CERTAIN PROPERTIES OF A NEW SUBCLASS OF MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

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Communicated by Ayman Badawi

Abstract In this paper we introduce and investigate an interesting subclass $\mathcal{MK}^{(k)}(\lambda, h)$ of functions which are analytic in the punctured unit disk and meromorphically close-to-convex. By using the principle of subordination, we establish several properties such as inclusion relationship and distortion theorems for functions in our function class. The results presented here would provide extensions of those given in earlier works.

1 Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathcal{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} =: \mathcal{U} \backslash \{0\}$$

where $\mathcal{U}$ is an open unit disk.

Let $\mathcal{P}[A, B]$ denote the class of functions p given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathcal{U}) \tag{1.2}$$

which are analytic and convex in $\mathcal{U}$ and satisfy the condition

$$p(z) \prec \frac{1 + Az}{1 + Bz} \qquad (z \in \mathcal{U}; -1 \le B < A \le 1). \tag{1.3}$$

Let $f, g \in \Sigma$, where f is given by (1.1) and g is defined by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n. \tag{1.4}$$

A function $f \in \Sigma$ is said to be in the class $\mathcal{MS}^*(\alpha)$ of meromorphic starlike of order α if it satisfies the inequality

$$R\left(\frac{-zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}; 0 \le \alpha < 1). \tag{1.5}$$

Moreover, a function $f \in \Sigma$ is said to be in the class $\mathcal{MC}$ of meromorphic close-to-convex functions if it satisfies the condition

$$R\left(\frac{zf'(z)}{g(z)}\right) < 0 \qquad (z \in \mathcal{U}; g \in \mathcal{MS}^*(0) \equiv \mathcal{MS}^*).$$

Recently, Wang et al. [9] introduced and investigated the class $\mathcal{MK}$ of meromorphic close-to-convex functions which satisfy the following inequality

$$R\left(\frac{f'(z)}{g(z)g(-z)}\right) > 0 \qquad (z \in \mathcal{U}). \tag{1.6}$$

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In a recent paper [2], we introduced and investigated the following class $\mathcal{MK}^{(k)}[A, B]$ of meromorphic functions:

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $\mathcal{MK}^{(k)}[A, B]$ if it satisfies the inequality

$$\frac{-f'(z)}{z^{k-2}g_k(z)} \prec \frac{1 + Az}{1 + Bz} \qquad (z \in \mathcal{U}; -1 \le B < A \le 1) \qquad (1.7)$$

where $g \in \mathcal{MS}^*(\frac{k-1}{k})$, $k \ge 1$ is fixed positive integer and $g_k(z)$ is defined by the following equality

$$g_k(z) = \prod_{\nu=0}^{k-1} \rho^\nu g(\rho^\nu z) \qquad \left(\rho = e^{2\pi i/k} \right). \qquad (1.8)$$

.

Remark 1.2. For k=2, we get the subclass defined by Sim and Kwon [8] which includes the subclass $\mathcal{MK}$ studied by Wang et al. [9] for A=1 and B=-1.

Remark 1.3. By simple calculations we see that the subordination (1.7) is equivalent to

$$\left| \frac{f'(z)}{z^{k-2}g_k(z)} + 1 \right| < \left| \frac{Bf'(z)}{z^{k-2}g_k(z)} + A \right| \qquad (z \in \mathcal{U}; -1 \le B < A \le 1).$$

Definition 1.4. (see, e.g. [5]). For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g, and write $f(z) \prec g(z)$, if there exists a schwarz function w, which (by definition) is analytic in $\mathcal{U}$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathcal{U}$, such that
$$f(z) = g(w(z)), z \in \mathcal{U}.$$
In particular, if the function g is univalent in $\mathcal{U}$, then above subordination is equivalent to

$$f(0) = g(0) \ \ and \ \ f(\mathcal{U}) \subset g(\mathcal{U}).$$

Motivated by the aforementioned works, we now introduce the following new subclass of meromorphic close-to-convex functions :

Definition 1.5. Let $h : \mathcal{U} \to \mathbb{C}$ be a convex function such that

$$h(0) = 1, \qquad h(\bar{z}) = \overline{h(z)}, \qquad \text{Re}h(z) > 0, \qquad z \in \mathcal{U}. \qquad (1.9)$$

Suppose also that the function h satisfies the following conditions for all $r \in (0, 1)$:

$$\left. \begin{array}{l} \min \{|h(z)| : |z| = r\} = \min \{h(r), h(-r)\}, \\ \max \{|h(z)| : |z| = r\} = \max \{h(r), h(-r)\} \end{array} \right\}. \qquad (1.10)$$

Let the function f be analytic in $\mathcal{U}^*$, we say that $f \in \mathcal{MK}^{(k)}(\lambda, h)$, if there exists a function $g \in \mathcal{MS}^*(\frac{k-1}{k})$, $k \in \mathbb{N}$ such that

$$\frac{-\left[(1 + 2\lambda) f'(z) + \lambda z f''(z)\right]}{z^{k-2}g_k(z)} \in h(\mathcal{U}), \ (z \in \mathcal{U}, 0 \le \lambda \le 1) \qquad (1.11)$$

Remark 1.6. There are many choices of the function h which would provide interesting subclasses of analytic functions.
(i) If we let

$$h(z) = \frac{1 + Az}{1 + Bz} \qquad (-1 \le B < A \le 1) \qquad (1.12)$$

then it is easy to verify that h is convex in $\mathcal{U}$, and satisfies the hypothesis of Definition 1.5. Thus if f be an analytic function in $\mathcal{U}^*$ and $g \in \mathcal{MS}^*(\frac{k-1}{k})$, $k \in \mathbb{N}$ such that

$$\frac{-\left[(1 + 2\lambda) f'(z) + \lambda z f''(z)\right]}{z^{k-2}g_k(z)} \prec \frac{1 + Az}{1 + Bz} \qquad (0 \le \lambda \le 1, -1 \le B < A \le 1)$$

then we say that $f \in \mathcal{MK}^{(k)}(\lambda, A, B)$. For $\lambda = 0$, we get the class $\mathcal{MK}^{(k)}[A, B]$ which contains the subclasses $\mathcal{MK}^{(2)}[A, B]$ and $\mathcal{MK}$ as special cases.
(ii) For

$$h(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} \qquad (0 \le \gamma < 1)$$

we get the new class $\mathcal{MK}^{(k)}(\lambda, \gamma)$ which satisfies the inequality

$$\text{Re}\left(\frac{-\left[(1 + 2\lambda) f'(z) + \lambda z f''(z)\right]}{z^{k-2}g_k(z)} \right) > \gamma, \ z \in \mathcal{U}. \qquad (1.13)$$

For $\lambda = 0$, we get the class studied by Yi et al. [10]. Also for $\gamma = 0$, we get the new class $\mathcal{MK}^{(k)}(\lambda, 0) \equiv \mathcal{MK}^{(k)}(\lambda)$ satisfying the condition

$$\text{Re}\left(\frac{-\left[(1 + 2\lambda)f'(z) + \lambda z f''(z)\right]}{z^{k-2}g_k(z)}\right) > 0, \ z \in \mathcal{U}.$$

Recently authors and other research workers ([3],[4],[7]) have established certain new results for subclasses of close-to-convex function. In this paper, by using the principle of subordination, we establish inclusion theorem and distortion theorems for functions in the class $\mathcal{MK}^{(k)}(\lambda, h)$. Our results unify and extend the corresponding results obtained earlier by Shi et al. [7], Sim and Kwon [8], Wang et al. [9] and Yi et al. [10].

2 Results Required

To prove our main results given in the next section, we shall require the results contained in following Lemmas:

Lemma 2.1. *(see [10]). Let* $g(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} b_n z^n \in \mathcal{MS}^*\left(\frac{k-1}{k}\right)$, *then*

$$G_k(z) = z^{k-1}g_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} B_n z^n \in \mathcal{MS}^*. \tag{2.1}$$

Lemma 2.2. *(see [1]). Suppose that*

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{MS}^*. \tag{2.2}$$

Then

$$|c_n| \leq \frac{2}{n+1} \ (n \in N). \tag{2.3}$$

Each of these inequality is sharp, with the extremal function given by

$$h(z) = z^{-1}\left(1 + z^{n+1}\right)^{\frac{2}{n+1}}. \tag{2.4}$$

These bounds are sharp.

Lemma 2.3. *(see [6]). Suppose that* $g \in \mathcal{MS}^*$, *then*

$$\frac{(1-r)^2}{r} \leq |g(z)| \leq \frac{(1+r)^2}{r} \qquad (|z| = r; 0 < r < 1). \tag{2.5}$$

Lemma 2.4. *If* $\text{Re}\, c \geq 0$, *then* $f \in \mathcal{MC}$ *implies*

$$H(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt \in \mathcal{MC}. \tag{2.6}$$

We now state and prove the main results of our present investigation:

3 Main Results

Theorem 3.1. *Let* f *be analytic in* $\mathcal{U}^*$. *Then* $f \in \mathcal{MK}^{(k)}(\lambda, h)$ *if and only if there exists a function* $g \in \mathcal{MS}^*\left(\frac{k-1}{k}\right)$, *such that*

$$\frac{-\left[(1 + 2\lambda)f'(z) + \lambda z f''(z)\right]}{z^{k-2}g_k(z)} \prec h(z), \qquad (0 \leq \lambda \leq 1) \tag{3.1}$$

where g_k *is given by* (1.8).

Proof. This result can be proven fairly easily by using the Definition 1.4 combined with the equation (1.11).
In view of Remark 1.2, if we set $\lambda = 0$, and

$$h(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} \qquad (0 \leq \gamma < 1)$$

in theorem 3.1, we deduce the following result.   □

Corollary 3.2. *Let f be analytic in $\mathcal{U}^*$. Then $\mathcal{MK}^{(k)}(\gamma)$ $(0 \leq \gamma < 1)$ if and only if there exists $g \in \mathcal{MS}^*(\frac{k-1}{k})$, such that*

$$\frac{-f'(z)}{z^{k-2}g_k(z)} \prec \frac{1 + (1 - 2\gamma)z}{1 - z}$$

where g_k is given by (1.8).

Note that Corollary 3.1 was proven by Yi et al. [10]. However by using Theorem 3.1 we are able to deduce this result as an easy consequence of the theorem.

Theorem 3.3. *If $0 \leq \lambda \leq 1$, then*

$$\mathcal{MK}^{(k)}(\lambda, h) \subset \mathcal{MC} \subset \Sigma$$

Proof. Let $f \in \mathcal{MK}^{(k)}(\lambda, h)$ be an arbitrary function, and let we define the corresponding functions F and G_k by $F(z) = (1 + \lambda)f(z) + \lambda z f'(z)$ and $G_k(z) = z^{k-1}g_k(z)$ respectively.

Then the condition (3.1) can be written as

$$\frac{-zF'(z)}{G_k(z)} \prec h(z).$$

By Lemma 2.1, we have $G_k \in \mathcal{MS}^*$, and from the above subordination combined with the fact that $\mathrm{Re}\, h(z) > 0$ for all $z \in \mathcal{U}$, we deduce that

$$F(z) = (1 + \lambda)f(z) + \lambda z f'(z) \in \mathcal{MC}.$$

Now we will consider the following two cases:

Case 1: If $\lambda = 0$, then it is obvious that $f = F \in \mathcal{MC}$.

Case 2: If $0 < \lambda \leq 1$, according to the definition of F, we have

$$f(z) = \frac{1}{\lambda} z^{-1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}} F(t)dt.$$

Denoting $\frac{1}{\lambda} = c$, then $\mathrm{Re}\, c \geq 0$ and by using Lemma 2.4, we conclude that $f \in \mathcal{MC}$, which completes the proof of the Theorem. $\square$

Theorem 3.4. *Suppose that the convex function $h : \mathcal{U} \to C$ satisfy the conditions* (1.9) *and* (1.10) *and $g \in \mathcal{MS}^*(\frac{k-1}{k})$, where g_k is given by* (1.8).

Let f be an analytic function in $\mathcal{U}^*$ of the form (1.1). *Then $f \in \mathcal{MK}^{(k)}(\lambda, h)$ if and only if*

$$1 - \sum_{n=1}^{\infty} \frac{n + \lambda n(n+1)}{1 - h(e^{i\theta})} a_n z^{n+1} - \sum_{n=1}^{\infty} \frac{h(e^{i\theta})}{1 - h(e^{i\theta})} B_n z^{n+1} \neq 0 \qquad (3.2)$$

for all $z \in \mathcal{U}$ and $\theta \in [0, 2\pi)$ where B_n are given by (2.1)

Proof. Since $g \in \mathcal{MS}^*(\frac{k-1}{k})$, then g is univalent in $\mathcal{U}^*$, hence it follows that $g_k(z) = \prod_{\nu=0}^{k-1} \rho^\nu g(\rho^\nu z) \neq 0$ for all $z \in \mathcal{U}^*$, that is

$$z^k g_k(z) \neq 0, \qquad z \in \mathcal{U}. \qquad (3.3)$$

i) First, supposing that $f \in \mathcal{MK}^{(k)}(\lambda, h)$, from (3.1), we have

$$\frac{-[(1 + 2\lambda)f'(z) + \lambda z f''(z)]}{z^{k-2}g_k(z)} \prec h(z). \qquad (3.4)$$

Then there exists a function w, which is analytic in $\mathcal{U}$ with $w(0) = 0$, $|w(z)| < 1$, for all $z \in \mathcal{U}$, such that

$$\frac{-[(1 + 2\lambda)f'(z) + \lambda z f''(z)]}{z^{k-2}g_k(z)} = h(w(z))$$

and thus

$$\frac{-[(1 + 2\lambda)f'(z) + \lambda z f''(z)]}{z^{k-2}g_k(z)} \neq h(e^{i\theta}), \; z \in \mathcal{U}, \; \theta \in [0, 2\pi).$$

According to (3.3) and using the fact that h is univalent in $\mathcal{U}$, the previous subordination is equivalent to

$$- \left[(1 + 2\lambda) z^2 f'(z) + \lambda z^3 f''(z) \right] \neq h(e^{i\theta}) z^k g_k(z), \; z \in \mathcal{U}. \tag{3.5}$$

According to Lemma 2.1, above relation leads to (3.2), hence it proves the first part of our result.

    ii) Reversely, since it was previously shown that assumption (3.2) is equivalent to (3.5), using (3.3) we obtain that

$$\frac{- \left[(1 + 2\lambda) f'(z) + \lambda z f''(z) \right]}{z^{k-2} g_k(z)} \neq h(e^{i\theta}), \; z \in \mathcal{U}, \; \theta \in [0, 2\pi). \tag{3.6}$$

If we denote

$$\varphi(z) = \frac{- \left[(1 + 2\lambda) f'(z) + \lambda z f''(z) \right]}{z^{k-2} g_k(z)}$$

the relation (3.6) shows that $\varphi(U) \cap h(\partial U) = \varphi$. Thus, the simply-connected domain $\varphi(U)$ is included in a connected component of $C \backslash h(\partial U)$. From here, using the fact that $\varphi(0) = h(0)$ together with the univalence of the function h, it follows that $\varphi(z) \prec h(z)$, which represents in fact the subordination (3.4), i.e. $f \in \mathcal{MK}^{(k)}(\lambda, h)$. $\square$

For the special case when the function h is given by (1.12), From theorem 3.3, we obtain the following results:

Corollary 3.5. *Suppose that $g \in \mathcal{MS}^*(\frac{k-1}{k})$ and g_k is given by* (1.8). *If f is analytic function in $\mathcal{U}^*$ of the form* (1.1), *such that*

$$(1 + |B|) \sum_{n=1}^{\infty} [n + \lambda n(n+1)] |a_n| + (1 + |A|) \sum_{n=1}^{\infty} \frac{2}{n+1} < A - B \tag{3.7}$$

where the coefficient B_n are given by (2.1), *then $f \in \mathcal{MK}^{(k)}(\lambda, A, B)$.*

Substituting $A = 1 - 2\gamma \; (0 \leq \gamma < 1)$ and $B = -1$ in above corollary, we obtain the following special case:

Corollary 3.6. *Suppose that $g \in \mathcal{MS}^*(\frac{k-1}{k})$ and g_k is given by* (1.8). *If f is analytic function in $\mathcal{U}^*$ of the form* (1.1), *such that*

$$\left[\sum_{n=1}^{\infty} [n + \lambda n(n+1)] |a_n| + (1 + |1 - 2\gamma|) \sum_{n=1}^{\infty} \left(\frac{1}{n+1} \right) \right] < (1 - \gamma)$$

where the coefficient B_n are given by (2.1), *then $f \in \mathcal{MK}^{(k)}(\lambda, \gamma)$.*

Remark 3.7. Letting $\lambda = 0$ in Corollary 3.5, we get the result obtained by Goyal et al. [2].

Theorem 3.8. *If $f \in \mathcal{MK}^{(k)}(\lambda, h)$, then*
(i) $0 \leq \lambda \leq 1$, for $|z| \leq r \; (0 < r < 1)$, we have

$$\frac{(1-r)^2}{r^2} \min \{ h(r), h(-r) \} \leq |(1 + 2\lambda) f'(z) + \lambda z f''(z)| \leq \frac{(1+r)^2}{r^2} \max \{ h(r), h(-r) \}. \tag{3.9}$$

(ii) For $|z| \leq r \; (0 < r < 1)$, we have

$$\int_0^r \frac{\min \{ h(t), h(-t) \}}{t^2} (1-t)^2 \, dt \leq |f(z)| \leq \int_0^r \frac{\max \{ h(t), h(-t) \}}{t^2} (1+t)^2 \, dt. \tag{3.10}$$

(iii) If $0 < \lambda \leq 1$, for $|z| \leq r \; (0 < r < 1)$, we have

$$\frac{1}{\lambda z^{1 + \frac{1}{\lambda}}} \int_0^r \int_0^s \frac{\min \{ h(t), h(-t) \}}{t^2} (1-t)^2 s^{\frac{1}{\lambda}} \, ds \, dt \leq |f(z)|$$

$$\leq \frac{1}{\lambda z^{1 + \frac{1}{\lambda}}} \int_0^r \int_0^s \frac{\max \{ h(t), h(-t) \}}{t^2} (1+t)^2 s^{\frac{1}{\lambda}} \, ds \, dt. \tag{3.11}$$

Proof. Since $f \in \mathcal{MK}^{(k)}(\lambda, h)$, there exists a function $g \in \mathcal{MS}^*(\frac{k-1}{k})$ such that (1.11) holds. From lemma 2.1, it follows that the function G_k given by (2.1) is starlike, and according to the well known inequality (2.5), we have

$$\frac{(1-r)^2}{r} \leq |G_k(z)| \leq \frac{(1+r)^2}{r}, |z| \leq r \, (0 < r < 1). \tag{3.12}$$

From the equations, (1.11) combined with (1.10), we deduce that

$$\min\{h(r), h(-r)\} \leq \left| \frac{-\left[(1+2\lambda) zf'(z) + \lambda z^2 f''(z)\right]}{G_k(z)} \right| \leq \max\{h(r), h(-r)\}. \tag{3.13}$$

Letting

$$F(z) = (1+\lambda) f(z) + \lambda z f'(z). \tag{3.14}$$

then $F'(z) = (1+2\lambda) f'(z) + \lambda z f''(z)$, and the inequality (3.13) may be written as

$$\min\{h(r), h(-r)\} \leq \left| \frac{-zF'(z)}{G_k(z)} \right| \leq \max\{h(r), h(-r)\}, |z| \leq r \tag{3.15}$$

From (3.12) and (3.15), we obtain that

$$\frac{(1-r)^2}{r^2} \min\{h(r), h(-r)\} \leq |F'(z)| \leq \frac{(1+r)^2}{r^2} \max\{h(r), h(-r)\}. \tag{3.16}$$

which proves (3.9).

If l denotes the semi-closed line-segment that connects the points 0 and $z = re^{i\theta}$ $(0 < r < 1)$, i.e. $l = (0, re^{i\theta}]$, then

$$|F(z)| = \left| \int_l F'(\zeta) d\zeta \right| = \left| \int_0^r F'\left(te^{i\theta}\right) e^{i\theta} dt \right| \leq \int_0^r \left|F'\left(te^{i\theta}\right)\right| dt$$

and from the right-hand side part of (3.16), we deduce that

$$|F(z)| \leq \int_0^r \frac{\max\{h(t), h(-t)\}}{t^2} (1+t)^2 \, dt, |z| = r. \tag{3.17}$$

Since $f \in \mathcal{MK}^{(k)}(\lambda, h)$, then

$$\frac{-zF'(z)}{G_k(z)} \prec h(z)$$

where $G_k \in \mathcal{MS}^*$, and $\mathrm{Re} h(z) > 0$ for all $z \in \mathcal{U}$. Thus, we deduce that $F \in \mathcal{MC}$, hence the function F is univalent in $\mathcal{U}^*$.

To prove the corresponding left-hand side inequality, let $z_0 \in \mathcal{U}^*$ with $|z_0| = r$, such that

$$|F(z_0)| = \min\{|F(z)| : |z| = r\}$$

for some $0 < r < 1$. It is sufficient to prove that the left-hand side inequality holds for this point z_0, because, otherwise, we have $|F(z)| \geq |F(z_0)|$ for all $|z| = r$. Since the function F is univalent in $\mathcal{U}^*$, the image of the semi-closed line segment $\sigma = (0, F(z_0)]$ by F^{-1} is a simple jordan curve Λ included in the closed disk $\{z \in \mathbb{C} : |z| \leq r, 0 < r < 1\}$ i.e. $\Lambda = F^{-1}(\sigma) \subset \{z \in \mathbb{C} : |z| \leq r, 0 < r < 1\}$.

Let denote $z_0 = re^{i\theta}$, and $F(z_0) = Re^{i\Phi}$. If $w \in \sigma$ is an arbitrary point, then $w = se^{i\Phi}$, where $s \in (0, R]$, hence $|dw| = ds$. Denoting $\zeta = F^{-1}(w)$, $\zeta = te^{i\varphi}$, hence $d\zeta = e^{i\varphi} dt + tie^{i\varphi} d\varphi$, and thus $|d\zeta| \geq dt$.

From here and from the left-hand side inequality of (3.16), it follows that

$$|F(z_0)| = \left|F(re^{i\theta})\right| = \int_0^R ds = \int_\sigma |dw| = \int |F'(\zeta)| |d\zeta|$$

$$\geq \int_0^r \left|F'(te^{i\theta})\right| dt \geq \int_0^r \frac{\min\{h(t), h(-t)\}}{t^2} (1-t)^2 \, dt$$

hence

$$|F(z)| \geq \int_0^r \frac{\min\{h(t), h(-t)\}}{t^2} (1-t)^2 \, dt. \tag{3.18}$$

Combining the inequalities (3.17) and (3.18), together with the maximum modulus principle, we have

$$\int_0^r \frac{\min\{h(t), h(-t)\}}{t^2} (1-t)^2 \, dt \leq |F(z)| \leq \int_0^r \frac{\max\{h(t), h(-t)\}}{t^2} (1+t)^2 \, dt. \tag{3.19}$$

To complete our proof, we will discuss the following two cases for the parameter $\lambda \in [0, 1]$
Case I. For $\lambda = 0$, from (3.19), we will easily get (3.10).
Case II. For $0 < \lambda \leq 1$, from (3.14), we obtain

$$f(z) = \frac{1}{\lambda z^{1+\frac{1}{\lambda}}} \int_0^z t^{\frac{1}{\lambda}} F(t) dt$$

hence we easily conclude that (3.11) holds. □

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