Partial Stabilization of a Coupled Wave Equations

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Abstract. We consider a stabilization problem for a coupled wave equations on a compact Riemannian manifold $\Omega$ with or without boundary. We prove the exponential stability result in the energy space, under a geometrical control condition (BLR). Without any geometrical assumption and for all regular initial data, we give a logarithmic decay result of the energy.

1 Introduction

In this paper we study the stabilization of a coupled wave equations. More precisely, we consider the following initial and boundary value problem:

\begin{align*}
\partial_t^2 u_1 - \Delta u_1 + \beta \partial_t u_2 + 2a(x) \partial_t u_1 &= 0, \quad \Omega \times (0, +\infty), \\
\partial_t^2 u_2 - \alpha \Delta u_2 - \beta \partial_t u_1 &= 0, \quad \Omega \times (0, +\infty), \\
u_1 \big|_{\partial \Omega} &= 0, \quad \Omega \times (0, +\infty), \\
u_2 \big|_{\partial \Omega} &= 0, \quad \Omega \times (0, +\infty), \\
u_1(x,0) &= u_0^1(x), \quad \partial_\nu u_1(x,0) = u_1^1(x), \quad x \in \Omega, \\
u_2(x,0) &= u_0^2(x), \quad \partial_\nu u_2(x,0) = u_1^2(x), \quad x \in \Omega,
\end{align*}

where $\Omega$ is a compact connected Riemannian manifold, $a(x) \in C(\overline{\Omega}, \mathbb{R}^+)$ and $\alpha, \beta$ are positives constants.

If we set $u = (u_1, u_2)$ then the system of equations (1.1)-(1.6) is equivalent to the following system

\begin{equation}
\begin{cases}
\partial_t^2 u - D_\alpha u + K_\beta^a \partial_t u = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \\
u = 0 \quad \text{on} \quad \partial \Omega \times (0, +\infty), \quad u(\cdot, 0) = u_0, \quad \partial_\nu u(\cdot, 0) = u_1, \quad \text{in} \quad \Omega,
\end{cases}
\end{equation}

where

\begin{equation}
D_\alpha = \begin{pmatrix} \alpha \Delta & 0 \\ 0 & \Delta \end{pmatrix}, \quad K_\beta^a = \begin{pmatrix} 2a(x) & \beta \\ -\beta & 0 \end{pmatrix}, \quad u_0 = (u_0^1, u_0^2) \quad \text{and} \quad u_1 = (u_1^1, u_1^2).
\end{equation}

The problem (1.7) has an unique solution $u(x, t) \in C^0 \left(\mathbb{R}, (H_0^1(\Omega))^2\right) \cap C^1 \left(\mathbb{R}, (L^2(\Omega))^2\right)$ for all initial data $u_0 \in (H_0^1(\Omega))^2 \oplus (L^2(\Omega))^2$, obtained by using the Hille-Yosida theorem for an unbounded operator.

We consider the Hilbert space $H = (H_0^1(\Omega))^2 \oplus (L^2(\Omega))^2$, we define
Theorem 1.1. \(A_{a}^{\alpha,\beta} = \begin{pmatrix} 0 & id \\ D_{a} & -K_{a} \end{pmatrix}\), \(\mathcal{D}(A_{a}^{\alpha,\beta}) = (H_{0}^{1}(\Omega) \cap H^{2}(\Omega))^{2} \oplus (H_{0}^{1}(\Omega))^{2}\). (1.8)

Let \(u(x,t) = (u_{1}, u_{2})(x,t)\) solution of (1.7), we define the energy functional at the time \(t\) by
\[
E(u, t) = \frac{1}{2} \int_{\Omega} \left( \left| \partial_{t}u_{1} \right|^{2} + |\nabla u_{1}|^{2} \right) dx
\]
that satisfy the following estimation
\[
E(u, 0) - E(u, t) = \int_{0}^{t} \int_{\Omega} a(x)|\partial_{t}u_{1}(x,s)|^{2} dx ds,
\]
where \(\nabla u = (\nabla_{x} u_{1}, \sqrt{2} \nabla_{x} u_{2})\). We recall the following results,

**Theorem 1.1.** Assume that \(a \neq 0\). Then, we have

(i) If \(\partial \Omega \neq \emptyset\), we have \(\text{Re} \lambda < 0\) for \(\lambda \in \text{sp}(A_{a}^{\alpha,\beta})\) (spectra set of \(A_{a}^{\alpha,\beta}\)). If \(\partial \Omega = \emptyset\), \(\lambda = 0\) is the only eigenvalue with null real part.

(ii) For any initials data \((u_{1}^{0}, u_{2}^{0}), (u_{1}^{1}, u_{2}^{1})\) \(\in (H_{0}^{1}(\Omega))^{2} \oplus (L^{2}(\Omega))^{2}\), the solution \(u = (u_{1}, u_{2})\) of (1.7) satisfies \(\lim_{t \to +\infty} E(u(t)) = 0\).

(iii) Moreover, assume that \(a \neq 1\) and that the geodesic of \(\Omega\) hasn’t contact of infinite order with \(\partial \Omega\) and there exists a time \(T_{0}\) such that any generalized geodesics of \(\Omega\) with its length larger than \(T_{0}\) meet \((a(x) > 0)\). Then, there exists \(c_{0}, c_{1} > 0\) such that
\[
E(u(t)) \leq c_{0} e^{-c_{1}t} E(u(0)), \quad \forall u \in H, \quad \forall t \geq 0.
\]

**Proof.**

(i) If \(\lambda = i\omega \in \text{sp}(A_{a}^{\alpha,\beta}), \omega \in \mathbb{R}\) there exists \(f = (f_{1}, f_{2}) \neq 0\) in \((H_{0}^{1}(\Omega))^{2}\) such that
\[
-D_{a}f + \lambda K_{a} f + \lambda^{2} f = 0,
\]
which implies
\[
\omega \left( \int_{\Omega} a|f_{1}|^{2} + \beta \text{Re} \int_{\Omega} f_{2} \overline{f_{1}} \right) = 0,
\]
\[
\omega \beta \text{Re} \int_{\Omega} f_{1} \overline{f_{2}} = 0,
\]
and
\[
\int_{\Omega} |\nabla f_{1}|^{2} - \omega^{2} \int_{\Omega} |f_{1}|^{2} - \omega \beta \text{Im} \int_{\Omega} f_{1} \cdot \overline{f_{2}} = 0,
\]
\[
\alpha \int_{\Omega} |\nabla f_{2}|^{2} - \omega^{2} \int_{\Omega} |f_{2}|^{2} + \omega \beta \text{Im} \int_{\Omega} f_{2} \cdot \overline{f_{1}} = 0.
\]

If \(\omega = 0\) then we have \(f_{1} = \text{cst}\) and \(f_{2} = \text{cst}\); if \(\omega \neq 0\), we have \(\sqrt{a} f_{1} = 0\) in \(L^{2}(\Omega)\), since \(\Omega = \{a(x) > 0\}\) is non empty open set. Then, \(f_{1} = 0\) and
\[
\left\{ \begin{array}{l}
-\Delta f_{1} + \lambda^{2} f_{1} + \beta \lambda f_{2} = 0, \\
-\alpha \Delta f_{2} + \lambda^{2} f_{2} - \beta \lambda f_{1} = 0,
\end{array} \right.
\]
this implies that \((f_{1}, f_{2})_{\Omega} \equiv (0, 0)\), using that \(\Omega\) is connected set, thus \((f_{1}, f_{2}) \equiv (0, 0)\).
(ii) We deduce 2. by 1. because $\partial E\lambda = H$, using [7].

(iii) If $\partial \Omega = \emptyset$, we can see [10] and the general case, following Bardos, Lebeau and Rauch [1], using the propagation Theorem of Melrose- Sjöstrand which will be the goal of the proof of point 2. of Theorem 1.3.

**Theorem 1.2.** Assume that $a \not\equiv 0$. Then, there exists $C > 0$ such that

$$\forall \lambda \in \text{sp}(A^{\alpha,\beta}_a) \setminus \{0\}, \quad \text{Re} \lambda < -\frac{1}{C} e^{-C|\text{Im} \lambda|}. \tag{1.12}$$

For $\lambda = -\sigma + i\omega$, $\omega \in \mathbb{R}$, $|\omega| \geq 1$ and $0 \leq \sigma \leq \frac{1}{C} e^{-C|\omega|}$ we have

$$\left\| (\lambda - A^{\alpha,\beta}_a)^{-1} \right\|_{L(H)} \leq Ce^{C|\omega|} \tag{1.13}$$

(Here the norm of the resolvent is the norm of the operator on $H$.) Moreover, for any $k > 0$, there exists $C > 0$ such that for all $(u_0, u_1) \in D \left( (A^{\alpha,\beta}_a)^k \right)$, we have

$$\forall t \geq 0, \quad E(u, t)^\frac{1}{2} \leq \frac{C}{(\ln(2 + t))^k} \left\| (u_0, u_1) \right\|_{D \left( (A^{\alpha,\beta}_a)^k \right)}. \tag{1.14}$$

Let $R > 0$, we set

$$D(R) = \sup \{ \text{Re} \lambda_j \mid \lambda_j \in \text{Sp}(A^{\alpha,\beta}_a), \ |\lambda_j| \geq R \} \tag{1.15}$$

that is a negative function, decreasing when $R > 0$. We denote $D(\infty) = \lim_{R \to \infty} D(R)$ and $D(0) = \lim_{R \to 0^+} D(R)$.

Assuming that there have no contacts of infinite order between the bicharacteristic of $\overline{\Omega}$ and its boundary $\partial \Omega$ (the geometric control condition (GCC)). First, we notice that determinant of the symbol is given by

$$p^{\alpha,\beta}_a(t, x; \tau, \xi) = (|\xi|^2 - \tau^2) (\alpha |\xi|^2 - \tau^2)$$

this leads to two bicharacteristic families in the characteristic set of $P^{\alpha,\beta}_a$, $\text{Char} P^{\alpha,\beta}_a = \{(x, t; \xi, \tau); \ p^{\alpha,\beta}_a(t, x; \tau, \xi) = 0\}$, namely those of the symbols

$$p_1 = |\xi|^2 - \tau^2 \quad \text{and} \quad p_0 = \alpha |\xi|^2 - \tau^2,$$

if $\alpha \not\equiv 1$, the wave front sets propagate independently along the null bicharacteristic of each one of the two families. Let $\rho_0 = (x_0, u_0) \in \overline{\Omega}$, with $|u_0| = 1$ (where $u_0$ is in a half closed space defined by $\overline{\Omega}$ if $x_0 \in \partial \Omega$ ) there exists a unique geodesic generalized

$$s \to x_1(s, \rho_0) \quad \text{in} \overline{\Omega} \quad \text{resp.} \ s \to x_2(s, \rho_0) \quad \text{in} \overline{\Omega}$$

issued to $\rho_0$ i.e. satisfy

$$x_1(0, \rho_0) = x_0, \quad \lim_{s \to 0^-} \frac{x_1(s, \rho_0) - x_0}{s} = u_0 \quad \text{resp.} \quad \lim_{s \to 0^-} \frac{x_2(s, \rho_0) - x_0}{s} = \sqrt{\alpha} u_0.$$

Let $t > 0$, we set

$$C_i(t) = \inf_{\rho_0} \frac{1}{t} \int_0^t a(x_1(s, \rho_0))ds, \quad C_i(t) = \inf_{\rho_0} \frac{1}{t} \int_0^t a(x_2(s, \rho_0))ds.$$

that satisfies

$$tC_i(t) + sC_i(s) \leq (t + s)C_i(t + s), \quad i = 1, 2.$$
We denote
\[
C(t) = \min(C_1(t), C_2(t)) = \min \left( \inf_{\rho_0} \frac{1}{t} \int_0^t a(x_1(s, \rho_0))ds, \inf_{\rho_0} \frac{1}{t} \int_0^t a(x_2(s, \rho_0))ds \right) \tag{1.16}
\]
that is a additive function and we set \( C(\infty) = \lim_{t \to +\infty} C(t) \). We have \( C(t) \leq C(\infty) \) for all \( t \).

Let \( \varrho = \sup \{ \gamma \geq 0 / \exists B > 0, \forall u \in H, E(u, t) \leq Be^{-\gamma t}E(u, t) \} \). \tag{1.17}

**Theorem 1.3.** Assume that \( \alpha \neq 1 \), then we have

(i) \( \varrho = \min \{ -D(0), C(\infty) \} \).

(ii) \( C(\infty) \leq -D(0) \).

### 2 Proof of Theorem 1.2

We denote \( H = (H_0^1(\Omega))^2 \bigoplus (L^2(\Omega))^2, \) \( H^* \) the dual space of \( H \) and the duality product is given by

\[
\langle u_1, u_2 \rangle = \int_{\Omega} u_1 \cdot u_2 \, d\Omega, \quad u_1 = (u_1^1, u_2^1), \quad u_2 = (u_1^2, u_2^2) \in H^*, \quad u_2 = (u_1^2, u_2^2) \in H. \tag{2.1}
\]

We decompose \( A_0^\alpha, \beta \) in the following form

\[
A_0^\alpha, \beta = A_0^{\alpha, 0} + B_0^\beta = A_0^{\alpha} + B_0^\beta, \quad A_0^{\alpha} = \begin{pmatrix} 0 & \text{id} \\ D_\alpha & 0 \end{pmatrix}; \quad B_0^\beta = \begin{pmatrix} 0 & 0 \\ 0 & K_0^\beta \end{pmatrix} \tag{2.2}
\]

\( B_0^\beta \) is a bounded operator in \( H \) and compact as an operator of \( \mathcal{L}(H, H^*) \).

\( (\lambda - A_0^{\alpha, \beta}) u = v \) equivalent to

\[
\begin{cases}
    u_2 = \lambda u_1 - v_1 \\
    \mathcal{P}_\alpha u_1 = v_2 + K_0^\beta v_1 + \lambda v_1; \quad \mathcal{P}_\alpha u_1 = \lambda^2 id + \lambda K_0^\beta - D_\alpha.
\end{cases} \tag{2.3}
\]

\( D(A_0^{\alpha, \beta}) = (H_0^1(\Omega \cap H^2(\Omega))^2 \bigoplus (H_0^1(\Omega))^2 \) endowed with the graph norm is an Hilbert space and we define the resolvent set

\[
\mathcal{R}(A_0^{\alpha, \beta}) = \{ \lambda \in \mathbb{C} : (\lambda - A_0^{\alpha, \beta}) \text{ is bijective from } D(A_0^{\alpha, \beta}) \text{ onto } H \}. \nonumber
\]

The operator \( \lambda - A_0^{\alpha, \beta} \) is a Fredholm operator of zero index from \( H \) onto \( H^* \) this implies that \( \lambda - A_0^{\alpha, \beta} \) is too and we have

\[
\mathcal{R}(A_0^{\alpha, \beta}) = \{ \lambda \in \mathbb{C} : (\lambda - A_0^{\alpha, \beta}) \text{ is bijective from } H \text{ onto } H^* \}. \nonumber
\]

[ Indeed, if \( (\lambda - A_0^{\alpha, \beta}) \) is bijective from \( H \) onto \( H^* \), that injective onto \( D(A_0^{\alpha, \beta}) \) and for \( v \in H \subset H^* \) and \( u \in H \) such that \( (\lambda - A_0^{\alpha, \beta}) u = v \) we have \( A_0^{\alpha, \beta} u = \lambda u - v \) then \( u \in D(A_0^{\alpha, \beta}) \). inversely, if \( (\lambda - A_0^{\alpha, \beta}) \) is bijective of \( D(A_0^{\alpha, \beta}) \) onto \( H \), if \( u \in H \) satisfy \( (\lambda - A_0^{\alpha, \beta}) u = 0 \) we... ]
Lemma 2.1. Let \( \exists \}\eta \\in \Omega \) we extended the metric on \( \Omega \) onto \( H^* \). We obtain that

\[
\mathcal{R}(A_\alpha^\beta) = \left\{ \lambda \in \mathbb{C} | P_\alpha^\beta \text{ is injective from } (H^0_\alpha(\Omega))^2 \text{ onto } (H^{-1}(\Omega))^2 \right\}
\]  

(2.4)

and let \( \lambda \in \mathcal{R}(A_\alpha^\beta) \), we have

\[
(\lambda - A_\alpha^\beta)^{-1} = \left( \lambda P_\alpha^\beta(K_\alpha^\beta + \lambda i) \right)^{-1} \left( \lambda P_\alpha^\beta(K_\alpha^\beta + \lambda i) - \lambda id \right) \left( \lambda P_\alpha^\beta(K_\alpha^\beta + \lambda i) \right)^{-1}
\]

(2.5)

where \( P_\alpha^\beta = (P_\alpha^\beta)^{-1} \). In the following, we assume that \( a(x) \) is not identically zero functions.

**Lemma 2.1.** Let \( C > 0 \). There exists \( C_1, C_0 > 0 \) such that for all \( \lambda = -\sigma + i\omega, \omega \in \mathbb{R}, |\sigma| \leq C \) we have

\[
\forall f = (f_1, f_2) \in (H^0_\alpha(\Omega) \cap H^2(\Omega))^2,
\]

\[
\|f\|_{H^2(\Omega)}^2 \leq C_0 e^{C|\omega|} \left[ \|P_\lambda f\|_{L^2(\Omega)}^2 + f a(x)|f_1|^2 \right].
\]

(2.6)

**Proof.** Let \( \Omega' \) be a small neighborhood of \( \overline{\Omega} \). We extended \( \Delta \) onto \( \Omega' \) as the following: we extended the metric on \( \Omega \) onto \( \Omega' \) and we denoted so \( \Delta \) the Laplacian on \( \Omega' \). On neighborhood of \( \partial \Omega \) in \( \Omega' \), we choose the coordinates geodesic systems \( x = (x', x_n), x' \in \partial \Omega = \{x_n = 0\} \), \( |x_n| = dist(x, \partial \Omega), x_n > 0 \) located define the interior of \( \Omega \). We assume \( \Omega' \) is an open set with \( \partial \Omega = \{x = (x', x_n), -\epsilon_0 < x_n < 0\} \) with \( \epsilon_0 \) small, in a neighborhood of \( \partial \Omega \), we have \( \Delta = \partial_{x'}^2 + S(x_n, x', \partial_x) + L(x, \partial_x) \) where \( L \) (resp. \( S \)) is one order (resp. second order). There exists \( \eta \in C^\infty(\overline{\Omega}), \eta > 0 \) such that for \( |x_n| < \epsilon_0 \) we have \( \eta^{-1} \circ \Delta \circ \eta = \partial_{x_n}^2 + R(x_n, x', \partial_x) \) where \( R \) is two order operator. We set \( \hat{\Delta} = \eta^{-1} \circ \Delta \circ \eta \) in \( \Omega, \hat{\Delta} = \partial_{x_n}^2 + R(-x_n, x', \partial_x) \) in \( x_n < 0 \) and we denote \( \hat{a} \) the extension of \( a \) on \( \Omega' \) define by \( \hat{a}(x', x_n) = a(x', -x_n) \) for \( x_n < 0 \).

Let \( Q \) the elliptic operator with Lipschitz coefficients on \( \mathbb{R} \times \Omega' \) of matrix principal symbol

\[
Q = -(\partial_{x_n}^2 + \hat{\Delta}) I_n - iK_\alpha^\beta \partial_x.
\]

(2.7)

Let \( U \neq \emptyset \) is an open set with \( U \) is compact, \( s_0 > 2, \Omega = [-s_0, s_0] \times U \) and \( \varphi \in C_0^\infty(\Omega'), \varphi \equiv 1 \) in a neighborhood of \( \overline{\Omega} \). According to [8], we have the following lemma.

**Lemma 2.2.** There exists \( \theta \in [0, 1[ \) and \( c > 0 \) such that for all \( v \in (H^2([-s_0, s_0] \times \Omega'))^2 \), we have the following estimate

\[
\|\varphi v\|_{H^1((-1,1) \times \Omega')} \leq \epsilon \|v\|_{H^1(V)} \left( \|Q v\|_{L^2(V)}^2 + \|v\|_{H^0(\Omega)}^2 \right)^{1-\theta}
\]

(2.8)

where \( V = [-s_0, s_0] \times \Omega' \).

**Proof.** The proof is a simple adaptation of the proof of the result given in [9]. For \( f = (f_1, f_2) \in (H^0_\alpha(\Omega \cap H^2(\Omega))^2) \), we set \( g(s, x) = e^{is\lambda} g^{-1}(x) \) if \( x \in \Omega \), and \( g = -g(s, x', -x_n) \) if \( x_n < 0 \). We have \( g \in (H^2(\Omega))^2 \) and \( Q(g)(s, x) = \eta^{-1} e^{is\lambda} P_\lambda(f)(x) \) if \( x \in \Omega \) and \( Q(g)(s, x', x_n) = -Q(g)(s, x', -x_n) \) if \( x_n < 0 \). We have

\[
\|f\|_{H^0(\Omega)}^2 \leq C \|\varphi g\|_{H^0((-1,1) \times \Omega')},
\]

\[
\|Q g\|_{L^2(V)}^2 \leq C \epsilon \|v\|_{H^0(\Omega)}^2,
\]

\[
\|g\|_{H^1(V)^2} \leq (1 + |\omega|) \epsilon \|v\|_{H^0(\Omega)}^2,
\]

\[
\|g\|_{H^1(V)^2} \leq (1 + |\omega|) \epsilon \|v\|_{H^0(\Omega)}^2.
\]
then (2.8) implies, with $s_1 > s_0$

\[
\|f\|_{(H^1(\Omega))} \leq \text{Cte} e^{\frac{C}{\omega}} \left[ \|P_\lambda f\|_{(L^2(\Omega))}^2 + \|f\|_{(H^1(\Omega))} \right].
\]  

(2.9)

Choosing an open set $U' \subset \subset \Omega$ such that $a_{1,\nu} > 0$, $U' \subset \subset U$ and $\chi \in C^\infty_0(U')$, equal to $1$ near of $\overline{U}$. We have $(-\text{Id} + D_\alpha)[\chi f] = \chi[(\lambda^2\text{Id} - \text{Id} + K_\alpha^2)f - P_\lambda f] + [D_\alpha, \chi]f$ then

\[
\|f\|_{(H^1(U'))^2} \leq \text{Cte}\|(-\text{Id} + D_\alpha)[\chi f]\|_{(H^{-1}(\Omega))^2} \leq \text{Cte} \left[ \|P_\lambda f\|_{(L^2(\Omega))}^2 + (1 + |\lambda|^2)\|f\|_{(H^1(U'))}^2 \right].
\]  

(2.10)

and we obtain (2.6) by writing (2.9) in (2.10).

**Proof of Theorem 1.2**

Let $\omega \in \mathbb{R}$, $|\omega| \leq 1$, $\sigma \in [0, \frac{1}{\sqrt{\epsilon}} e^{-C_1|\omega|}]$. By (2.6), for all $f = (f_1, f_2) \in (H^1_0(\Omega) \cap H^2(\Omega))^2$, we have

\[
\|f\|_{(H^1(\Omega))}^2 \leq C e^{C_1|\omega|}\|P_\lambda^\alpha f\|_{(L^2(\Omega))}^2,
\]  

(2.11)

or

\[
\|f\|_{(H^1(\Omega))}^2 \leq C e^{C_1|\omega|} \int |a[f_1]|^2.
\]  

(2.12)

In the second case, the identity

\[
(P_\lambda^\alpha \beta f, f) = \lambda^2 \left( \|f_1\|_{(L^2(\Omega))}^2 + \|f_2\|_{(L^2(\Omega))}^2 \right) + \int_\Omega |\nabla f_1|^2 + \alpha \int_\Omega |\nabla f_2|^2 + 2\lambda \int_\Omega |a[f_1]|^2
\]

that implies

\[
2\omega \left[ \int_\Omega |a[f_1]| \right] - 2\omega |\sigma| \|f\|_{(L^2(\Omega))}^2 \leq \|f\|_{(L^2(\Omega))}^2 \|P_\lambda^\alpha \beta f\|_{(L^2(\Omega))}^2,
\]

using (2.12), we get

\[
\|f\|_{(H^1_0(\Omega))}^2 \leq \frac{A e^{C_1|\omega|}}{2|\omega|} \left[ \|P_\lambda^\alpha \beta f\|_{(L^2(\Omega))}^2 \|f\|_{(H^1_0(\Omega))}^2 + 2|\omega| |\|f\|_{(L^2(\Omega))}^2 \right].
\]

As (2.11) implies that the norm of $P_\lambda^{-1}$ from $(L^2(\Omega))^2$ onto $(H^1_0(\Omega))^2$ is bounded by $C e^{C_1|\omega|}$ and we obtain the results (1.12) and (1.13) from (2.5).

Let $\hat{H} = \oplus E_\lambda$, the space of finite linear combination of $H$ in the characteristic subspace $E_\lambda$. We know that $\hat{H}$ is dense in $H$. Let $\hat{H}_0 = \oplus_{\lambda_0 \neq 0} E_{\lambda_0}$, we have $\hat{H}_0 = \hat{H}$ if and only if $\partial M \neq \emptyset$ and $E_0 = \{(1, 0) / u_1 = cte, u_2 = 0\}$ if $\partial M = \emptyset$.

Let $S = \frac{1}{2} A_\omega^\alpha \beta$, $D = \{z \in \mathbb{C} / \text{Im} z \notin [0, 2\|a\|_\infty]\}$. We define on $\hat{H}$ an inner product

\[
\langle u, v \rangle = \int_\Omega \nabla^\alpha u_1 \cdot \nabla^\alpha v_1 + \int u_2 \cdot v_2
\]

induct a norm equivalent to $\|\cdot\|_H$ and we have

\[
\text{Re} \left( \langle (z - A_\omega^\alpha \beta)u, u \rangle \right) = \text{Re} z \int_\Omega (|\nabla u_1|^2 + |\nabla v_1|^2) dx + \int_\Omega ((\text{Re} z + 2a)|u_2|^2 + \text{Re} z \|v_2\|^2) dx
\]

hence result to

\[
\exists C > 0, \forall u \in \hat{H}_0, \forall z \in D, \quad \|z - S\|^{-1}(u) \|u\|_H \leq \frac{C}{\text{dist}(z, D^c)}\|u\|_H.
\]  

(2.13)
Moreover, for $u \in \tilde{H}_0$ we have $z \mapsto (z - S)^{-1}(u)$ is a meromorphic map with the asymptotic behavior $O(\frac{1}{|t|})$ as $|z| \to +\infty$ and by the Theorem 1.2 (1.13), if $x \in \tilde{H}_0$ we have $(\xi - S)^{-1}(x)$ is holomorphic at $\xi \in \{z \in \mathbb{C}; \text{Im } z < 2\epsilon_0 e^{-cz}\text{Re } z\}$ with $\epsilon_0$, $c_2^{-1} > 0$ small enough and satisfies on

$$
\Gamma = [0, -d + 2i\epsilon_0 e^{-cz}] \cup \left\{ \xi \in \mathbb{C} / \xi = \eta + 2i\epsilon_0 e^{-c|\eta|}, |\eta| \geq d \right\} \cup [0, +d + 2i\epsilon_0 e^{-cz}]
$$

Then, there exists $d > 0$ such that for $x \in \tilde{H}_0$ the operator $(z - S)^{-1}(x)$ is analytic in the region below the outline $\Gamma$. We consider $\psi \in C^\infty(\mathbb{R}_+)$, equal to 0 for $t < \frac{1}{\delta}$ and to 1 for $t > \frac{2}{\delta}$ and we set $u = \frac{1}{(1 - S)^t} (\psi v)$ solution of

$$
(\partial_t - S) u = \psi'(t) \frac{1}{(1 - S)^t} v(t).
$$

Let

$$
u(t) = \int_0^t e^{(t-s)\xi} \psi'(s) \frac{1}{(1 - S)^t} v(s) ds.
$$

Let $c_0$ and $c_1$ are a later choose. We have

$$
u(t) = \int_0^t \int_{|\lambda| < c_1 \sqrt{\text{Im } t}} \int_{|\lambda| > c_1 \sqrt{\text{Im } t}} \frac{c_0}{\pi} \psi'(s) e^{(t-s)\xi} \frac{1}{(1 - S)^t} v(s) d\lambda d\xi ds - \frac{1}{(1 - S)}
$$

$$= I_1 + I_2.
$$

We remark that the decomposition is similar to that of Lebeau [8] and Burq [2].

**Estimation of $I_1$**

The idea is to estimate $I_1$, we deform the outline of integration in $\xi$ on the outline $\Gamma$. This requires to verify that the operator $(\xi - S)^{-1} e^{it\xi}$ is holomorphic with respect to $\xi$ in the field is below the contour and it verifies an estimate of type

$$
\left\| (\xi - S)^{-1} \right\|_{\mathcal{L}(H)} \leq C_1 e^{C_2 |\text{Re } \xi|}.
$$

What can be deduced from (2.13). We know that for $\text{Im } \xi < 0$, the two families of operators

$$e^{it\xi} \left((\xi - S)^{-1} - i \int_0^s e^{i(\xi - \xi') \sigma} \right) \text{ and } (\xi - S)^{-1} e^{its}\xi$$

coincide for $s = 0$ and satisfy the same differential equation

$$\partial_s t = i\xi t - e^{its}.$$
\(\xi\) on the contour \(\Gamma\). By the fact \(e^{is\xi}\) is bounded for all \(s \geq 0\) and since the operator \((\xi - S)\) \(e^{is\xi}\) is uniformly bounded in \(H\) with respect to \(\xi\) and \(s \in [0, 1]\), for \(t \geq 0\) we have

\[
\left\| \int_{\xi \in \Gamma} \int_{|\xi| < c_1 \sqrt{\ln t}} \right\| \leq C\|u_0\| \int_{z=0} e^{-(t-1)\delta t - a|\xi|} dz \leq \frac{C\|u_1\|}{t-1}. \tag{2.18}
\]

By (2.17), we have for \(t > 1\),

\[
\left\| \int_{|\xi| < c_1 \sqrt{\ln t}} \int_{\xi \in \Gamma} \int_{|\xi| < c_1 \sqrt{\ln t}} \right\| \leq C\sqrt{c_0} \int_{-\infty}^{+\infty} \int_{|\xi| < c_1 \sqrt{\ln t}} e^{-(t-1)\delta t - a|\xi| + A|\eta| - c_0(\xi - \frac{\eta}{\sqrt{\ln t}})} d\eta d\lambda. \tag{2.19}
\]

Let \(c_2\) such that \(c_2 \alpha < 1\) and \(\varphi = -(t-1)c_0 e^{-a|\eta|} + A|\eta| - c_0(\xi - \frac{\eta}{\sqrt{\ln t}})^2\). Then, we have

\[
|\eta| \leq c_2 \ln t \Rightarrow \varphi \leq c_2 A \ln t - (t-1)c_0 e^{-c_2 \alpha}. \tag{2.20}
\]

We choose \(c_1 \in [0, c_2]\). Then, there exists \(\delta > 0\) such that if \(|\lambda| < c_1 \sqrt{\ln t}\) and if \(|\eta| > c_2 \ln t\) then

\[
(\lambda - \frac{\eta}{\sqrt{\ln t}})^2 \geq \delta(\lambda^2 + \frac{\eta^2}{\ln t}),
\]

let

\[
\varphi \leq A|\eta| - c_0 \delta(\lambda^2 + \frac{\eta^2}{\ln t}). \tag{2.21}
\]

We choose \(c_0 > \frac{A}{c_2} + 1\). For \(\epsilon > 0\) we have

\[
\int_{|\eta| > c_2 \ln t} e^{A|\eta| - c_0\delta(\frac{\eta^2}{\ln t})} = \mathcal{O}\left(e^{-\epsilon \ln t}\right). \tag{2.22}
\]

By (2.18), (2.19), (2.20) and (2.23),

\[
\|I_1\| \leq C t^{-\epsilon} \|u_1\|. \tag{2.24}
\]

**Estimation of \(I_2\)**

Let

\[
J(u) = \int_{\xi \in \Gamma} \int_{|\xi| \geq c_1 \sqrt{\ln t}} v(s) \sqrt{n_i} e^{-c_0(\lambda - \frac{n_i}{\sqrt{\ln t}})^2} ds d\xi d\lambda.
\]

For \(t \geq 1\), we have \(J(t) = I_2(t)\) and for all \(u \in \mathbb{R}\),

\[
(\partial_{t} - iS)J(u) = \int_{0}^{1} \int_{|\lambda| \geq \sqrt{\ln t}} v(s) \sqrt{n_i} e^{-c_0(\lambda - \frac{n_i}{\sqrt{\ln t}})^2} ds d\lambda = K(u),
\]

that implies

\[
J(t) = e^{itS}J(0) + \int_{0}^{t} e^{i(t-s)S}K(s)ds. \tag{2.26}
\]
Now we are going that \( J(t) \) is bounded in norm in \( H \), we use that \( e^{is} \) is a contraction of \( H \) for \( s \geq 0 \) and separately \( K(u) \) for \( u \geq 0 \), \( J(0) \) and \( \int_0^1 \|K(u)\|du \) (see [2]).

For \( u \in [1, t] \), we show that the outline in \( \xi \) given in (2.26), is deformed in the outline given by \( \text{Im} \xi = \sqrt{\ln t} \), that give for \( k > 1 \) and \( \text{supp} \psi \subset [\frac{1}{2}, \frac{3}{2}] \),

\[
\|K(u)\| \leq \int_{\xi > \Gamma} e^{-(\xi - \frac{1}{4})\sqrt{\ln t}} \frac{1}{(1 + |\xi|)^k} d\xi \|u_0\| \leq C_k e^{-\sqrt{\ln t}/3} \|u_0\|.
\]

(2.28)

Then we bound \( J(0) \). We treat such a contribution (2.25) of the region. For that is deformed according to [8], the integral in \( \xi \) on the contour \( \Gamma = \Gamma^+ \cup \Gamma^- \), where

\[
\begin{align*}
\Gamma^+ &= \left\{ z = 1 + \eta - i\frac{1}{2} \ln t; \eta > 0 \right\} \\
\Gamma^- &= \left\{ z = 1 + \eta - \frac{1}{2} i; \eta \leq 0 \right\} \cup \left[ 1 - \frac{1}{2} i, 1 - i \sqrt{\ln t} \right].
\end{align*}
\]

For \( \xi \in \Gamma^- \), by (2.13), we have for all \( s \in [0, 1] \) and for all \( \lambda \in [c_1 \sqrt{\ln t}, +\infty] \) there exist \( \delta > 0 \)

\[
\|e^{-is\xi} \frac{v(s)}{(1 - i\xi)^k} \frac{1}{(\xi - B)} \sqrt{\frac{e^{c_0}}{2\pi}} e^{-c_0(\lambda - \frac{1}{4}\sqrt{\ln t})^2} \| \leq \frac{C}{(1 + |\xi|)^k} e^{-\delta(\lambda^2 + \frac{e^C}{\pi t^2})} \|u_0\|.
\]

The contribution de \( \Gamma^- \) to \( J(0) \) is bounded in norm by

\[
C\sqrt{\ln t} \int_{\lambda \geq c_1 \sqrt{\ln t}} e^{-\delta \lambda^2} \|u_0\| = \mathcal{O}(e^{-c\ln t}) \|u_0\|.
\]

(2.29)

For \( \xi \in \Gamma^+ \) and \( s \in [\frac{1}{2}, \frac{3}{2}] \) we have

\[
\|e^{-is\xi} \frac{V(s)}{(1 - i\xi)^k} \frac{1}{(\xi - B)} \sqrt{\frac{e^{c_0}}{2\pi}} e^{-c_0(\lambda - \frac{1}{4}\sqrt{\ln t})^2} \| \leq e^{-\sqrt{\ln t}/3} \frac{C}{(1 + |\eta|)^k} e^{-\delta(\lambda^2 + \frac{e^C}{\pi t^2})} \|u_0\|,
\]

So, since the contribution of \( \Gamma^+ \) to \( J(0) \) is bounded in norm by

\[
C e^{-\sqrt{\ln t}/3} \|u_0\|.
\]

(2.30)

The contribution to \( J(0) \) of the region \( \lambda < -c_1 \sqrt{\ln t} \) is bounded by the same way.

Finally, it remains to bounding

\[
\int_0^1 \|K(u)\|du \leq \left( \int_0^1 \|K(u)\|^2du \right)^{\frac{1}{2}}.
\]

(2.31)

By the Plancherel identity,

\[
\int_{-\infty}^{\infty} \|K(u)\|^2du = C \int_{-\infty}^{\infty} \left\| \frac{i}{(1 - i\xi)^k} \psi^{\prime} (\xi) \int_{|\lambda| \geq c_1 \sqrt{\ln t}} e^{c_0(\lambda - \xi) \sqrt{\ln t})^2} d\lambda \right\|^2 d\xi
\]

(2.32)

for \( \xi > \frac{1}{2} c_1 \ln t \).
\[ \|H(\xi)\| = \left\| \int_{|\lambda| > C_1 \eta} \frac{1}{(1 - i \xi)^k} e^{-\alpha(\lambda - \xi/\sqrt{\ln t})^2} d\lambda \right\| \leq \frac{C}{(\ln t)^k} \|\hat{v}\| \] \hspace{1cm} (2.33)

and for \( \xi \leq \frac{1}{2} C_1 \ln t \),

\[ \|H(\xi)\| \leq \int_{|\lambda| > c_1 \sqrt{\ln t}} e^{-\delta(\lambda^2 + \xi^2/\ln t)} d\lambda \|\hat{v}\| \leq C e^{-\epsilon \ln t} \|\hat{v}\| . \] \hspace{1cm} (2.34)

Then, by (2.31), (2.32), (2.33), (2.34) and

\[ \int_{\infty}^{\infty} \|\hat{v}\| = \int \|v\|^2 ds \leq C \int_0^1 |\psi(s)|^2 ds \leq C \left( \frac{1}{(\ln t)^k} + e^{-\epsilon \ln t} \right) \|U_0\|. \] \hspace{1cm} (2.35)

By (2.27), (2.28), (2.29), (2.30) and (2.35) we obtain

\[ \|I_2\| \leq \frac{C}{(\ln t)^k} \|u_1\| , \]

hence the estimate of \( I_2 \).

3 Proof of Theorem 1.3

First, we prove \( \rho \leq 2 \min(-D(0), C(\infty)) \). Let \( \lambda_j \in \text{Sp}(A_0^{\alpha, \beta}) \setminus \{0\} \) there exists \( u = (u_0, u_1) = ((u_0^0, u_0^1), (u_1^0, u_1^1)) \in E_{\lambda_1} \) such that \( A_0^{\alpha, \beta} u = \lambda_j u \) and \( u(t, x) = e^{i\lambda_j t} u_0 \) satisfy (1.1)-(1.6).

As \( E(u, t) = e^{i\Re \lambda_j} E(u, 0) \) and \( E(u, 0) = \frac{1}{2} \int_0^1 |\lambda|^2 |u_0|^2 + |\nabla x u_0|^2 + \alpha |\nabla x u_0|^2 \neq 0 \), we have \( \rho \leq -2\Re \lambda_j \) then \( \rho \leq -2\Re(D(0)) \). We assume that \( \rho = 2C(\infty) + 4\eta \) with \( \eta > 0 \) there exists \( B > 0 \) such that for all \( u \in H \) and for all \( t \geq 0 \) we have the following estimate

\[ E(u, t) \leq B e^{-(\rho - \eta) t} E(u, 0). \] \hspace{1cm} (3.1)

Let \( t \) fixed such that \( B e^{-(\rho - \eta) t} < e^{-(\rho - 2\eta) t} \), we have \( C(t) \leq C(\infty) = \frac{\rho}{2} - 2\eta \), then there exists \( i \in \{1, 2\} \) such that \( \frac{1}{2} \int_0^1 a(x(s, \rho)), ds \leq C(\infty) = \frac{\rho}{2} - 2\eta \), and there exists \( \rho_0 \in \partial T \) with \( C(t) < \frac{\rho}{2} - \eta \) has left a little disturbing \( \rho_0 \), we can assume that the outcome of generalized geodesic \( \rho_0 \) did as points of intersection with transverse \( \partial T \) on \([-2t, +2t] \). By constructing geometric standard optical near \( \gamma \), we can construct a solution \( u_0 \) of (1.1) - (1.6) such that \( E(u, 0) = 1 \) and \( E(u, t) > e^{-(\rho - 2\eta) t} \) which contradicts (3.1), so we have \( \rho \leq 2C(\infty) \). To check \( \rho \geq 2 \min \{-D(0), C(\infty)\} \), we prove the following lemma:

Lemma 3.1. For all \( T > 0 \) and for \( \epsilon > 0 \) there exists \( C(\epsilon, T) \) such that for all solution of the evolution equation (1.7) we have

\[ E(u, T) \leq (1 + \epsilon)e^{-2TC(T)} E(u, 0) + C(\epsilon, T)(\|u_0, u_1\|) \|L^2(\Omega) \times H^{-1}(\Omega)\|^2 \] \hspace{1cm} (3.2)
\textbf{Proof}: If (3.2) is false then there exists $T > 0$ and $\varepsilon > 0$ such that for all $k \geq 1$ there exists $u_k$ satisfy
\[
E(u_k, T) \geq (1 + \varepsilon)e^{-2T(C(T))}e(u_k, 0) + k\|(u_k^k, u_1^k)\|_{(L^2(\Omega) \times H^{-1}(\Omega))^2},
\]
then $u_k$ is bounded in $(H^1(I \times \Omega))^2$, $I = [-2T, 2T]$ converges weakly to zero because $\|(u_k^k, u_1^k)\|_{(L^2(\Omega) \times H^{-1}(\Omega))^2} \leq \frac{1}{k} E(u_k, T) \leq \frac{1}{k} E(u_k, 0) = \frac{1}{k}$.

Let $\mu$ the measure positive onto $SZ$ (see section 4 (4.6)) associated to extract sequence of $u_k$. Let $\eta \in [0, T]$. As the energy function is decreasing, for all $\varphi \in C_0^\infty([0, \eta])$ we have by (3.2)
\[
\int_{T-\eta}^T \varphi(T-t)E(u_k, t)dt \geq (1 + \varepsilon)e^{-2T(C(T))}\int_0^\eta \varphi(t)E(u_k, t)dt
\]
then $\mu((SZ) \cap (t \in [T-\eta), T]) \geq (1 + \varepsilon)e^{-2T(C(T))}\mu((SZ) \cap (t \in [0, \eta])$. (3.5)

Gold by the propagation Theorem we have
\[
\mu((SZ) \cap (t \in [T-\eta), T]) \leq e^{-2(T-\eta)C(T-\eta)}\mu((SZ) \cap t \in [0, \eta]).
\]

Since $\mu((SZ) \cap (t \in [0, \eta])) > 0$ (because if $u_k \to 0$ in $(H^1([0, \eta] \times \Omega))^2$ that implies $u_k \to 0$ in $(H^1(J \times \Omega))^2$ for all $J$ this give a contradiction with the fact $E(u_k, 0) = 1$). Since $C(t)$ defined in (1.16) as an infimum over a compact of a continuous function is continuous at $t > 0$, (3.6) contradicts (3.5) to $\eta$ small, hence the Lemma.

Let $A_{\alpha, \beta, *}^\delta$ the adjoint of $A_{\alpha, \beta}^\delta$, we denote by $E_{\lambda_j}^\delta$ the characteristic subspace of $A_{\alpha, \beta, *}^\delta$ associated of the eigenvalue $\lambda_j$. Let $H = (H^0(\Omega))^2 \oplus (L^2(\Omega))^2$ and for $N \geq 1$

\[
H_N = \left\{ x \in H / (x, y)_H = 0, \forall y \in \oplus_{|\lambda| \leq N} E_{\lambda_j}^\delta \right\}.
\]

Then $H_N$ is invariant under $e^{TA_{\alpha, \beta}^\delta}$ (indeed let $x \in H_N$, $(y_k)$ a basis of the vectorial space $\oplus_{|\lambda| \leq N} E_{\lambda_j}^\delta$ we have $\frac{d}{dt} \left( e^{TA_{\alpha, \beta}^\delta}x | A_{\alpha, \beta, *}^\delta y_k \right) = \sum c_{k,l} \left( e^{TA_{\alpha, \beta}^\delta}x | y_l \right)$ then $e^{TA_{\alpha, \beta}^\delta}x | y_l \equiv 0$). Let $H^* = (L^2(\Omega))^2 \oplus (H^{-1}(\Omega))^2$ and $\Phi_N$ the norm of injection from $H_N$ onto $H^*$. We have $\lim_{N \to +\infty} \Phi_N = 0$, indeed, we assume that there exists $u_N \in H_N$, $\|u_N\|_H = 1$ and $\|u_N\|_{H^*} \leq \lim_{N \to +\infty} \Phi_N = \rho > 0$. We can assume that $u_N$ converges weakly to $u$ in $H$, and strongly in $H^*$. We have $\|u\|_{H^*} \geq \rho$ and $(u, y)_H = 0$, $\forall y \in E_{\lambda_j}^\delta$, $\forall j$. This is impossible by the fact that $\oplus_{\lambda_j \leq N} E_{\lambda_j}^\delta$ is a perturbation bounded of self-adjoint $A_0$.

We can assume $2 \min \{ -D(0), C(\infty) \} > 0$, let $\eta > 0$ small and $\beta$ define by $\tilde{\beta} + \eta = 2 \min \{ -D(0), C(\infty) \}$. Choosing $T > 0$ such that $4|C(\infty) - C(T)| < \eta$, $2 \log 3 < \eta T$ and $N$ such that $C(1, T)\Phi_N \leq e^{-2T(C(T))}$. By Lemma 3.1, identifying $u \in H$ to the solution of (1.1)-(1.6) with initial data $u$ \(\forall u \in H_N, \ E(u, T) \leq 3e^{-2T(C(T))}E(u, 0)\)

then $H_N$ is stable by the evolution \(\forall u \in H_N, \ \forall k \ E(u, kT) \leq 3e^{-kT[2C(T)-\frac{\beta}{2}]}E(u, 0) \leq e^{-kT\tilde{\beta}}E(u, 0)\)
as the energy decreases

\[ \forall u \in H, \, \forall t \geq 0, \, E(u, t) \leq B e^{-\tilde{\beta} t} E(u, 0), \quad B = e^{\tilde{\beta} T}. \]  

(3.10)

Let \( \tilde{\gamma} \) the contour encircling \( \{ \lambda_j \mid |\lambda_j| \leq N \} \) in the direct sense and \( \Pi = \frac{1}{2\pi i} \oint_{\tilde{\gamma}} \frac{d\lambda}{\lambda - \lambda_j} \) the spectral projector on \( \oplus\{ \lambda_j \leq N \} E_{\lambda_j} = W_N \); then \( \Pi^* \) is the spectral projector of \( A_n^{\alpha, \beta, \ast} \) on \( \oplus\{ \lambda_j \leq N \} E_{\lambda_j}^* \).

Then for all \( u \in H \), we have

\[ u = v + w, \quad v = \Pi u \in W_N, \quad w = (id - \Pi) u \in H. \]  

(3.11)

As \( W_N \) is a finite dimensional and \( \tilde{\beta} < -2D(0) \), we have

\[ \exists C, \, \forall u \in W_N, \, \forall t \geq 0, \, E(u, T) \leq C e^{-\tilde{\beta} t} E(u, 0). \]  

(3.12)

The decomposition (3.11) is continuous, there exists \( C_0 \) such that \( E(v, 0) + E(w, 0) \leq C_0 E(u, 0) \) and by (3.10), (3.11) and (3.12) implies that \( \tilde{\beta} \geq \beta \) this achieve the Proof of 1. and 2. of Theorem 1.3 result the fact that \( E_{\lambda_j} \subset H_N \) if \( |\lambda_j| > N \) (since the projector \( \Pi \) is equal to zero on \( E_{\lambda_j} \) and by (3.10), if \( C(\infty) > 0 \) and \( \beta < 2C(\infty) \))

for \( N \) large enough

\[ |\lambda_j| > N \Rightarrow 2Re\lambda_j \leq -\tilde{\beta}. \]  

(3.13)

Then \( D(\infty) \leq -C(\infty), \) hence 2. (since \( D(\infty) \leq 0 \) treats the case \( C(\infty) = 0 \)).

4 Geometric and construction of measure

Near \( \partial M = (M = \Omega \times \mathbb{R}_+) \), we choose the geodesic coordinate system : \( (x', x_n) \in \partial M \times [0, r_0] \rightarrow x : x_n = \text{dist}(x, \partial M) = \text{dist}(x, x') \) where \( r_0 > 0 \) small enough. In the system, the principal symbol of \( -\Delta \) is \( \xi_n^2 + R(x_n, x', \xi') \) and \( R_0(x', \xi') = R|_{x_n=0} \) is the metric form on \( T^*\partial M \). We denote \( \mathcal{G} \) the operator space \( Q \) of the form \( Q = Q_1 + Q_0 \) where \( Q_i \) is a classical pseudo-differential operator onto \( \mathbb{R}_t \times \Omega \) with compact support in \( \mathbb{R}_t \times \text{int}\Omega \) and \( Q_0 \) is a tangential pseudo differential operator with compact support near \( \mathbb{R}_t \times \partial \Omega \) i.e \( Q_0(t, x', x_n) = Q_0(x_n)(\psi) (t, x_n) \) where \( Q_0(x_n) \) is a \( C^\infty \) p.d.o onto \( \mathbb{R}_t \times \partial \Omega \) and \( Q_0 = \psi Q_0 \psi \) with \( \psi(t, x_n) \in C_0^\infty (\mathbb{R} \times (-r_0, r_0)) \). We denote \( \mathcal{G}^{(s)} \) the element of degree \( s \) in \( \mathcal{G} \) and \( \mathcal{G}_{\text{sym}} \) the subset of element in \( \mathcal{G} \) with self-adjoint principal symbol.

Let \( X = \mathbb{R}_t \times \Omega, bTX \) of the tangent bundle of rung \( \text{dim}X \), the sections of which are the tangent vector fields to \( \mathbb{R} \times \partial \Omega, bT^*X \) the dual bundle (of the cotangent compressed bundle of Melrose) and \( j : T^*X \rightarrow bT^*X \) the canonical maps. Near the \( \partial X, bTX \) is generate by the fields \( \partial_t, \partial_{x'}, x_n \partial_{x_n} \) and

\[ j(t, x', x_n; \tau, \xi', \xi_n) = (t, x', x_n; \tau, \xi', v = x_n \xi_n). \]

We denote

\[ P_n^{\alpha, \beta} = \partial_t^2 - D_\alpha + R_n^\beta \]

with principal symbol

\[ P_0 = \begin{pmatrix} -\tau^2 + |\xi|^2 & 0 \\ 0 & -\tau^2 + \alpha |\xi|^2 \end{pmatrix} \]

we notice that the determinate of the principal symbol is given by \([11]:: \]

\[ p(t, x; \tau, \xi) = (|\xi|^2 - 1) (\alpha |\xi|^2 - 1). \]  

(4.1)
This leads to two bicharacteristic families in the characteristic set of $\mathcal{P}_a^{\alpha,\beta}$, $\text{Char}\mathcal{P}_a^{\alpha,\beta}$, namely those of the symbols

\[ p_1(t, x; \tau, \xi) = |\xi|^2 - \tau^2 \quad \text{and} \quad p_\alpha(t, x; \tau, \xi) = \alpha|\xi|^2 - \tau^2, \]

with respect to the projection $\Pi : T^*(\mathbb{R} \times \Omega) \to T^*(\partial M)$. We will illustrate what happens at the boundary point $(t, x) \in \partial M$. Let $(\tau, \eta) \neq (0, 0)$ be a tangential direction to $\partial M$ at $(t, x)$; i.e., $\eta \cdot \nu(x) = 0$, $\nu(x)$ being the exterior normal to $\Omega$ at $x$. With the assumption $\alpha \neq 1$, we can consider $(\tau, \eta)$ as an element of $T^*_{(t, x)}(\partial M)$, and to look for its inverse image is both characteristic sets means to look for $\lambda \in \mathbb{R}$ such that

\[ p_{1, \alpha}(t, x; \tau, \eta + \lambda \nu(x)) = 0. \quad (4.2) \]

Because of

\[ p_{1, \alpha}(t, x; \tau, \eta + \lambda \nu(x)) = c_{1, \alpha}^2 (|\eta|^2 + \lambda^2) - \tau^2, \]

this requires

\[ \lambda = \pm \sqrt{\tau^2 - |\eta|^2} \quad \text{or} \quad \lambda = \pm \sqrt{\frac{\tau^2}{\alpha} - \eta^2}. \quad (4.3) \]

Hence, for the existence of such real $\lambda$, one of the two relations

\[ r_1 = \tau^2 - \eta^2 \geq 0 \quad \text{or} \quad r_\alpha = \tau^2 - \alpha \eta^2 \geq 0 \]

must be fulfilled. From the geometrical point of view there are some possibilities for a tangential direction $\xi = (\tau, \eta) \neq (0, 0)$, with different number of inverse images with respect to the projection. We can introduce the transversal manifold:

\[
\text{Char}\mathcal{I} = \text{Char}\mathcal{I}_\Omega \cup \text{Char}\mathcal{I}_{\partial \Omega},
\]

\[
\text{Char}\mathcal{I}_\Omega = \{(x, t; \xi, \tau) : \tau^2 - c_{1, \alpha}^2 |\xi|^2 = 0, \ t > 0\},
\]

\[
\text{Char}\mathcal{I}_{\partial \Omega} = \{(y, t; \xi, \tau) : y \in \partial \Omega, \ y \in \partial \Omega, \ t > 0, \ r_\alpha \geq 0\}
\]

and the longitudinal manifold of the wave coupled system is

\[
\text{Char}\mathcal{L} = \text{Char}\mathcal{L}_\Omega \cup \text{Char}\mathcal{L}_{\partial \Omega},
\]

\[
\text{Char}\mathcal{L}_\Omega = \{(x, t; \xi, \tau) : \tau^2 - c_{1, \alpha}^2 |\xi|^2 = 0, \ t > 0\},
\]

\[
\text{Char}\mathcal{L}_{\partial \Omega} = \{(y, t; \xi, \tau) : y \in \partial \Omega, \ y \in \partial \Omega, \ t > 0, \ r_1 \geq 0\},
\]

the characteristic manifold of the system is

\[
\text{Char}\mathcal{P} = \text{Char}\mathcal{P}_\Omega \cup \text{Char}\mathcal{P}_{\partial \Omega}
\]

and the assumption on the coupled wave $(\alpha \neq 1)$ one obtains

\[
\text{Char}\mathcal{P}_\Omega = \text{Char}\mathcal{I}_\Omega \cup \text{Char}\mathcal{L}_\Omega
\]
where  

\[ G \]  

such that  

be the outgoing (resp. incoming) bicharacteristic of  

\( P \). 

We shall consider separately the case where  

\( \rho \in [8] \) for more details. The generalized bicharacteristic flow lives in  

\( \text{Char} \)  

Definition 4.1. Let  

\( \eta \in T^*\partial \Omega \). We say that  

(i) \( \eta \) is a elliptic (or  \( \eta \in \mathcal{E} \)) if and only if  

\( \eta \not\in (\text{Char} P)_{\partial \Omega} \).  

(ii) \( \eta \) is a hyperbolic for the longitudinal wave (or  \( \eta \in \mathcal{H}_L \)) if and only if  

\( r_1 > 0 \).  

(iii) \( \eta \) is a glancing for the longitudinal wave (or  \( \eta \in \mathcal{G}_L \)) if and only if  

\( r_1 = 0 \).  

(iv) \( \eta \) is a hyperbolic for the transversal wave (or  \( \eta \in \mathcal{H}_T \)) if and only if  

\( r_\alpha > 0 \).  

(v) \( \eta \) is a elliptic (or  \( \eta \in \mathcal{E} \)) if and only if  

\( r_\alpha = 0 \).  

We are going now to make a description of a generalized bicharacteristic path and refer to  

[8]  

for more details. The generalized bicharacteristic flow lives in  

\( \text{Char} P \subset T^*\overline{\Omega} \) and for  

\( \rho \in \text{Char} P \), we denote by  

\( G(s, \rho) \) the generalized bicharacteristic path starting from  

\( \rho \). Since  

\( \text{Char} P \) is the disjoint union of  

\( \text{Char} P_\Omega \),  

\( \mathcal{H}_T \) and  

\( \mathcal{G}_T \) if  

\( \alpha > 1 \) or  

\( \text{Char} P_\Omega \),  

\( \mathcal{H}_L \) and  

\( \mathcal{G}_L \) if  

\( \alpha < 1 \).  

We shall consider separately the case where  

\( \rho \) belongs to each one of these sets. Moreover all the description below holds for  \(|s|\) small.

**Case 1.** \( \rho \in \text{Char} P_\Omega \)

Here  

\( \rho = (x, t; \xi, \tau) \) where  

\( x \in \Omega, \ t \in (0, T), \ p(x, t; \xi, \tau) = 0 \). Then for  \(|s|\) small, we have  

\[ G(s, \rho) = (x(s), t(s), \tau, \xi) \subset T^*(\mathbb{R} \times \Omega) \]  

where  

\( (x(s), \xi) \) is the characteristic starting from the point  

\( (x, \xi) \) of

-  

  \( P_i \) if  

  \( \rho \in \text{Char} \mathcal{L}_\Omega \),

-  

  \( P_o \) if  

  \( \rho \in \text{Char} \mathcal{T}_\Omega \).

**Case 2.**  

\( \rho \in (\text{Char} P)_{\partial \Omega} \) (i.e  

\( 0 \leq r_\alpha \)) Here  

\( \rho = (x(s), t(s), \eta(s), \tau(s)) \) where  

\( x \in \partial \Omega, \ t \in (0, T) \) and the equation  

\( p(x, t, \eta + \xi_n, \tau) = 0 \) has roots  

\( \xi_n = \lambda \nu(x) \) described in (4.3).

For  

\( s > 0 \) (resp.  

\( s < 0 \)) let  

\( G^+(s, \rho) = (x^+(s), t(s), \xi^+, \tau(s)) \) (resp.  

\( G^+(s, \rho) = (x^-(s), t(s), \xi^-, \tau(s)) \) be the outgoing (resp. incoming) bicharacteristic of  

\( P \). The generalized bicharacteristic path is such that  

\( G(0, \rho) = \rho \) and

\[ G(s, \rho) = \begin{cases} 
G^+(s, \rho) & 0 < s < \epsilon \\
G^-(s, \rho) & -\epsilon < s < 0 
\end{cases} \]

Four possibilities may occur  

(i)  

\[ \begin{cases} 
x^+(s) &= x + 2c_\alpha^2 s \xi^+, \quad 0 < s < \epsilon, \\
x^-(s) &= x + 2c_\alpha^2 s \xi^-, \quad -\epsilon < s < 0, 
\end{cases} \]

where  

\( \xi^+ = \eta - \frac{\sqrt{\tau}}{c_T} \nu(x) \) and  

\( \xi^- = \eta + \frac{\sqrt{\tau}}{c_T} \nu(x) \).

In particular, if  

\( 0 < r_\alpha \), one has  

\( x(s) \in \Omega \) for small  

\(|s| \neq 0 \).
(ii) If $0 \leq r_1$ (i.e. $\eta \in G_L \cup \mathcal{H}_L \subset \mathcal{H}_T$):

\begin{itemize}
  \item[i] -
  \begin{align*}
  x^+(s) &= x + 2c_1^2 s \xi^+ , \quad 0 < s < \epsilon, \\
  x^-(s) &= x + 2c_1^2 s \xi^- , \quad -\epsilon < s < 0,
  \end{align*}

  where $\xi^+ = \eta - \frac{\sqrt{\tau}}{c_1} \nu(x)$ and $\xi^- = \eta + \frac{\sqrt{\tau}}{c_1} \nu(x)$.

\item[ii] -
  \begin{align*}
  x^+(s) &= x + 2c_1^2 s \xi^+ , \quad 0 < s < \epsilon, \\
  x^-(s) &= x + 2c_1^2 s \xi^- , \quad -\epsilon < s < 0,
  \end{align*}

  where $\xi^+ = \eta - \frac{\sqrt{\tau}}{c_1} \nu(x)$ and $\xi^- = \eta + \frac{\sqrt{\tau}}{c_1} \nu(x)$.

\item[iii] -
  \begin{align*}
  x^+(s) &= x + 2c_1^2 s \xi^+ , \quad 0 < s < \epsilon, \\
  x^-(s) &= x + 2c_1^2 s \xi^- , \quad -\epsilon < s < 0,
  \end{align*}

  where $\xi^+ = \eta - \frac{\sqrt{\tau}}{c_1} \nu(x), \xi^- = \eta + \frac{\sqrt{\tau}}{c_1} \nu(x)$.
\end{itemize}

and the manifold characteristic $\text{Char}(P_{\alpha,\beta}^\eta) = \{(t, x', x_n; \tau, \xi', \xi_n); \det p = 0\}$. We set

\[ Z = \text{j}(\text{Char}(P_{\alpha,\beta}^\eta)), \quad \hat{Z} = Z \cup j(T^*X_{x_n=0}) \quad (4.4) \]

We have $Z_{x_n=0} = \{(t, x', 0; \tau, \xi', 0); |\xi'| \leq |\tau| \text{ or } \sqrt{\alpha} |\xi'| \leq |\tau| \}$ and $\hat{Z}_{x_n=0} = \{(t, x', 0; \tau, \xi', v = 0)\} = T^*(\mathbb{R} \times \partial M) = Z_{x_n=0} \cup \mathcal{E}$ where $\mathcal{E}$ is the boundary of elliptic region.

As $x_n \in [0, r_0]$ we have $p = \xi_n^2 I_n + R - \tau^2 \text{id}$, $R$ is nondegenerate positive matrix we have

\[ (t, x', x_n; \tau, \xi', v) \in \hat{Z}, \quad x_n \in [0, r_0] \Rightarrow \begin{cases} |v| \leq x_n |\tau| \quad \text{or} \\ \sqrt{\alpha |v|} \leq x_n |\tau| \end{cases} \quad (4.5) \]

We obtain that $Z$ and $\hat{Z}$ are closed conic sets in $T^*X$. We denote $S\hat{Z}$ and $SZ$ the spherical quotients spaces

\[ S\hat{Z} = (\hat{Z} \setminus X)/\mathbb{R}^*_+, \quad SZ = (Z \setminus X)/\mathbb{R}^*_+ \quad (4.6) \]

which are a locally compact metric spaces. For $Q \in C^0$ with principal symbol $q = \sigma(Q)$ and we define the function

\[ \begin{cases} \kappa(q) \in C^0 \left( S\hat{Z}, \text{end}(\mathbb{C}) \right) \\ \rho \in \hat{Z} \setminus X \quad \kappa(q)(\rho) = q(j^{-1}(\rho)) \end{cases} \quad (4.7) \]

( which is well defined because $q$ is homogeneous and has $\kappa(q)(x', x_n, \xi', \xi_n) = q(x', x_n, \xi', \frac{\xi_n}{x_n})$ for $x \neq 0$ and $q$ is independent of $\xi$ for $x$ sufficiently small.) By (4.7) the set

\[ \{ \kappa(q), \quad q = \sigma(Q), \quad Q \in \mathcal{G}^0 \} \]

is locally dense in $C^0 \left( S\hat{Z}, \text{end}(\mathbb{C}^2) \right)$ where $C^0 \left( S\hat{Z}, \text{end}(\mathbb{C}^2) \right)$ is provided with the topology of uniform convergence on compact. For $G \in \mathcal{G}^0$, and $I$ is an open bounded real interval and $u(x, t) \in (H^1(I \times \Omega))^2$ solution of $P_{\alpha,\beta}^\eta u = 0$ near the boundary, we have $u \in C^k \left( x_n \leq 0; H^{1-k} \right)$ with $k \in \mathbb{N}$. If $Q \in \mathcal{G}^0$ (i.e, supported in $I$ and zero degree), $Q$ is a bounded operator onto $(L^2(I \times \Omega))^2$, $(H^1(I \times \Omega))$ and the commutators $[\nabla^\alpha_x, Q], [\partial_t, Q]$ are in $\mathcal{G}^0$. We set
\[ \varphi(Q, u) = (Qu, u)_{(H^1)} = (\nabla_x^a Qu, \nabla_x^a u)_{(L^2)} + (\partial_t Qu, \partial_t u)_{(L^2)}. \] (4.8)

By the integration by parts
\[ \varphi(Q, u) = \int_{\mathbb{R}^d \times \partial \Omega} Qu \cdot \partial_x^a \tau + 2(\partial_t Qu, \partial_t u)_{(L^2)} - (Qu, K^0_a \partial_t u)_{(L^2)} + (Qu, u)_{(L^2)}. \] (4.9)

where \( \partial_x^a u = (\partial_x u_1, \alpha \partial_x u_2) \)

According [3], we recall some results useful in this work. We denote \( \mathcal{M}^+ \) the spaces of Borel measure \( \mu \) onto \( S\hat{Z} \) with \( C^* \) value Hermitian positive on \( \mathbb{C}^2 \), a measure \( \mu \) of \( \mathcal{M}^+ \) is an element of the dual space \( C_0^0(S\hat{Z} \text{end}) \) satisfy
\[ \langle \mu, q \rangle \geq 0, \forall q \in C^0(S\hat{Z}, \text{end}(\mathbb{C}^2)), \] (4.10)

where \( (\mathbb{C}^2) \) denotes the set of positive Hermitian matrices \( 2 \times 2 \).

Let \( (u_k) \) a bounded sequence in \( (H^1(I \times \Omega))^2 \), solutions of \( Pu = 0 \) converges weakly to 0. Then \( u^k \) (resp. \( \partial_\nu u^k \) ) is bounded in \( H^2_{\text{loc}}(I \times \partial \Omega) \) (resp. \( H^1_{\text{loc}}(I \times \partial \Omega) \) ) has zero weakly limits.

**Proposition 4.2.** There exists a subsequences of \( (u_k) \) and \( \mu \in \mathcal{M}^+ \) such that
\[ \forall Q \in \mathcal{G}, \lim_{k \to \infty} \varphi(Q, u_k) = \langle \mu, \kappa(q) \rangle \] (4.11)

where \( q \) the principal symbol of \( Q \) and \( \mu = \left( \begin{array}{cc} \mu_1 & \mu_{12} \\ \mu_{12} & \mu_2 \end{array} \right) \).

testing the measure \( \mu \) on different operators \( Q \), the limit equation (4.11) can be written as
\[ \begin{cases} \lim_{k \to \infty} (\nabla_x Qu^k_1, \nabla_x u^k_1)_{L^2} + (\partial_t Qu_1, \partial_t u_1) + (Qu_1, u_1) = \langle \mu_1, \kappa(q) \rangle \\ \lim_{k \to \infty} \alpha(\nabla_x Qu^k_2, \nabla_x u^k_2)_{L^2} + (\partial_t u_2, \partial_t u_2) + (Qu_2, u_2) = \langle \mu_2, \kappa(q) \rangle \\ \lim_{k \to \infty} (\nabla_x Qu^k_2, \nabla_x u^k_1)_{L^2} + (\partial_t Qu^k_2, \partial_t u_1) + (Qu^k_2, u_1) = \langle \mu_{12}, \kappa(q) \rangle \end{cases} \] (4.12)

**Proof.** According to [3] and we follow the method given by [6]. \( u^k \) (resp. \( \partial_\nu u^k \) ) has zero weakly limits that implies
\[ \forall Q \in \mathcal{G}^{-1}, \lim_{k \to \infty} \varphi(Q, u_k) = 0. \] (4.13)

Let \( \chi \in C^\infty_0(|x| < \varepsilon), 0 \leq \chi \leq 1, \chi(x) = 1 \) for \( |x| \leq \frac{\varepsilon}{2} \) and \( E \) is a pseudo-differential operator matrix supported near \( \text{Char}(P^{a,b}) \) such that
\[ \text{id} - \sigma(E) = \begin{cases} 0 \text{ near neighborhood } \text{Char}(P) \cap \text{supp}(1 - \chi), \\ \text{non negative}, \end{cases} \]
for all \( \psi \in C^\infty_0(I) \) we have
\[ (\text{id} - \chi)(\text{id} - E)\psi u_k \to 0, \ H^1. \] (4.14)
If $Q = Q_i + Q_\partial \in \mathcal{G}_1^0$ choosing $\epsilon$ small we have $\chi Q_i \equiv 0$ and we write

$$Q = \chi Q + (\text{id} - \chi)Q = \chi Q_\partial + (\text{id} - \chi)QE + (\text{id} - \chi)Q(\text{id} - E)$$

then $Q_\partial$ is tangential pseudo differential operator, $(\text{id} - \chi)QE$ is interior pseudo differential operator and $(\text{id} - \chi)Q(\text{id} - E)\varphi_{u_k} \to 0$ in $(H^1)^2$ for all $\varphi \in C^\infty_0(\overline{X}, \text{End}(\mathbb{C}^2))$.

$$\forall Q \in \mathcal{G}_\text{sym}^0, \quad \sigma(Q) + M \text{ id} \text{ positive} \Rightarrow -M \liminf_{k \to \infty} \varphi(Q, u_k) \leq -M \limsup_{k \to \infty} \|u_k\|^2_{H^1}. \quad (4.15)$$

Indeed, $[\sigma(Q) + M \text{ id}]$ nonnegative matrix implies $[\sigma(\chi Q) + M \text{ id}]$ and $[\sigma(\text{id} - \chi)QE + M \text{ id}]$ are nonnegative matrix and it is sufficient to study independently these cases $Q = Q_\partial$, $Q = Q_i$.

In the first, $Q = Q_\partial$ there exists $\varphi \in C^\infty_0(I)$ such that

$$a_k = (\nabla^\alpha_{x} Q\partial u_k, \nabla^\alpha_{x} u_k)_{L^2} = (Q\partial \nabla^\alpha_{x} \varphi u_k, \nabla^\alpha_{x} \varphi u_k) + b_k$$

with $b_k = ([\nabla^\alpha_{x} Q\partial u_k, \nabla^\alpha_{x} u_k])_{L^2} \to 0$. For all $\epsilon > 0$ there exists $B_\partial$ of zero degree, $C_\partial$ of $-1$ degree tangential d.p.o such that $Q_\partial + (M + \epsilon) \text{ id} = B^* B_\partial + C_\partial$. As $C_\partial \nabla^\alpha_{x} \varphi u_k \to 0$ in $(L^2)^2$ (because $(\varphi u_k)$ is a bounded sequence near the boundary in $C^1(\{x_n \geq 0, \left(\frac{1}{1 + \epsilon_0^2}\right)\}$), we have $\liminf_{k \to \infty} u_k \leq -(M + \epsilon) \limsup \|\nabla^\alpha_{x} \varphi u_k\|$, the same method to $\limsup \|\partial x \varphi u_k\| \leq \limsup \|u_k\|^2_{H^1}$.

So we have

$$Q \in \mathcal{G}_1^0, \quad \sigma(Q)|_{\text{Char} P} = 0 \quad \text{and} \quad \sigma(Q)|_{x_n \leq \epsilon} \Rightarrow \lim_k \varphi(Q, u_k) = 0. \quad (4.16)$$

Let $\sigma(\mathcal{G}) = \{q = \sigma(Q); Q \in \mathcal{G}\}$, that is a vectorial subspace of functions space $C^0$ homogeneities of zero degree onto $T^* X \setminus X$ with value in End$(\mathbb{C}^2)$ endowed with the $L^\infty$ and there exists a subset dense of $\sigma(\mathcal{G})$. By (4.15) and (4.16), there exists a subsequence of $(u_k)$ and a linear map $\tilde{\varphi}$ from $\sigma(\mathcal{G})$ onto $\mathbb{C}$ such that

$$\forall Q \in \mathcal{G}_1^0, \quad \lim_{k \to \infty} \varphi(Q, u^k) = \tilde{\varphi}(\sigma(Q)), \quad (4.17)$$

$$|\tilde{\varphi}(q)| \leq \|q\|_{L^\infty} \limsup \|u_k\|^2_{H^1}. \quad (4.18)$$

Moreover, we have

$$q \in \sigma(\mathcal{G}^0) \quad \text{and} \quad \kappa(q) = 0 \Rightarrow \tilde{\varphi}(q) = 0 \quad (4.19)$$

because if $\kappa(q) = 0$, for all $\epsilon > 0$, there exists $\chi \in C^\infty_0(\mathbb{R}, \text{End}(\mathbb{C}^2))$ supported near $x = 0$ such that $|\chi q|_{L^\infty} \leq \epsilon$ and $(\text{id} - \chi)q = \sigma(Q)$ where $Q \in \mathcal{G}^0$ satisfies (4.16). By Riesz Theorem there exists a Radon measure $\mu$ in the dual of $C^\infty_0(\overline{S\mathbb{Z}}, \text{End}(\mathbb{C}^2))$ such that

$$\forall Q \in \mathcal{G}_1^0, \quad \lim_k \varphi(Q, u_k) = (\mu, \kappa(\sigma(Q))) \quad (4.20)$$

with $\mu$ is positive Hermitian by (4.15) and a measure $\mu_\partial$ on $S(T^* \partial X)$ such that

$$\forall Q \in \mathcal{G}_1^0, \quad \lim_k \int_{\partial X} Q u_k \partial_x u_k = \int \sigma(Q)|_{x_n = 0} d\mu_\partial. \quad (4.21)$$
and by (4.9) we have

$$\mu = \mu_\partial + \mu_{\text{cin}}$$

(4.22)

where $\mu_\partial$ is considered to measure on $SZ$ through the injection $S(T^\ast \partial X) \hookrightarrow SZ$. If the sequence $u_k$ satisfies the Dirichlet condition $u_{k|\partial X} = 0$ then $\mu_\partial \equiv 0$ and if $Q = Q_\partial \in G^0_\ell$ with compact support near $x_n = 0$, $(t, x^i, \xi^i) \in T^\ast \partial X$ we have $Qu_k$ bounded in $C^\infty(\tilde{X})$.

We have $Z_{x=0} = T^\ast Y$, since the sequence $u^k$ satisfies the Dirichlet $u_{k|\partial X} \equiv 0$ then $\mu_\partial \equiv 0$.

### 4.1 Propagation Theorem to boundary

We assume that there is no contact of infinity order between the geodesics of $\overline{\Omega}$ and the boundary $\partial \Omega$. In this section we recall some concepts and properties to the boundary value problem of coupled waves system. Let $u_k(t, x)$ a sequence of solution of the following problem

$$
\begin{cases}
    ( \partial_t^2 - D_\alpha + K_\alpha^2 \partial_t ) u_k = 0, & u_{k|_{\partial X}} = 0 \\
    (u_{k|_{\partial \omega}}, \partial_t u_{k|_{\partial \omega}}) \text{ bounded in } (H^0_\Omega)^2 \times (L^2(\Omega))^2
\end{cases}
$$

(4.23)

has null weak limits, $\mu = 2\mu_{\text{cin}}$ associated measures on $(SZ)$, $\mu^+ = 2\mu_{\text{cin}}^{-}$ their restrictions to $(SZ)^+$.  

**Theorem 4.3.** For all $s \in \mathbb{R}$ we have

$$
G(s)^\ast(\mu) = (\exp \left( - \int_0^s K_{a(G(\sigma)(\rho))}^\beta d\sigma \right), \mu).
$$

(4.24)

Precisely, for all $B$ a Borel set of $SZ$, we have

$$
\mu(G(s)(B)) = \int_B H(s, \rho) d\mu = \sum_{i,j} \int_B H_{ij} d\mu_{ij}
$$

with $H(s, \rho) = \exp \left( - \int_0^s K_{a(G(\sigma)(\rho))}^\beta d\sigma \right)$.

**Proof.** We set $\mu_s = H(s, \rho)\mu$. As $\{G(s)\}$ is a $C^\alpha$-homeomorphic group of $SZ$ and change $t$ to $-t$ returns change $a$ to $-a$. Then it is sufficient to prove that

$$
G(s)^\ast(\mu^+) \leq \mu^+_s \text{ for all } s > 0.
$$

(4.25)

If $K$ is a compact of $(SZ)^+ \cap (t = 0)$ and $J$ a compact of $\mathbb{R}$. We denote

$$
K_J = \{G(\sigma)(\rho); \rho \in K, \sigma \in J\}.
$$

The fact that $G(s)(t, x, \xi) = (t + s, G(s)(x, \xi))$, the map $\Theta : ((SZ)^+ \cap (t = 0)) \times \mathbb{R} \rightarrow (SZ); \Theta(\rho, \sigma) = G(\sigma)\rho$ is a homomorphic that redress the flow $(G(\rho, \sigma + s) = G(s)\Theta(\rho, \sigma))$. To prove (4.25) it is sufficient to verify the following properties

$$
\begin{cases}
    \forall \alpha_1 > 0, \exists \beta_1 > 0 \text{ such that } \\
    \text{for all } K' \subset \subset \text{int}(K) \subset K \subset ((SZ)^+ \cap (t = 0)), \text{ diam}K \leq \beta_1 \\
    \text{and for all } b_0 < b_0 < b_1 < b_1, b_1 - b_0 \leq \beta_1 \\
    \text{with } J = [b_0, b_1], J' = [b_0', b_1']
\end{cases}
$$

(4.26)

we have $G(s)^\ast(\mu(K'_{J'})) \leq (1 + \alpha_1)\mu_s(K_J)$. 

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Indeed, by the redress flow, we can consider the measures $\mu^+$ and $\mu^+_*$ onto product $((SZ)^+ \cap (t = 0)) \times \mathbb{R}$, we denote by $\tilde{\nu}_t^*$, $\tilde{\mu}_t^*$ and $\tilde{\nu}_t^* = G(s)^*(\tilde{\mu}_t^*)$. By (4.26) we deduce that

$$\tilde{\nu}_t^*(E') \leq (1 + \alpha_1)\mu_+^*(E)$$

(4.27)

for $E' = K' \times I, E = K \times I, K' \in K$, $\text{diam}(K) \leq \beta_1, I = [b_0, b_1], b_1 - b_0 \leq \beta_1$ by increasing limits, and for $E = O \times I, O$ open set with $\text{diam}(O) \leq \beta_1$ with $\text{diam}(O) \leq \beta_1$ and by decreasing limits for $E = E' = O \times L$ for any interval $L$ with $\text{diam}(L) \leq \beta_1$ then we have additivity of measure and increasing limits we have

$$\tilde{\nu}_t^*(V) \leq (1 + \alpha_1)\mu_+^*(V), \forall V \text{ open}$$

then $\tilde{\nu}_t^* \leq (1 + \alpha_1)\mu_+^*$, for all $\alpha_1 > 0$, hence (4.25).

Now we prove (4.26), we have $G(s)^*(\mu^+) = \mu(K_{j,r+s})$ and we can assume $0 < \beta_1 \ll s$.

We set $u^h = u$ and we identify $u(x, t)$ to

$$u(x, t) = (u(x, t), \partial_t u(x, t)) \in (C^0(\mathbb{R}, (H_0^1(\Omega))^2) \cap C^1(\mathbb{R}, (L^2(\Omega))^2)) \oplus C^0(\mathbb{R}, (L^2(\Omega)))^2).$$

We set

$$H = (H_0^1(\Omega))^2 \oplus (L^2(\Omega))^2, \ H' = (L^2(\Omega))^2 \oplus (H^{-1}(\Omega))^2, \ H_1 = L^2(\mathbb{R}, H) \text{ and } H_0 = L^2(\mathbb{R}, H')$$

and for $\varphi = ((u_0, v_0), (u_1, v_1)) \in \mathcal{H}_i$

$$\|\varphi\| = \|\mu\|_{\mathcal{H}_i}.$$ (4.28)

We recall that the operator $A_{\alpha,\beta}^+$ with boundary Dirichlet and that $e^{tA_{\alpha,\beta}^+}$ is bounded on $H$ and $H'$, we denote by $C$ some independent constants of $k$ index concerning the sequence $u^h$ and by $C_0$ some independent constants of $k$, $K'$, $K$, $J$, $J'$ and $b_0$, $b_1$ given in a fixed compact of $\mathbb{R}$.

Let $\varphi \in C_0^\infty(\mathbb{R})$, equal to 1 on $[b_0 - 1, b_1' + s + 1], \psi(t) \in C^\infty(\mathbb{R}), 0 \leq \psi \leq 1$, in a neighborhood of $[b_1, +\infty[, \psi = 1$ in a neighborhood of $]-\infty, b_0[, \Psi(t) \in C_0^\infty([b_0, b_1]), 0 \leq \Psi \leq 1$ and $-\Psi$, $\Psi \equiv 1$ in a neighborhood of $\text{supp} \varphi$. If $Q \in G^0$ and $\rho \in Z \setminus X$, we write $\rho \notin \text{ES}(Q)$ if $j^{-1}(\rho) \cap \text{Car}P_{\alpha,\beta}^0$ not meet the essential supported of $Q$ that is define because if $\rho$ is an interior point, $Q$ is a d.p.o. near the point $\rho' = J^{-1}(\rho) \in P_{\alpha,\beta}^\circ$.

So we write for $K$ compact of $Z \setminus X, Q = 1d$ near of $K$ if $K \cap \text{ES}(Q - 1d) = \emptyset$. Let $Q_0 \in C^0$ with its principal symbol $q_0 = \sigma(Q_0), \text{id} - q_0$ positive, such that $Q_0 \subset (G(\sigma)(\rho); \rho \in \text{int}(K), b_0 - \epsilon < \sigma < b_1' + s + \epsilon)$ with $\epsilon > 0$ small and $Q_0 = 1d$ near of $K_{[b_0,b_1' + s]}$, and let $Q_1 \in G^0$ with $q_1$ its principal symbol with $q_1$ and $\text{id} - q_1$ are nonnegative and such that $Q_1 = 1d$ near of $K_{j,r+s}$, $\text{ES}(Q_1)$ include a neighborhood of $K_{j,r+s}$, $Q_0 = 1d$ near of ES($Q_1$).

Let $Q \in G^0$ and $\varphi = ((u_0, v_0), (u_1, v_1)) \in \mathcal{H}_i$ we set $Q\varphi = (Q(u_0, v_0), Q(u_1, v_1))$.

We have $(\partial_t - A_{\alpha,\beta}^+) u = 0$, then $(\partial_t - A_{\alpha,\beta}^+) \psi u = \psi'(t) u$.

Let

$$w = -\int_{-\infty}^t e^{(t-\sigma)A_{\alpha,\beta}^+} \psi'(\sigma) u(\sigma) d\sigma,$$

we have $(\partial_t - A_{\alpha,\beta}^+) w = -\psi'(t) u$, then $(\partial_t - A_{\alpha,\beta}^+) [u - \psi(t) u - w] = 0$, since $u - \psi(t) u - w = 0$ for $t < b_0$ that result

$$u = \psi(t) u + w.$$ (4.29)

We have $(\partial_t - A_{\alpha,\beta}^+) Q_0 w = -Q_0 \psi' u - [\partial_t - A_{\alpha,\beta}^+, Q_0] w$ and we let

$$h = -\int_{-\infty}^t e^{(t-\sigma)A_{\alpha,\beta}^+} Q_0 \psi'(\sigma) u(\sigma) d\sigma.$$
hence \((\partial_t - A_{\alpha,\beta}^\circ)\mathfrak{h} = -Q_0\psi'(t)\mathfrak{u}\)

\[(\partial_t - A_{\alpha,\beta}^\circ)[Q_0\mathfrak{u} - \mathfrak{h}] = -[\partial_t - A_{\alpha,\beta}^\circ, Q_0]\mathfrak{u}.
\]

(4.30)

The key point is the following estimation

\[|Q_1(Q_0\mathfrak{u} - \mathfrak{h})| \leq C|\varphi\mathfrak{u}|_0.
\]

(4.31)

that result by the propagation Theorem of Melrose-Sjöstrand.

Indeed, let \(F = \{ u \in L^2_{\partial_0(t)}(X) \mid \mathcal{P}_u^{\alpha,\beta} u = 0, u|_{\partial X} = 0 \}\) inner the norm \(|\varphi\mathfrak{u}|_0\) and \(WF_{\mathfrak{b}}\) the wavefront at the boundary. Let \(\mathfrak{w}, \mathfrak{h}\) associate to \(u\) as given below, we have \(WF_{\mathfrak{h}}(u) \subset Z\) that implies \(WF_{\mathfrak{h}}(u) \subset Z, WF_{\mathfrak{h}}(\mathfrak{h}) \subset Z\) and \(WF_{\mathfrak{h}}([\partial_t - A_{\alpha,\beta}^\circ]u) \subset Z \setminus \{x, Q_0 = \text{id near } x\}\). As \(WF_{\mathfrak{h}}(Q_0\mathfrak{u}) \subset (b_0, +\infty)\) by the propagation theorem (see [11]), we have \(WF_{\mathfrak{h}}(Q_0\mathfrak{u} - \mathfrak{h}) \cap ES(Q_1) = \emptyset\) then \(Q_1(Q_0\mathfrak{u} - \mathfrak{h}) \in C^\infty(X)\). As \(\mathfrak{w} \mapsto Q_1(Q_0\mathfrak{u} - \mathfrak{h})\) is continuous from \(F\) onto \(\mathcal{H}_0\) and (4.31) result of closed graph theorem.

We have

\[\mathfrak{h} = -\int_{-\infty}^t e^{(t-\sigma)A_{\alpha,\beta}^\circ} \psi'(\sigma)\Psi(\sigma)Q_0\mathfrak{u}d\sigma - \int_{-\infty}^t e^{(t-\sigma)A_{\alpha,\beta}^\circ} [Q_0, \psi'\Psi]_0d\sigma,
\]

then \(\mathfrak{h} \in C^0(\mathbb{R}, H)\) and for \(t \in [b_0 - 1, b_1' + s + 1]\),

\[||\mathfrak{h}||_H \leq C_0||\psi'||_{L^2}\Psi Q_0\mathfrak{u}|_1 + C|\varphi\mathfrak{u}|_0.
\]

(4.33)

because \([Q_0, \psi'\Psi]_0 = (Q_{-1}u(t,x), Q_{-1}\partial u(x,t))\) with \(Q_{-1} \in \mathcal{G}^{-1}\) then

\[\left|\mathfrak{h}_{\mathfrak{u}}\right|_1 \leq \left|\nabla_x Q_{-1}u\right|_{(L^2(\mathbb{R} \times \Omega))} + \left|\partial_t u\right|_{(L^2(\mathbb{R} \times \Omega))}
\leq C|\varphi\mathfrak{u}|_{(L^2(\mathbb{R} \times \Omega))}
\]

(4.34)

Let \(d\) a real constant, \(A_{\alpha,\beta}^\circ = \begin{pmatrix} 0 & \text{id} \\ D_\alpha & -K_{\beta}^\circ \end{pmatrix}\). We have \((\partial_t - A_{\alpha,\beta}^\circ)\mathfrak{h} = -Q_0\psi'\mathfrak{u} + (A_{\alpha,\beta}^\circ - A_{\alpha,\beta}^\circ)\mathfrak{h}\)

(4.35)

Then results for all \(t \in [b_0, b_1' + s + \varepsilon]\), \(t' > 0, \varepsilon' \ll \varepsilon\)

\[\left|\mathfrak{h}(t)\right|_H \leq (e^{-d(t-b_1')}\varepsilon'd^\circ + C_0\|a(x) - d\|_{L^\infty(T_f)}(t - b_0)} + \left|\psi'\right|_{L^2}\Psi Q_0\mathfrak{u}|_1 + C|\varphi\mathfrak{u}|_0
\]

(4.36)

where \(T_f = K_{[b_0,\varepsilon,b_1,\varepsilon'+c]}\). Indeed, we write by (4.36)

\[\mathfrak{h} = (1) + (2) + (3)
\]

We have \(WF_{\mathfrak{h}}(\mathfrak{h}) \subset \{t > t_0\}\) and \(WF_{\mathfrak{h}}([\partial_t - A_{\alpha}^\circ]\mathfrak{h}) = WF_{\mathfrak{h}}(Q_0\psi'(t)\mathfrak{u}) \subset (SE(\mathfrak{u}) \cap (t > b_0))\).

By Cauchy Schwartz we obtain
that give the term (3). We can see the term (2) by (4.34).

Finally for the term (1), we see that if \((e_j, u_j)\) is the orthonormal basis of eigenfunctions of \(H_0^1(\Omega)\), \(-\Delta e_j = \omega_j^2 e_j, \omega_j \geq 0\), we denote by \(\lambda_{j_i} \), \(i = 1, 2\) roots of \(\lambda^4 + 2d\lambda^3 + (\beta^2 + \omega_j^2 + \omega_j^2)\lambda^2 + 2d\omega_j^2\lambda + \omega_j^4 = 0\). The family \((e_j, \alpha_e j), \lambda_{j_i} (e_j, \alpha_e j)\), \(i = 1, 2\) constitute an orthonormal basis in \(H\) of eigenfunctions of \(A^\alpha_{\omega}B^\beta\). For \(j\) large, we have \(\text{Re}(\lambda_{j_i}^\pm) = -\frac{\delta}{2}\), we obtain

\[
\begin{aligned}
(1) & \leq \int_{b_0}^{b_1} e^{-(t-t')\frac{\delta}{2}} \left| \psi'(\sigma) \right| \left| \Psi_{Q_0 u} \right|_H d\sigma + C \left| \varphi_{u} \right|_0 \\
& \leq e^{-(t-t')\frac{\delta}{2} + \frac{\delta}{2} t_0} \left| \psi' \right|_{L^2} \left| \Psi_{Q_0 u} \right|_1 + C \left| \varphi_{u} \right|_0,
\end{aligned}
\]

this give (4.36). We have \(\lim_k \left| \varphi_{u}^{k} \right|_0 = 0\), and since \(\sigma(Q_1^1 Q_1) = \text{Id}\) on \(K_{j' + s}\)

\[
\mu^+(K_{j' + s}) \leq \limsup_k \left| Q_1 u_k \right|_1^2,
\]

Let \(\chi \in C_0^\infty ([b_0' + s - \epsilon, b_0' + s + \epsilon], \text{ with } \chi \equiv 1\) on \(SE(Q_1)\). By (4.29), (4.31) and \(q_1 = \sigma(Q_1) \in [0, 1]\), we have

\[
\limsup_k \left| Q_1 u_k \right|_1^2 \leq \limsup_k \left| (h^k)^2 \right|_1
\]

and by (4.36)

\[
\limsup_k \left| (h^k)^2 \right|_1 \leq (b_1' - b_0 + 2\epsilon) \left| \psi' \right|_{L^2}^2 \left( e^{-\frac{\delta}{2}(b_1' + s - b_1 - \epsilon)} e^{\frac{\delta}{2}(b_1' + s + \epsilon)} \right)
\]

\[
+ C_0 \|a(x) - d\|_{L^\infty(T)} (b_1' + s + \epsilon - b_0)^2 \limsup_k \left| \Psi_{Q_0 U} \right|^2.
\]

As \(b_1' - b_0 < b_1 - b_0\), we can assume \((b_1' - b_0 + 2\epsilon) \|\psi'\|_{L^2} \leq 1\). Moreover, \(\text{Id} - \sigma(\Psi_{Q_0})\) non negative and supported in \(K_j\), then we deduce from (4.39), (4.40), (4.40) with \(T = T_0 = 0\)

\[
\mu^+(K_{j' + s}) \leq \mu^+(K) \left[ e^{-\delta s} e^{\frac{\delta}{2} b_1} + C_0 \|a(x) - d\|_{L^\infty(T)} (b_1' + s - b_0)^2 \right].
\]

This estimation is valid for \(K' \in K, J' \in J, b_1 - b_0 \leq \beta\) and \(s > \beta\) where \(b_0, b_1, d\) are bounded constant. For all \(r \in K_j\), we have

\[
\left| e^{dx} G(s, \rho) - \text{Id} \right| \leq C_0 s \|a(x) - d\|_{L^\infty(T)}.
\]

Let \(\delta > 0\) small enough, there exist \(s_\delta > 0\) and \(\beta_\delta\) such that \(\rho_0 \text{ in } K_j\) we have \(\text{diam}(K) < \beta_\delta, 0 < s \leq s_\delta\) and \(b_1 - b + 0 \leq \delta_\delta\), and choosing \(d = a(\rho_0), \|a(x) - d\|_{L^\infty(T)} \leq C_0 \delta\) and by (4.42)

\[
\mu^+(K_{j' + s}) \leq \mu^+(K) H(s, \rho_0)(1 + C_0 \delta s).
\]

Proving (4.26), let \(s > 0\) and \(\beta < \inf(\beta_\delta, s_\delta)\). By iterating at most \(N = \frac{s}{\delta}\) times the inequality with \(s = s_\delta\) and a sequence \(J' = J_1 \subset J_2 \subset \ldots \subset J_N = J\) of intervals and a sequence of compacts \(K' = K_1 \subset K_2 \subset K_3 \subset \ldots \subset K_N = K\), we obtain for any \(\rho_0 \in \text{int}(K)\)

\[
\mu^+(K_{j' + s}) \leq \mu^+(K) H(s, \rho_0)(1 + C_0 \delta) s.
\]
Since \((1 + C_0 \delta s) \frac{d}{ds} \leq 1 + C_0 \delta\). As we have \(\mu^+_{s}(KJ) = \int_{KJ} H(s, \rho)\) and for \(\beta\) small \(|H(s, \rho) - H(s, \rho_0)| \leq C_0 \delta\) for \(\rho \in KJ\), hence the function \(H\) the function \(H\) remaining in a compact \((0, + \infty)\)

\[
|\mu^+_{s}(KJ) - \mu^+_{s}(KJ)H(s, \rho)| \leq C_0 \delta \mu^+_{s}(KJ)
\]

(4.45)

and (4.26) deduced from (4.44) and (4.45).

References


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