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ON F-SUPPLEMENTS OF GROUPS

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Abstract In 2000, L. C. Kappe and J. Kirtland obtained useful results for arbitrary, normal, characteristic subgroups to have supplements with some properties in their work 'Supplementation in Groups'. In this work, we will discuss the existence of such supplement that we will call F-supplement, and we will obtain some properties of an F-group.

1 Introduction

The $Frattini\ subgroup$ of an arbitrary group G is defined to be the intersection of all the maximal subgroups, with the stipulation that it shall equal G if G should prove to have no maximal subgroups. This subgroup, which is evidently characteristic, is written as $Frat(G)$

The Frattini subgroup has the remarkable property that it is the set of all nongenerators of the group; here an element g is called a $nongenerator$ of G if $G = \langle g, X\rangle$ always implies that $G = \langle X\rangle$ when X is a subset of G [1].

A subgroup H of a group G is $supplemented$ in G if there is a subgroup K of G such that $G = HK$. If $H \cap K = \{1\}$ then K is said to be $complement$ of H in G.

A group G is called $xP - group$ if every nontrivial $x - subgroup$ satisfies the condition P , where x and P can have the following values:

$x = a\ \ (arbitrary\ subgroup);$
$\ \ = n\ \ (normal\ subgroup);$
$\ \ = c\ \ (characteristic\ subgroup);$
$P = D\ \ (is\ a\ direct\ factor);$
$\ \ = C\ \ (has\ a\ complement);$
$\ \ = S\ \ (has\ a\ proper\ supplement);$
$\ \ = PNS\ \ (has\ a\ proper\ normal\ supplement);$
$\ \ = CS\ \ (has\ a\ proper\ characteristic\ supplement)$ [3].

We say a group is $elementary$ if the group and all its subgroups have trivial Frattini Subgroup.

We call $G' = \langle a^{-1}b^{-1}ab \mid a, b \in G\rangle$ the commutator subgroup of a group G.

Definition 1.1. Let G be a group , N be a subgroup of G and let $G = NS$ for some $S \leq G$. If $N \cap S \leq Frat(S)$, then S is called an $F - supplement$ of N in G. G is called an $F - group$ if every subgroup of G has $F - supplement$ in G.

Example 1.2. (1) G itself is $F - supplement$ of 1_G since $G = 1_G G$ and $1_G \cap G \leq Frat(G)$. So the $F - supplement$ of 1_G is G.

(2) Let G be a group and N be a minimal subgroup of G. Then G is $F - supplement$ of N.

(3) For generalized quaternion group $Q_{2^n} = \left\langle x, y \mid x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}},\ \ y^{-1}xy = x^{-1}\right\rangle$ $(n \geq 3)$, $\langle y\rangle$ is $F - supplement$ of the normal subgroup $\langle x\rangle$ in Q_{2^n} .

2 F-groups

Corollary 2.1. *Let G be a finite group. If G is nilpotent then G is the $F - supplement$ of G'*

Proof. Since G is nilpotent $G \cap G' = G' \leq Frat(G)$ by [1, 5.2.16] and so, G is the $F - supplement$ of G'. □

Proposition 2.2. *Let G be an elementary group. If G is an $F - group$ then for every $H \leq G$, H has a complement in G .*

Proof. Since G is an $F - group$, for every $H \leq G$ there exists a subgroup S of G such that $G = HS$ and $H \cap S \leq Frat(S)$. Hence $H \cap S = \{1\}$ because G is elementary and $Frat(S) = \{1\}$. Therefore H has a complement in G . □

Proposition 2.3. *Let G be a torsion $aCS - group$. Then every subgroup of G has a complement in G.*

Proof. G is abelian by [2, Theorem 4.4] and [2, Proposition 1.3].Then by [2, Theorem 5.1] G is the direct sum of cyclic groups of prime order for distinct primes p and by [1, 3.3.12] if $G = Dr_{\lambda \in \Lambda} G_\lambda$ where G_λ is simple then for a normal subgroup N of G , $G = N \times Dr_{\mu \in M} G_\mu$ for some $M \subseteq \Lambda$.Hence N has a complement in G. □

Corollary 2.4. *Let G be a finite group. If G is an $aCS - group$ then G is an $F - group$..*

Proof. By [2, Theorem 5.1] G is the direct sum of cyclic groups of prime order for distinct primes p and by Proposition 5, G is an $F - group$. □

Theorem 2.5. *Let G be an abelian $aD - group$ then;*

i) For every $N \leq G$, N is an $F - group$. In particullar if φ in a homomorphism of G then $\varphi(G)$ is an $F - group$.

ii) If H is an abelian $aD - group$ then $G \times H$ is an $F - group$.

Proof.(i) By [2, Proposition 7.1(i)] every subgroup of G is an $aD - group$. Then for every $N \leq G$ if M is a subgroup of N then M has a $F - supplement$ in N. Hence N is an $F - group$. For a homomorphism φ of G , then $\varphi(G) \leq G$ and so $\varphi(G)$ is an $F - group$.

(ii) Since G and H are $nD - groups$, $G \times H$ is an $nD - group$ by [2, Proposition 7.1(v)]. Therefore $G \times H$ is an $F - group$ by (i). □

Proposition 2.6. *Let G be an $aC - group$. Then every subgroup of G is an $F - group$.*

Proof. By [2,Proposition 7.1(ii)] every subgroup of G is an $aC - group$. Hence obviously for each $H \leq G$, every subgroup of H has an $F - supplement$ in H and H is an $F - group$. □

Theorem 2.7. *Let G be a group and H be a subgroup of G. If G is an $F - supplement$ of H in G then H has no other $F - supplement$ in G.*

Proof. Let S be an $F - supplement$ of H in G. Then $G = HS$ and $H \cap S \leq Frat(S)$. Since G is an $F - supplement$ of H in G , $H = H \cap G \leq Frat(G)$. This means that H is a group of some nongenerators of G. So we get $G = \langle S \rangle = S$.
□

Proposition 2.8. *Let G be an $aS - group$ and N be a nontrivial subgroup of G. Then G can not be an $F - supplement$ of N in G.*

Proof.$Frat(G) = \{1\}$ by [2, Proposition 3.4]. Suppose G is the $F - supplement$ of N in G.Then $N \cap G = N \leq Frat(G) = \{1\}$ and $N = \{1\}$ which is a contradiction.
□

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