is an altering distance.*

Using the method from [28], we prove the following theorem:

Theorem 5.4. *Let (X, G) be a complete metric space $f, g : X \to X$ be two mappings satisfying the inequality:*

$$\begin{aligned} &F\Big(\textstyle\int_0^{G(fx,fy,fy)} h(t)dt, \int_0^{G(gx,gy,gy)} h(t)dt, \int_0^{G(gy,fy,fy)} h(t)dt, \\ &\textstyle\int_0^{G(fx,gy,gy)} h(t)dt, \int_0^{G(fy,gx,gy)} h(t)dt, \int_0^{G(fx,gx,gy)} h(t)dt\Big) \leq 0 \end{aligned} \tag{5.2}$$

for all $x, y \in X$, $F \in \mathfrak{F}_G$ and $h(t)$ is as in Theorem 5.1. If
1) f and g satisfy $G - (E.A)$ - property,
2) $g(X)$ is a closed subspace of X, then, f and g have a point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique fixed point.

Proof. As in Lemma 5.3,

$$\begin{aligned} \psi(G(fx, fy, fy)) &= \textstyle\int_0^{G(fx,fy,fy)} h(t)dt, \psi(G(gx, gy, gy)) = \int_0^{G(gx,gy,gy)} h(t)dt, \\ \psi(G(gy, fy, fy)) &= \textstyle\int_0^{G(gy,fy,fy)} h(t)dt, \psi(G(fx, gy, gy)) = \int_0^{G(fx,gy,gy)} h(t)dt, \\ \psi(G(fy, gx, gy)) &= \textstyle\int_0^{G(fy,gx,gy)} h(t)dt, \psi(G(fx, gx, gy)) = \int_0^{G(fx,gx,gy)} h(t)dt. \end{aligned}$$

By (5.2) we obtain

$$\begin{aligned} &F(\psi(G(fx, fy, fy)), \psi(G(gx, gy, gy)), \psi(G(gy, fy, fy)), \\ &\psi(G(fx, gy, gy)), \psi(G(fy, gx, gy)), \psi(G(fx, gx, gy))) \leq 0. \end{aligned}$$

Hence, the conditions of Theorem 4.2 are satisfied and the proof follows from Theorem 4.2. □

Remark 5.5. If $h(t) = 1$, then by Theorem 5.4 we obtain Theorem 4.3.

By Theorem 5.4 and Examples 3.6 - 3.13 we obtain new particular results.

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