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On A Class Of Multivalent Starlike Functions With A Bounded Positive Real Part

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Abstract By introducing a new subcalss of p-valent functions with respect to $(j,\ k)$-symmetric points, we have obtained the integral representations and conditions for starlikeness by using differential subordination. Some already known results have been, incidentally, shown to be particular cases of the main results of the paper.

1 Introduction, Definitions And Preliminaries

Let $\triangle = \{z \in \mathbb{C} : \mid z \mid < 1\}$ be the open unit disc. Let $\mathcal{H}$ be the class of functions analytic in $\triangle$ and let $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$.
Let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1,\, 2,\, 3, \dots\}), \tag{1.1}$$

and let $\mathcal{A} = \mathcal{A}_1$.

Let the functions $f(z)$ and $g(z)$ be members of $\mathcal{A}$. We say that the function f is subordinate to g (or g is superordinate to f), written $f \prec g$, if there exists a Schwarz function w analytic in $\triangle$, with $w(0) = 0$ and $|w(z)| < 1$ and such that $f(z) = g(w(z))$. In particular, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\triangle) \subset g(\triangle)$.

We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\triangle$. Also, we let $\mathcal{P}$ to denote the class of functions analytic in $\triangle$ having Taylor series expansion of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n,$$

and satisfy the condition $Re\,\{h(z)\} > 0$, $(z \in \triangle)$.

Motivated by the concept introduced by Sakaguchi [5], recently several subclasses of analytic functions with respect to k-symmetric points were studied by various authors. More prominently, Wang, Gao and Yuan [6] introduced the class $S_s^{(k)}(\varphi)$ of functions $f \in \mathcal{A}$ subject to satisfying the condition

$$\frac{zf'(z)}{f_k(z)} \prec \varphi(z) \quad (z \in \triangle),$$

where $\varphi(z) \in \mathcal{P}$, $k \geq 1$ is fixed positive integer and $f_k(z)$ is defined by

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z).$$

Similarly, $C_s^{(k)}(\varphi)$ denote the class of functions in $\mathcal{S}$ satisfying the condition

$$\frac{(zf'(z))'}{f_k'(z)} \prec \varphi(z) \quad (z \in \triangle),$$

where $\varphi(z) \in \mathcal{P}$, $k \geq 1$ is fixed positive integer.

Liczberski and Połubiński [2] introduced the notion $(j,\, k)$ symmetrical function ($k = 2,\, 3,\, \ldots;$ $j = 0,\, 1,\, \ldots,\, k-1$), which is a generalization of even, odd and k-symmetrical functions. A function $f \in \mathcal{A}$ is said to be $(j,\, k)$-symmetrical if for each $z \in \triangle$

$$f(\varepsilon z) = \varepsilon^j f(z), \tag{1.2}$$

$$(k = 1,\, 2,\, \ldots;\, j = 0,\, 1,\, 2,\, \ldots (k-1)),$$

where $\varepsilon = \exp(2\pi i/k)$. The family of $(j,\, k)$-symmetrical functions will be denoted by $\mathcal{F}_k^j$. We observe that $\mathcal{F}_2^1$, $\mathcal{F}_2^0$ and $\mathcal{F}_k^1$ are well-known families of odd functions, even functions and k-symmetrical functions respectively. It was further proved in [2] that each function defined on a symmetrical set can be uniquely represented as the sum of an even function and an odd function.

Also let $f_{j,\,k}(z)$ be defined by the following

$$f_{j,\,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{f(\varepsilon^\nu z)}{\varepsilon^{\nu p j}}, \tag{1.3}$$

$$(f \in \mathcal{A}_p;\, k = 1,\, 2,\, \ldots;\, j = 0,\, 1,\, 2,\, \ldots (k-1)).$$

In this paper, new subclass of $p-$valent functions with respect to $(j,\, k)$-symmetric points are introduced.

We now define the following:

Definition 1.1. The function $f(z) \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_{j,\,k}^p(b;\, \alpha,\, \beta)$ of p-valently functions of complex order $b \neq 0$ in $\triangle$ if and only if

$$\alpha < Re\left\{1 + \frac{1}{b}\left(\frac{zf^{(m+1)}(z)}{f_{jk}^{(m)}(z)} - p + m\right)\right\} < \beta,$$

$$(z \in \triangle, p \in \mathbb{N}, m \in \mathbb{N}\cup\{0\})$$

where $0 \leq \alpha < 1 < \beta$ and $f_{j,\,k}(z) \neq 0$ in $\triangle$.

Remark 1.2. If $m = 0$, $j = k = p = 1$ and $\alpha \geq 0$, then $f(z)$ reduces to the well-known class of starlike functions of complex order. Similarly, if we let $m = 1$, $j = k = p = 1$ and $\alpha \geq 0$, then $f(z)$ reduces to the well-known class convex functions of complex order. We let $\mathcal{S}^*(b)$ and $\mathcal{C}(b)$ to denote the class of starlike and convex of complex order $b \neq 0$ respectively. Note that starlike functions of complex order and convex functions of complex order are the classes considered by Nasr and Aouf [4] and by Wiatrowski [7].

We observe that for a given $\alpha,\, \beta\, (0 \leq \alpha < 1 < \beta)$, $f \in \mathcal{S}_{j,\,k}^p(b;\, \alpha,\, \beta)$ satisfies each of the following subordination equations

$$1 + \frac{1}{b}\left(\frac{zf^{(m+1)}(z)}{f_{jk}^{(m)}(z)} - p + m\right) \prec \frac{1 + (1-2\alpha)\, z}{1-z}$$

and

$$1 + \frac{1}{b}\left(\frac{zf^{(m+1)}(z)}{f_{jk}^{(m)}(z)} - p + m\right) \prec \frac{1 + (1-2\beta)\, z}{1-z}.$$

Both superordinate functions in the above expressions maps the unit disc onto right half plane, so it is obvious that $1 + \frac{1}{b}\left(\frac{zf^{(m+1)}(z)}{f_{jk}^{(m)}(z)} - p + m\right)$ is mapped on to a plane having real part greater than α but less than β.

From the equivalent subordination condition proved by Kuroki and Owa in [1], we have $f \in \mathcal{S}_{j,\,k}^p(b;\, \alpha,\, \beta)$ if and only if

$$1 + \frac{1}{b}\left(\frac{zf^{(m+1)}(z)}{f_{jk}^{(m)}(z)} - p + m\right) \prec 1 + \frac{\beta-\alpha}{\pi} \quad i\log\left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)}z}{1-z}\right).$$

Further, we note that

$$q(z) = 1 + \frac{\beta-\alpha}{\pi} i\log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, z}{1-z}\right) \tag{1.4}$$

maps $\triangle$ onto a convex domain conformally and is of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

where $c_n = \frac{\beta-\alpha}{n\pi}\, i\, \left(1 - e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right)$.

2 Integral Representations

Theorem 2.1. *Let $f \in \mathcal{S}^p_{j,\,k}(\alpha,\,\beta)$. Then we have*

$$f^{(m)}_{jk}(z) = z^{p-m} \exp\left\{ \frac{b}{k} \sum_{v=0}^{k-1} \int_0^{\varepsilon^v z} \frac{1}{t} \left(\frac{\beta-\alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} w(t)}{1 - w(t)} \right) \right) dt \right\}$$

where $f_{j,k}(z)$ defined by (1.3), $w(z)$ is analytic in $\triangle$ with $w(0) = 0$ and $|w(z)| < 1$.

Proof. Let $f \in \mathcal{S}^p_{j,\,k}(\alpha,\,\beta)$. In view of the equivalent subordination condition proved by Kuroki and Owa [1] for the class $\mathcal{S}^p_{j,\,k}(\alpha,\,\beta)$, we have

$$1 + \frac{1}{b} \left(\frac{z f^{(m+1)}(z)}{f^{(m)}_{j,\,k}(z)} - p + m \right) = 1 + \frac{\beta-\alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} w(z)}{1 - w(z)} \right), \tag{2.1}$$

where $w(z)$ is analytic in $\triangle$ and $w(0) = 0$, $|w(z)| < 1$. Substituting z by $\varepsilon^v z$ in (2.1) respectively($v = 0, 1, 2, \ldots k-1, \varepsilon^k = 1$), we have

$$1 + \frac{1}{b} \left(\frac{\varepsilon^v z f^{(m+1)}(\varepsilon^v z)}{f^{(m)}_{jk}(\varepsilon^v z)} - p + m \right) = 1 + \frac{\beta-\alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} w(\varepsilon^v z)}{1 - w(\varepsilon^v z)} \right).$$

Using $f_{j,k}(\varepsilon^v z) = \varepsilon^{vpj} f_{j,k}(z)$, we get

$$\frac{1}{b} \left(\frac{z \varepsilon^{(m+1)v - vj} \left(f(\varepsilon^v z) \right)^{(m+1)}}{\left(f_{jk}(z) \right)^{(m)}} - p + m \right) = \frac{\beta-\alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} w(\varepsilon^v z)}{1 - w(\varepsilon^v z)} \right). \tag{2.2}$$

Let $v = 0, 1, 2, \ldots k-1$ in (2.2) respectively and summing them, we get

$$\frac{1}{b} \left(\frac{z f^{(m+1)}_{j,\,k}(z)}{f^{(m)}_{j,\,k}(z)} - p + m \right) = \frac{1}{k} \sum_{v=0}^{k-1} \frac{\beta-\alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} w(\varepsilon^v z)}{1 - w(\varepsilon^v z)} \right).$$

On simplifying and integrating, we get

$$\log \left(\frac{f^{(m)}_{jk}(z)}{z^{p-m}} \right) = \frac{b}{k} \sum_{v=0}^{k-1} \int_0^{\varepsilon^v z} \frac{1}{t} \left(\frac{\beta-\alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} w(t)}{1 - w(t)} \right) \right) dt. \tag{2.3}$$

The difficulty to integrate the term with presence of the first order pole at the origin, has been avoided by integrating from z_0 to z with $z_0 \neq 0$ and then let $z_0 \to 0$. Further simplifying (2.3), we get

$$f^{(m)}_{jk}(z) = z^{p-m} \exp\left\{ \frac{b}{k} \sum_{v=0}^{k-1} \int_0^{\varepsilon^v z} \frac{1}{t} \left(\frac{\beta-\alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} w(t)}{1 - w(t)} \right) \right) dt \right\}.$$

This completes the proof of theorem. $\square$

Theorem 2.2. *Let $f \in \mathcal{S}^p_{j,\,k}(\alpha,\,\beta)$. Then we have*

$$f(z) = \int_0^z \int_0^z \cdots \int_0^z \zeta^{p-m-1} \exp\left\{ \frac{b}{k} \sum_{v=0}^{k-1} \int_0^{\varepsilon^v z} \frac{1}{t} \left(\frac{\beta-\alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} w(t)}{1 - w(t)} \right) \right) dt \right\}$$
$$\times \left[p - m + \frac{b(\beta-\alpha)}{\pi}\, i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} w(z)}{1 - w(z)} \right) \right] d\zeta \ldots dz\, dz.$$

Proof. From the Definition 1.1 and Theorem 2.1, it can be easily seen that,

$$zf^{(m+1)}(z) = z^{p-m}\, \exp\left\{\frac{b}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^v z}\frac{1}{t}\left(\frac{\beta-\alpha}{\pi}\quad i\log\left(\frac{1-e^{2\pi i(1-\alpha)\setminus(\beta-\alpha)}w(t)}{1-w(t)}\right)\right)dt\right\}$$
$$\times\left[p-m+\frac{b(\beta-\alpha)}{\pi}\,i\log\left(\frac{1-e^{2\pi i(1-\alpha)\setminus(\beta-\alpha)}w(z)}{1-w(z)}\right)\right].$$

Or equivalently,

$$f^{(m+1)}(z) = z^{p-m-1}\, \exp\left\{\frac{b}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^v z}\frac{1}{t}\left(\frac{\beta-\alpha}{\pi}\quad i\log\left(\frac{1-e^{2\pi i(1-\alpha)\setminus(\beta-\alpha)}w(t)}{1-w(t)}\right)\right)dt\right\}$$
$$\times\left[p-m+\frac{b(\beta-\alpha)}{\pi}\,i\log\left(\frac{1-e^{2\pi i(1-\alpha)\setminus(\beta-\alpha)}w(z)}{1-w(z)}\right)\right].$$

Integrating the above expression $m+1$ times, we have

$$f(z) = \int_0^z\int_0^z\ldots\int_0^z \zeta^{p-m-1}\exp\left\{\frac{b}{k}\sum_{v=0}^{k-1}\int_0^{\varepsilon^v z}\frac{1}{t}\left(\frac{\beta-\alpha}{\pi}\quad i\log\left(\frac{1-e^{2\pi i(1-\alpha)\setminus(\beta-\alpha)}w(t)}{1-w(t)}\right)\right)dt\right\}$$
$$\times\left[p-m+\frac{b(\beta-\alpha)}{\pi}\,i\log\left(\frac{1-e^{2\pi i(1-\alpha)\setminus(\beta-\alpha)}w(z)}{1-w(z)}\right)\right]d\zeta\,\ldots dz\,dz.$$

□

Corollary 2.3. *If $f\in\mathcal{A}_p$ satisfies the analytic criterion*

$$\alpha < Re\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-p\right)\right\} < \beta,$$

then the integral representation of $f(z)$ is given by

$$f(z) = z^p\exp\left\{\frac{i\,b\,(\beta-\alpha)}{\pi}\int_0^z\frac{1}{t}\left(\log\left(\frac{1-e^{2\pi i(1-\alpha)\setminus(\beta-\alpha)}w(t)}{1-w(t)}\right)\right)dt\right\}.$$

Proof. The proof of the corollary follows if we let $m=0, j=k=1$ in Definition 1.1 and following the steps analogous to the Theorem 2.1. □

Remark 2.4. For a case of $p=1$, the Corollary 2.3 reduces to the result proved by K. Kuroki and S. Owa [1].

If we let $m=1, j=k=1$ in Definition 1.1 and following the steps as in Theorem 2.1, we have the following result.

Corollary 2.5. *If $f\in\mathcal{A}_p$ satisfies the analytic criterion*

$$\alpha < Re\left\{1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}-p\right)\right\} < \beta,$$

then the integral representation of $f(z)$ is given by

$$f(z) = \int_0^z\xi^{p-1}\exp\left\{\frac{i\,b\,(\beta-\alpha)}{\pi}\int_0^\xi\frac{1}{t}\left(\log\left(\frac{1-e^{2\pi i(1-\alpha)\setminus(\beta-\alpha)}w(\zeta)}{1-w(\zeta)}\right)\right)d\zeta\right\}d\xi.$$

3 Conditions for Starlikeness with Respect to Symmetric Points

We now state the following result which will be used in the sequel.

Lemma 3.1. *[3] Let the function q be univalent in the open unit disc $\triangle$ and θ and ϕ be analytic in a domain D containing $q(\triangle)$ with $\phi(w)\neq 0$ when $w\in q(\triangle)$. set $Q(z)=zq^{'}(z)\phi(q(z))$, $h(z)=\theta(q(z))+Q(z)$. Suppose that*

1. Q is starlike univalent in $\triangle$, and

2. $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \triangle$.

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and q is the best dominant.

Theorem 3.2. *Let the function $h(z)$, analytic in $\triangle$, be defined by*

$$h(z) = \delta + (\delta+\gamma)\frac{\beta-\alpha}{\pi} \quad i\log\left(\frac{1-e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)}z}{1-z}\right)$$
$$+\gamma\left(\frac{\beta-\alpha}{\pi}\right) i\frac{z\left(1-e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)}\right)}{(1-z)\left(1-e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)}z\right)} - \gamma\left(\frac{\beta-\alpha}{\pi}\right)^2\left[\log\left(\frac{1-e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)}z}{1-z}\right)\right]^2, \tag{3.1}$$

where $\gamma > 0$, $\delta+\gamma > 0$. If $f \in \mathcal{A}$ with $\frac{f_{j,\,k}(z)}{z} \neq 0$ satisfies the condition

$$\delta + \frac{(\delta+\gamma)}{b}\left[\frac{zf^{(m+1)}(z)}{f^{(m)}_{j,\,k}(z)} - 1 + m\right] + \frac{\gamma}{b^2}\left[\frac{zf^{(m+1)}(z)}{f^{(m)}_{j,\,k}(z)} - 1 + m\right]^2 +$$
$$\frac{\gamma}{b}\left[\frac{zf^{(m+1)}(z)}{f^{(m)}_{j,\,k}(z)} + \frac{z^2f^{(m+2)}(z)}{f^{(m)}_{j,\,k}(z)} - \frac{z^2f^{m+1}(z)f^{m+1}_{j,\,k}(z)}{\left(f^{(m)}_{j,\,k}(z)\right)^2}\right] \prec h(z), \tag{3.2}$$

then $f \in \mathcal{S}^1_{j,\,k}(b;\,\alpha,\,\beta)$.

Proof. Let the function $p(z)$ be defined by

$$p(z) = 1 + \frac{1}{b}\left(\frac{zf^{(m+1)}(z)}{f^{(m)}_{j,\,k}(z)} - 1 + m\right) \qquad (z \in \triangle;\ z \neq 0;\ f \in \mathcal{A}),$$

where $p(z) = 1 + p_1z + p_2z^2 + \cdots \in \mathcal{P}$.
On simplification,

$$zp'(z) = \frac{1}{b}\left[\frac{zf^{(m+1)}(z)}{f^{(m)}_{j,\,k}(z)} + \frac{z^2f^{(m+2)}(z)}{f^{(m)}_{j,\,k}(z)} - \frac{z^2f^{m+1}(z)f^{m+1}_{j,\,k}(z)}{\left(f^{(m)}_{j,\,k}(z)\right)^2}\right].$$

Thus by (3.2), we have

$$\gamma zp'(z) + \gamma p^2(z) + (\delta-\gamma)p(z) \prec h(z). \tag{3.3}$$

Also let

$$g(z) = 1 + \frac{\beta-\alpha}{\pi} \quad i\log\left(\frac{1-e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)}w(z)}{1-w(z)}\right). \tag{3.4}$$

Set

$$\theta(w) := \gamma w^2 + (\delta-\gamma)w \qquad \text{and} \qquad \phi(w) := \gamma,$$

it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}$ with $\phi(0) \neq 0$ in the w-plane. Also, let

$$Q(z) = zg'(z)\phi(g(z)) = \gamma zg'(z)$$

and

$$h(z) = \theta(g(z)) + Q(z) = \gamma(g(z))^2 + (\delta-\gamma)g(z) + \gamma zg'(z).$$

Since $g(z)$ is convex univalent in $\triangle$ provided $\alpha \geq 0$(see [1]), it implies that $Q(z)$ is starlike univalent in $\triangle$. In view of the result proved in [1], $g(z)$ given by (3.4) is starlike for $\alpha \geq 0$, we have

$$Re\frac{zh'(z)}{Q(z)} = Re\left\{\gamma\left(\frac{g(z)}{zg'(z)}(g(z)-1)+1\right) + \delta\frac{g(z)}{zg'(z)}\right\} > 0.$$

The assertion of the Theorem 3.2 now follows by application of Lemma 3.1. □

If $m = 0$ and $b = 1 + 0i$ in Theorem 3.2, we have the following result.

Corollary 3.3. *Let the function $h(z)$ be defined as in (3.1). If $f \in \mathcal{A}$ with $\frac{f_{j,k}(z)}{z} \neq 0$ satisfies the condition*

$$\gamma \left\{ \frac{z^2 f''(z)}{f_{j,k}(z)} - \frac{z^2 f'(z) f'_{j,k}(z)}{(f_{j,k}(z))^2} + \frac{z^2 (f'(z))^2}{(f_{j,k}(z))^2} \right\} + \delta \frac{z f'(z)}{f_{j,k}(z)} \prec h(z), \quad (\gamma > 0, \gamma + \delta > 0),$$

then

$$\frac{zf'(z)}{f_{j,k}(z)} \prec 1 + \frac{\beta - \alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} z}{1 - z} \right).$$

If we $j = k = 1$ in the Corollary 3.3, we get the following result analogous to the result obtained by N. Xu and D. Yang [8].

Corollary 3.4. *Let the function $h(z)$ be defined as in (3.1). If $f \in \mathcal{A}$ with $\frac{f(z)}{z} \neq 0$ satisfies the condition*

$$\gamma \left\{ \frac{z^2 f''(z)}{f(z)} \right\} + \delta \frac{z f'(z)}{f(z)} \prec h(z), \quad (\gamma > 0, \gamma + \delta > 0),$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{\beta - \alpha}{\pi} \quad i \log \left(\frac{1 - e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)} z}{1 - z} \right).$$

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