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Partial Stabilization of a Coupled Wave Equations

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Abstract. We consider a stabilization problem for a coupled wave equations on a compact Riemanian manifold Ω with or without boundary. We prove the exponential stability result in the energy space, under a geometrical control condition (BLR). Without any geometrical assumption and for all regular initial data, we give a logarithmic decay result of the energy.

1 Introduction

In this paper we study the stabilization of a coupled wave equations. More precisely, we consider the following initial and boundary value problem :

$$\partial_t^2 u_1 - \Delta u_1 + \beta\, \partial_t u_2 + 2a(x)\partial_t u_1 = 0,\ \Omega \times (0,+\infty), \tag{1.1}$$

$$\partial_t^2 u_2 - \alpha\, \Delta u_2 - \beta\, \partial_t u_1 = 0,\ \Omega \times (0,+\infty), \tag{1.2}$$

$$u_1 = 0,\ \partial\Omega \times (0,+\infty), \tag{1.3}$$

$$u_2 = 0,\ \partial\Omega \times (0,+\infty), \tag{1.4}$$

$$u_1(x,0) = u_1^0(x),\ \partial_t u_1(x,0) = u_1^1(x),\ x \in \Omega, \tag{1.5}$$

$$u_2(x,0) = u_2^0(x),\ \partial_t u_2(x,0) = u_2^1(x),\ x \in \Omega, \tag{1.6}$$

where Ω is a compact connected Riemannian manifold, $a(x) \in C(\overline{\Omega}, \mathbb{R}_+)$ and $\alpha,\ \beta$ are positives constants.

If we set $u = (u_1, u_2)$ then the system of equations (1.1)-(1.6) is equivalent to the following system

$$\begin{cases} \partial_t^2 u - D_\alpha u + K_a^\beta \partial_t u = 0 \ \text{ in } \ \Omega \times (0,+\infty), \\ u = 0 \ \text{ on } \ \partial\Omega \times (0,+\infty), u(\cdot,0) = u_0 \ \ \partial_t u(\cdot,0) = u_1, \ \text{ in } \Omega, \end{cases} \tag{1.7}$$

where

$$D_\alpha = \begin{pmatrix} \Delta & 0 \\ 0 & \alpha\Delta \end{pmatrix}, \quad K_a^\beta = \begin{pmatrix} 2a(x) & \beta \\ -\beta & 0 \end{pmatrix},\ u_0 = (u_1^0, u_2^0)\ and\ u_1 = (u_1^1, u_2^1).$$

The problem (1.7) has an unique solution $u(x,t) \in C^0\left(\mathbb{R}, (H_0^1(\Omega))^2\right) \cap C^1\left(\mathbb{R}, (L^2(\Omega))^2\right)$ for all initial data $u_0 \in (H_0^1(\Omega))^2 \oplus (L^2(\Omega))^2$, obtained by using the Hille-Yosida theorem for an unbounded operator.

We consider the Hilbert space $H = (H_0^1(\Omega))^2 \oplus (L^2(\Omega))^2$, we define

$$A_a^{\alpha,\beta} = \begin{pmatrix} 0 & id \\ D_\alpha & -K_a^\beta \end{pmatrix}, \quad \mathcal{D}(A_a^{\alpha,\beta}) = (H_0^1(\Omega) \cap H^2(\Omega))^2 \oplus (H_0^1(\Omega))^2. \tag{1.8}$$

Let $u(x,t) = (u_1, u_2)(x,t)$ solution of (1.7), we define the energy functional at the time t by

$$\begin{aligned} E(u,t) &= \frac{1}{2} \int_\Omega \left(|\partial_t u|^2 + |\nabla_x^\alpha u|^2 \right) \\ &= \frac{1}{2} \int_\Omega \left(|\partial_t u_1|^2 + |\partial_t u_2|^2 + |\nabla_x u_1|^2 + \alpha |\nabla_x u_2|^2 \right) dx \end{aligned} \tag{1.9}$$

that satisfy the following estimation

$$E(u,0) - E(u,t) = \int_0^t \int_\Omega a(x) |\partial_s u_1(x,s)|^2 dx ds, \tag{1.10}$$

where $\nabla_x^\alpha u = (\nabla_x u_1, \sqrt{\alpha} \nabla_x u_2)$. We recall the following results,

Theorem 1.1. *Assume that $a \not\equiv 0$. Then, we have*

(i) If $\partial\Omega \neq \emptyset$, we have $Re\lambda < 0$ for $\lambda \in sp(A_a^{\alpha,\beta})$ (spectra set of $A_a^{\alpha,\beta}$) ; If $\partial\Omega = \emptyset$, $\lambda = 0$ is the only eigenvalue with null real part.

(ii) For any initials data $((u_1^0, u_2^0), (u_1^1, v_2^1)) \in (H_0^1(\Omega))^2 \oplus (L^2(\Omega))^2$, the solution $u = (u_1, u_2)$ of (1.7) satisfies $\lim_{t \to +\infty} E(u,t) = 0$.

(iii) Moreover, assume that $\alpha \neq 1$ and that the geodesic of $\overline{\Omega}$ hasn't contact of infinite order with $\partial\Omega$ and there exists a time T_0 such that any generalized geodesics of Ω with its length large than T_0 meet $(\{a(x) > 0\})$. Then, there exists $c_0,\ c_1 > 0$ such that

$$E(u)(t) \leq c_0 e^{-c_1 t} E(u)(0), \quad \forall u \in H, \quad \forall\, t \geq 0. \tag{1.11}$$

Proof.

(i) If $\lambda = i\omega \in sp(A_a^{\alpha,\beta})$, $\omega \in \mathbb{R}$ there exists $f = (f_1, f_2) \not\equiv 0$ in $\left(H_0^1(\Omega)\right)^2$ such that $-D_\alpha f + \lambda K_a^{\alpha,\beta} f + \lambda^2 f = 0$, which implies

$$\begin{aligned} &\omega \left(\int_\Omega a |f_1|^2 + \beta \operatorname{Re} \int_\Omega f_2 . \overline{f_1} \right) = 0, \\ &\omega\beta \operatorname{Re} \int_\Omega f_1 . \overline{f_2} = 0, \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega |\nabla f_1|^2 - \omega^2 \int_\Omega |f_1|^2 - \omega\beta \operatorname{Im} \int_\Omega f_1 \cdot \overline{f_2} = 0, \\ &\alpha \int_\Omega |\nabla f_2|^2 - \omega^2 \int_\Omega |f_2|^2 + \omega\beta \operatorname{Im} \int_\Omega f_2 \cdot \overline{f_1} = 0. \end{aligned}$$

If $\omega = 0$ then we have $f_1 = cst$ and $f_2 = cst$; if $\omega \neq 0$, we have $\sqrt{a} f_1 = 0$ in $L^2(\Omega)$, since $\mathcal{O} = \{a(x) > 0\}$ is non empty open set. Then, $f_{|\mathcal{O}} = 0$ and

$$\begin{cases} -\Delta f_1 + \lambda^2 f_1 + \beta\lambda f_2 = 0, \\ -\alpha\Delta f_2 + \lambda^2 f_2 - \beta\lambda f_1 = 0, \end{cases}$$

this implies that $(f_1, f_2)_{|\mathcal{O}} \equiv (0,0)$,using that Ω is connected set, thus $(f_1, f_2) \equiv (0,0)$.

(ii) We deduce 2. by 1. because $\overline{\oplus E_{\lambda_j}} = H$, using [7].

(iii) If $\partial\Omega = \emptyset$, we can see [10] and the general case, following Bardos, Lebeau and Rauch [1], using the propagation Theorem of Melrose- Sjöstrand which will be the goal of the proof of point 2. of Theorem 1.3.

Theorem 1.2. *Assume that $a \not\equiv 0$. Then, there exists $C > 0$ such that*

$$\forall \lambda \in sp(A_a^{\alpha,\beta}) \setminus \{0\}\,, \;\; Re\lambda < -\frac{1}{C} e^{-C|Im\lambda|}. \tag{1.12}$$

For $\lambda = -\sigma + i\omega$, $\omega \in \mathbb{R}$, $|\omega| \geq 1$ and $0 \leq \sigma \leq \frac{1}{C} e^{-C|\omega|}$ we have

$$\left\| \left(\lambda - A_a^{\alpha,\beta}\right)^{-1} \right\|_{\mathcal{L}(H)} \leq Ce^{C|\omega|} \tag{1.13}$$

(Here the norm of the resolvent is the norm of the operator on H). Moreover, for any $k > 0$, there exists $C > 0$ such that for all $(u_0, u_1) \in D\left(\left(A_a^{\alpha,\beta}\right)^k\right)$,

we have

$$\forall t \geq 0, \;\; E(u,t)^{\frac{1}{2}} \leq \frac{C}{(\ln(2+t))^k} \left\| (u_0, u_1) \right\|_{D\left(\left(A_a^{\alpha,\beta}\right)^k\right)}. \tag{1.14}$$

Let $R > 0$, we set

$$D(R) = \sup\left\{ \mathrm{Re}\lambda_j \mid \lambda_j \in \mathrm{S}p(A_a^{\alpha,\beta}), \; |\lambda_j| \geq R \right\} \tag{1.15}$$

that is a negative function, decreasing when $R > 0$. We denote $D(\infty) = \lim\limits_{R\to\infty} D(R)$ and $D(0) = \lim\limits_{R\to 0^+} D(R)$.

Assuming that there have no contacts of infinite order between the bicharacteristic of $\overline{\Omega}$ and its boundary $\partial\Omega$ (the geometric control condition (GCC)). First, we notice that determinant of the symbol is given by

$$p_a^{\alpha,\beta}(t, x; \tau, \xi) = \left(|\xi|^2 - \tau^2\right)\left(\alpha|\xi|^2 - \tau^2\right)$$

this leads to two bicharacteristic families in the characteristic set of $P_a^{\alpha,\beta}$, Char$P_a^{\alpha,\beta} = \left\{(x, t; \xi, \tau);\; p_a^{\alpha,\beta}(t, x; \tau, \xi) = 0\right\}$, namely those of the symbols

$$p_1 = |\xi|^2 - \tau^2 \;\text{ and }\; p_\alpha = \alpha|\xi|^2 - \tau^2,$$

if $\alpha \neq 1$, the wave front sets propagate independently along the null bicharacteristic of each one of the two families. Let $\rho_0 = (x_0, u_0) \in T\overline{\Omega}$, with $|u_0| = 1$ (u_0 is in a half closed space defined by $\overline{\Omega}$ if $x_0 \in \partial\Omega$) there exists a unique geodesic generalized

$s \to x_1(s, \rho_0)$ in $\overline{\Omega}$ (resp. $s \to x_2(s, \rho_0)$ in $\overline{\Omega}$) issued to ρ_0 i.e. satisfy

$x_1(0, \rho_0) = x_0$, $\lim\limits_{s\to 0^+} \frac{x_1(s,\rho_0)-x_0}{s} = u_0$ (resp. $\lim\limits_{s\to 0^+} \frac{x_2(s,\rho_0)-x_0}{s} = \sqrt{\alpha}u_0$).

Let $t > 0$, we set

$$C_1(t) = \inf_{\rho_0} \frac{1}{t} \int_0^t a(x_1(s, \rho_0))ds, \;\; C_2(t) = \inf_{\rho_0} \frac{1}{t} \int_0^t a(x_2(s, \rho_0))ds.$$

that satisfies

$$tC_i(t) + sC_i(s) \leq (t+s)C_i(t+s), \;\; i = 1, 2.$$

We denote

$$\begin{aligned} C(t) &= \min\left(C_1(t), C_2(t)\right) \\ &= \min\left(\inf_{\rho_0} \tfrac{1}{t}\int_0^t a(x_1(s,\rho_0))ds,\ \inf_{\rho_0} \frac{1}{t}\int_0^t a(x_2(s,\rho_0))ds\right) \end{aligned} \tag{1.16}$$

that is a additive function and we set $C(\infty) = \lim_{t\to+\infty} C(t)$. We have $C(t) \leq C(\infty)$ for all t.

Let

$$\varrho = \sup\left\{\gamma \geq 0 \ / \ \exists B > 0, \forall u \in H, E(u,t) \leq Be^{-\gamma t}E(u,t)\right\}. \tag{1.17}$$

Theorem 1.3. *Assume that $\alpha \neq 1$, then we have*

(i) $\varrho = \min\{-D(0), C(\infty)\}$.

(ii) $C(\infty) \leq -D(0)$.

2 Proof of Theorem 1.2

We denote $H = \left(H_0^1(\Omega)\right)^2 \bigoplus \left(L^2(\Omega)\right)^2$, H^* the dual space of H and the duality product is given by

$$\langle u_1, u_2\rangle = \int_\Omega u_1^1 \cdot u_2^2 - u_1^2 \cdot u_2^1, \ \ u_1 = (u_1^1, u_{21}) \in H^*, \ \ u_2 = (u_1^2, u_2^2) \in H. \tag{2.1}$$

We decompose $A_a^{\alpha,\beta}$ in the following form

$$A_a^{\alpha,\beta} = A_0^{\alpha,0} + B_a^\beta = A_0^\alpha + B_a^\beta; \ \ A_0^\alpha = \begin{pmatrix} 0 & \mathrm{id} \\ D_\alpha & 0 \end{pmatrix}; \ \ B_a^\beta = \begin{pmatrix} 0 & 0 \\ 0 & K_a^\beta \end{pmatrix} \tag{2.2}$$

B_a^β is a bounded operator in H and compact as an operator of $\mathcal{L}(H, H^*)$.

$\left(\lambda - A_a^{\alpha,\beta}\right) u = v$ equivalent to

$$\begin{cases} u_2 = \lambda u_1 - v_1 \\ \mathcal{P}_{a,\lambda}^{\alpha,\beta} u_1 = v_2 + K_a^\beta v_1 + \lambda v_1; \ \ \mathcal{P}_{a,\lambda}^{\alpha,\beta} = \lambda^2 \mathrm{id} + \lambda K_a^\beta - D_\alpha. \end{cases} \tag{2.3}$$

$D(A_a^{\alpha,\beta}) = (H_0^1(\Omega) \cap H^2(\Omega))^2 \oplus (H_0^1(\Omega))^2$ endowed with the graph norm is an Hilbert space and we define the resolvent set

$$\mathcal{R}(A_a^{\alpha,\beta}) = \left\{\lambda \in \mathbb{C}\,; (\lambda - A_a^{\alpha,\beta}) \text{ is bijective from } \mathcal{D}(A_a^{\alpha,\beta}) \text{ onto } H\right\}.$$

The operator $\lambda - A_0^\alpha$ is a Fredholm operator of zero index from H onto H^* this implies that $\lambda - A_a^{\alpha,\beta}$ is too and we have

$$\mathcal{R}(A_a^{\alpha,\beta}) = \left\{\lambda \in \mathbb{C} | (\lambda - A_a^{\alpha,\beta}) \text{ is bijective from } H \text{ onto } H^*\right\}.$$

[Indeed, if $(\lambda - A_a^{\alpha,\beta})$ is bijective from H onto H^*, that injective onto $D(A_a^{\alpha,\beta})$ and for $v \in H \subset H^*$ and $u \in H$ such that $(\lambda - A_a^{\alpha,\beta})u = v$ we have $A_a^{\alpha,\beta}u = \lambda u - v$ then $u \in D(A_a^{\alpha,\beta})$. inversely, if $(\lambda - A_a^{\alpha,\beta})$ is bijective of $D(A_a^{\alpha,\beta})$ onto H, if $u \in H$ satisfy $(\lambda - A_a^{\alpha,\beta})u = 0$ we have $u \in D(A_a^{\alpha,\beta})$ then $u = 0$, moreover $(\lambda - A_a^{\alpha,\beta})$ is a Fredholm operators of zero index and injective hence there is bijective from H onto H^*]. We obtain that

$$\mathcal{R}(A_a^{\alpha,\beta}) = \left\{\lambda \in \mathbb{C} | \mathcal{P}_{a,\lambda}^{\alpha,\beta} \text{ is bijective from } \left(H_0^1(\Omega)\right)^2 \textit{ onto } \left(H^{-1}(\Omega)\right)^2\right\} \tag{2.4}$$

and let $\lambda \in \mathcal{R}(A_a^{\alpha,\beta})$, we have

$$\left(\lambda - A_a^{\alpha,\beta}\right)^{-1} = \begin{pmatrix} \mathcal{P}_\lambda^{-1}\left(K_a^\beta + \lambda \mathrm{id}\right) & \mathcal{P}_\lambda^{-1} \\ \lambda \mathcal{P}_\lambda^{-1}\left(K_a^\beta + \lambda \mathrm{id}\right) - \mathrm{id} & \lambda \mathcal{P}_\lambda^{-1} \end{pmatrix} \tag{2.5}$$

where $\mathcal{P}_\lambda^{-1} = (\mathcal{P}_{a,\lambda}^{\alpha,\beta})^{-1}$. In the following, we assume that $a(x)$ is not identically zero functions.

Lemma 2.1. *Let $C > 0$. There exists $C_1, C_0 > 0$ such that for all $\lambda = -\sigma + i\omega$, $\omega \in \mathbb{R}$, $|\sigma| \leq C$ we have*

$$\begin{gathered} \forall\, f = (f_1, f_2) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right)^2, \\ \|f\|^2_{H_0^1(\Omega)} \leq \tfrac{C_0}{2} e^{C_1|\omega|}\left[\|\mathcal{P}_\lambda f\|^2_{(L^2(\Omega))^2} + \int a(x)|f_1|^2\right]. \end{gathered} \tag{2.6}$$

Proof. Let Ω' be a small neighborhood of $\overline{\Omega}$. We extended Δ onto Ω' as the following: we extended the metric on Ω onto Ω' and we denoted so Δ the Laplacian onto Ω'. On neighborhood of $\partial\Omega$ in Ω', we choose the coordinates geodesic systems $x = (x', x_n)$, $x' \in \partial\Omega = \{x_n = 0\}$, $|x_n| = dist(x, \partial\Omega)$, $x_n > 0$ located define the interior of Ω. We assume $\Omega' \setminus \overline{\Omega} = \{x = (x', x_n),\ -\epsilon_0 < x_n < 0\}$ with ϵ_0 small, in a neighborhood of $\partial\Omega$, we have $\Delta = \partial_{x_n}^2 + S(x_n, x', \partial_{x'}) + L(x, \partial_x)$ where L (resp. S) is one order (resp. second order). There exists $\eta \in C^\infty(\Omega')$, $\eta > 0$ such that for $|x_n| < \epsilon_0$ we have $\eta^{-1} \circ \Delta \circ \eta = \partial_{x_n}^2 + R(x_n, x', \partial_{x'})$ where R is two order operator. We set $\tilde{\Delta} = \eta^{-1} \circ \Delta \circ \eta$ in Ω, $\tilde{\Delta} = \partial_{x_n}^2 + R(-x_n, x', \partial_{x'})$ in $x_n < 0$ and we denote $\tilde{a}$ the extension of a on Ω' define by $\tilde{a}(x', x_n) = a(x', -x_n)$ for $x_n < 0$.

Let Q the elliptic operator with Lipschitz coefficients on $\mathbb{R} \times \Omega'$ of matrix principal symbol

$$Q = -\left(\partial_s^2 + \tilde{\Delta}\right) I_\alpha - iK_{\tilde{a}}^\beta \partial_s. \tag{2.7}$$

Let $U \neq \emptyset$ is an open set with $\overline{U}$ is compact, $s_0 > 2$, $\Omega =]-s_0, s_0[\times U$ and $\varphi \in C_0^\infty(\Omega')$, $\varphi \equiv 1$ in a neighborhood of $\overline{\Omega}$. According to [8], we have the following lemma.

Lemma 2.2. *There exists $\theta \in]0,1[$ and $c > 0$ such that for all $v \in \left(H^2(]-s_0, s_0[\times\Omega')\right)^2$, we have the following estimate*

$$\|\varphi v\|_{(H^1(]-1,1[\times\Omega'))^2} \leq c\|v\|^\theta_{(H^1(V))^2}[\|Qv\|_{(L^2(V))^2} + \|v\|_{(H^1(\Omega))^2}]^{1-\theta} \tag{2.8}$$

where $V =]-s_0, s_0[\times\Omega'$.

Proof. The proof is a simple adaptation of the proof of the result given in [9]. For $f = (f_1, f_2) \in \left(H_0^1(\Omega \cap H^2(\Omega)\right)^2$, we set $g(s,x) = e^{is\lambda}\eta^{-1}f(x)$ if $x \in \Omega$, and $g = -g(s, x', -x_n)$ if $x_n < 0$. We have $g \in (H^2(V))^2$ and $Q(g)(s,x) = \eta^{-1}e^{is\lambda}\mathcal{P}_\lambda(f)(x)$ if $x \in \Omega$ and $Q(g)(s, x', x_n) = -Q(g)(s, x', -x_n)$ if $x_n < 0$. We have

$$\begin{aligned} \|f\|_{(H^1(\Omega))^2} &\leq \text{Cte } \|\varphi g\|_{H^1((-1,+1)\times\Omega')}, \\ \|Qg\|_{L^2(V)} &\leq \text{C}te\ e^{s_0|\omega|}\|\mathcal{P}_\lambda f\|_{(L^2(\Omega))^2}, \\ \|g\|_{(H^1(V))^2} &\leq (1 + |\omega|)\ e^{s_0|\omega|}\|f\|_{(H^1(U))^2}, \\ \|g\|_{(H^1(V))^2} &\leq (1 + |\omega|)\ e^{s_0|\omega|}\|f\|_{(H^1(\Omega))^2}, \end{aligned}$$

then (2.8) implies, with $s_1 > s_0$

$$\|f\|_{(H^1(\Omega))^2} \leq \text{ Cte } e^{\frac{s_1}{1-\theta}|\omega|} \left[\|\mathcal{P}_\lambda f\|_{(L^2(\Omega))^2} + \|f\|_{(H^1(V))^2}\right]. \tag{2.9}$$

Choosing an open set $U' \subset\subset \Omega$ such that $a_{|_{U'}} > 0$, $\overline{U} \subset\subset U'$ and $\chi \in C_0^\infty(U')$, equal to id near of $\overline{U}$. We have $(-\mathrm{I}d + D_\alpha)[\chi f] = \chi[(\lambda^2 \mathrm{i}d - \mathrm{id} + K_a^\beta)f - \mathcal{P}_\lambda f] + [D_\alpha, \chi]f$ then

$$\begin{aligned} \|f\|_{(H^1(U))^2} &\leq \text{Cte}\|(-\mathrm{id} + D_\alpha)[\chi f]\|_{(H^{-1}(\Omega))^2} \\ &\leq \text{Cte}\left[\|\mathcal{P}_\lambda f\|_{(L^2(\Omega))^2} + (1+|\lambda|^2)\|f\|^2_{(L^2(U'))^2}\right]. \end{aligned} \tag{2.10}$$

and we obtain (2.6) by writing (2.9) in (2.10).

Proof of Theorem 1.2

Let $\omega \in \mathbb{R}$, $|\omega| \leq 1$, $\sigma \in [0, \frac{1}{C_1}e^{-C_1|\omega|}]$. By (2.6), for all $f = (f_1, f_2) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right)^2$, we have

$$\|f\|^2_{(H_0^1(\Omega))^2} \leq Ce^{C_0|\omega|}\|\mathcal{P}_\lambda^{\alpha,\beta} f\|^2_{(L^2(\Omega)^2}, \tag{2.11}$$

or

$$\|f\|^2_{(H_0^1(\Omega))^2} \leq Ce^{C_0|\omega|}\int a|f_1|^2. \tag{2.12}$$

In the second case, the identity

$$(\mathcal{P}_\lambda^{\alpha,\beta} f, f) = \lambda^2\left(\|f_1\|_{L^2(\Omega)} + \|f_2\|_{L^2(\Omega)}\right)^2 + \int_\Omega |\nabla f_1|^2 + \alpha\int_\Omega |\nabla f_2|^2 + 2\lambda\int_\Omega a|f_1|^2$$

that implies

$$\left|2\omega\left[\int_\Omega a|f_1|\right] - 2\omega\sigma\|f\|^2_{(L^2(\Omega))^2}\right| \leq \|f\|_{(L^2(\Omega))^2}\|\mathcal{P}_\lambda^{\alpha,\beta} f\|_{(L^2(\Omega))^2},$$

using (2.12), we get

$$\|f\|^2_{H_0^1(\Omega)} \leq \frac{Ae^{C_0|\omega|}}{2|\omega|}\left[\|\mathcal{P}_\lambda^{\alpha,\beta} f\|_{(L^2)(\Omega)^2}\|f\|_{(H_0^1(\Omega))^2} + 2\sigma|\omega\|f\|^2_{(L^2(\Omega))^2}\right].$$

As (2.11) implies that the norm of $\mathcal{P}_\lambda^{-1}$ from $(L^2(\Omega))^2$ onto $(H_0^1(\Omega))^2$ is bounded by $Ce^{C|\omega|}$ and we obtain the results (1.12)and (1.13) from (2.5).

Let $\widetilde{H} = \oplus E_{\lambda_j}$ the space of finite linear combination vector of H in the characteristic subspace E_{λ_j}. We know that $\widetilde{H}$ is dense in H. Let $\widetilde{H}_0 = \oplus_{\lambda_j \neq 0} E_{\lambda_j}$, we have $\widetilde{H}_0 = \widetilde{H}$ if and only if $\partial M \neq \emptyset$ and $E_0 = \{(u_1, u_2) \ / \ u_1 = \mathrm{c}te, \ u_2 = 0\}$ if $\partial M = \emptyset$

Let $S = \frac{1}{i}A_a^{\alpha,\beta}$, $D = \left\{z \in \mathbb{C} \ / \ \mathrm{I}mz \notin [0, 2\|a\|_\infty]\right\}$. We define on $\widetilde{H}$ an inner product

$$\langle u, v\rangle = \int_\Omega \nabla^\alpha u_1 \cdot \overline{\nabla^\alpha v_1} + \int u_2 \cdot \overline{v_2}$$

induct a norm equivalent to $\|\cdot\|_H$ and we have

$$\mathrm{R}e\left(\langle(z - A_a^{\alpha,\beta})u, u\rangle\right) = \mathrm{R}ez\int_\Omega \left(|\nabla u_1|^2 + \alpha|\nabla v_1|^2\right)dx + \int_\Omega \left((\mathrm{R}ez + 2a)|u_2|^2 + \mathrm{R}ez\|v_2\|^2\right)dx$$

hence result to

$$\exists C > 0, \ \forall u \in \widetilde{H}_0, \ \forall z \in D, \qquad \|(z - S)^{-1}(u)\|_H \leq \frac{C}{\mathrm{d}ist(z, D^c)}\|u\|_H. \tag{2.13}$$

Moreover, for $u \in \widetilde{H}_0$ we have $z \mapsto (z-S)^{-1}(u)$ is a meromorphic map with the asymptotic behavior $O(\frac{1}{|z|})$ as $|z| \to +\infty$ and by the Theorem 1.2 (1.13), if $x \in \widetilde{H}_0$ we have $(\xi - S)^{-1}(x)$ is holomorphic at $\xi \in \left\{z \in \mathbb{C};\ \mathrm{I}mz < 2\epsilon_0 e^{-c_2|\mathrm{Re}z|}\right\}$ with $\epsilon_0,\ c_2^{-1} > 0$ small enough and satisfies on

$$\Gamma = [0, -d + 2i\epsilon_0 e^{-c_2 d}] \cup \left\{\xi \in \mathbb{C} \ / \ \xi = \eta + 2i\epsilon_0 e^{-c_2|\eta|},\ |\eta| \geq d\right\} \cup [0, +d + 2i\epsilon_0 e^{-c_2 d}]$$

$$\left\| (\xi - S)^{-1}(x) \right\|_H \leq C e^{c_3|\mathrm{Re}\xi|} \|x\|_H. \tag{2.14}$$

Then, there exists $d > 0$ such that for $x \in \widetilde{H}_0$ the operator $(z-S)^{-1}(x)$ is analytic in the region below the outline Γ. We consider $\psi \in C^\infty(\mathbb{R}_t)$, equal to 0 for $t < \frac{1}{3}$ and to 1 for $t > \frac{2}{3}$ and we set $u = \frac{1}{(1-S)^k}(\psi v)$ solution of

$$(\partial_t - S)\, u = \psi'(t) \frac{1}{(1-S)^k} v(t). \tag{2.15}$$

Let

$$u(t) = \int_0^t e^{(t-s)S} \psi'(s) \frac{1}{(1-S)^k} v(s) ds. \tag{2.16}$$

Let c_0 and c_1 are a later choose. We have

$$\begin{aligned} u(t) &= \int_0^t \int_\Gamma \int_{-\infty}^{+\infty} \sqrt{\frac{c_0}{\pi}} \psi'(s) e^{(t-s)\xi} \frac{1}{(1-\xi)^k} e^{-c_0(\lambda - \frac{\xi}{\sqrt{\ln t}})^2} v(s) d\lambda d\xi ds \frac{1}{(1-S)} \\ &= \int_s \int_\xi \int_{|\lambda| < c_1\sqrt{\ln t}} + \int_s \int_\xi \int_{|\lambda| \geq c_1\sqrt{\ln t}} \\ &= I_1 + I_2. \end{aligned}$$

We remark that the decomposition is similar to that of Lebeau [8] and Burq [2].

Estimation of I_1

The idea is to estimate I_1, we deform the outline of integration in ξ on the outline Γ. This requires to verify that the operator $(\xi - S)^{-1} e^{itS}$ is holomorphic with respect to ξ in the field is below the contour and it verifies an estimate of type

$$\left\| (\xi - S)^{-1} \right\|_{\mathcal{L}(H)} \leq C_1 e^{C_2|\mathrm{Re}\xi|}. \tag{2.17}$$

What can be deduced from (2.13). We know that for $Im\xi < 0$, the two families of operators

$$e^{is\xi}\left((\xi - S)^{-1} - i\int_0^s e^{i\sigma(S-\xi)} d\sigma\right) \ \text{ and } \ (\xi - S)^{-1} e^{isS}$$

coincide for $s = 0$ and satisfy the same differential equation

$$\partial_s \omega = i\xi\omega - e^{isS}.$$

Then, by the Gronowell lemma, the two families coincide for $Im\xi < 0$. The family in the left gives the analytic announcement and it is therefore in the integral defining I_1 deform the outline

ξ on the contour Γ. By the fact e^{isS} is bounded for all $s \geq 0$ and since the operator $(\xi - S)\, e^{isS}$ is uniformly bounded in H with respect to ξ and $s \in [0,1]$, for $t \geq 0$ we have

$$\Big\| \int_s \int \int_{\xi \in \Gamma_1 \cup \Gamma_2,\ |\lambda| < c_1\sqrt{\ln t}} \Big\| \leq C\|u_0\| \int_{z=0}^{1} e^{-(t-1)\epsilon_0 e^{-ad}z} dz \leq \frac{c\|u_1\|}{t-1}. \tag{2.18}$$

By (2.17), we have for $t > 1$,

$$\begin{aligned} &\Big\| \int_s \int_{\xi \in \Gamma_3} \int_{|\lambda| < c_1\sqrt{\ln t}} \Big\| \\ \leq\ & C\sqrt{c_0} \int_{-\infty}^{+\infty} \int_{|\lambda| < c_1\sqrt{\ln t}} e^{-(t-1)\epsilon_0 e^{-a|\eta|} + A|\eta| - c_0(\lambda - \frac{\eta}{\sqrt{\ln t}})^2} d\eta d\lambda \|u_1\|. \end{aligned} \tag{2.19}$$

Let c_2 such that $c_2 a < 1$ and $\varphi = -(t-1)\epsilon_0 e^{-a|\eta|} + A|\eta| - c_0(\lambda - \frac{\eta}{\sqrt{\ln t}})^2$. Then, we have

$$|\eta| \leq c_2 \ln t \Rightarrow \varphi \leq c_2 A \ln t - (t-1)\epsilon_0 t^{-c_2 a}. \tag{2.20}$$

We choose $c_1 \in]0, c_2[$. Then, there exists $\delta > 0$ such that if $|\lambda| < c_1\sqrt{\ln t}$ and if $|\eta| > c_2 \ln t$ then

$$(\lambda - \frac{\eta}{\sqrt{\ln t}})^2 \geq \delta(\lambda^2 + (\frac{\eta}{\sqrt{\ln t}})^2), \tag{2.21}$$

let

$$\varphi \leq A|\eta| - c_0\delta(\lambda^2 + \frac{\eta}{\sqrt{\ln t}})^2). \tag{2.22}$$

We choose $c_0 > \frac{A}{\delta c_2} + 1$. For $\epsilon > 0$ we have

$$\int_{|\eta| > c_2 \ln t} e^{A|\eta| - c_0\delta(\frac{\eta}{\sqrt{\ln t}})^2} = \mathcal{O}\left(e^{-\epsilon \ln t}\right). \tag{2.23}$$

By (2.18), (2.19), (2.20) and (2.23),

$$\|I_1\| \leq Ct^{-\epsilon}\|u_1\|. \tag{2.24}$$

Estimation of I_2

Let

$$\begin{aligned} J(u) \ =\ & \int_0^1 \int \int_{\substack{\mathrm{Im}\xi = -\frac{1}{2} \\ |\lambda| \geq c_1\sqrt{\ln t}}} \psi'(s) e^{i(u-s)\xi} \frac{1}{(1-i\xi)^k} . \frac{1}{\xi - B} \\ . \ & v(s)\sqrt{\tfrac{c_0}{\pi}} e^{-c_0(\lambda - \frac{\xi}{\sqrt{\ln t}})^2} ds d\xi d\lambda. \end{aligned} \tag{2.25}$$

For $t \geq 1$, we have $J(t) = I_2(t)$ and for all $u \in \mathbb{R}$,

$$(\partial_t - iS)J(u) = \int_0^1 \int \int_{\substack{\mathrm{Im}\xi = -\frac{1}{2} \\ |\lambda| \geq \sqrt{\ln t}}} \psi'(s) \frac{ie^{i(u-s)\xi}}{(1-i\xi)^k} v(s) \sqrt{\frac{c_0}{\pi}} e^{-c_0(\lambda - \frac{\xi}{\sqrt{\ln t}})^2} ds d\xi d\lambda = K(u), \tag{2.26}$$

that implies

$$J(t) = e^{itS} J(0) + \int_0^t e^{i(t-s)S} K(s) ds. \tag{2.27}$$

Now we are going that $J(t)$ is bounded in norm in H, we use that e^{isS} is a contraction of H for $s \geq 0$ and separately $K(u)$ for $u \geq 0$, $J(0)$ and $\int_0^1 \|K(u)\|du$ (see [2]).

For $u \in [1, t]$, we show that the outline in ξ given in (2.26), is deformed in the outline given by I$m\xi = \sqrt{\ln t}$, that give for $k > 1$ and s$upp\psi \subset [\frac{1}{3}, \frac{2}{3}]$,

$$\|K(u)\| \leq \int_{-\infty}^{+\infty} e^{-(u-\frac{2}{3})\sqrt{\ln t}} \frac{1}{(1+|\xi|)^k} d\xi \|u_0\| \leq C_k e^{-\sqrt{\ln t}/3} \|u_0\|. \tag{2.28}$$

Then we bound $J(0)$. We treat such a contribution (2.25) of the region. For that is deformed according to [8], the integral in ξ on the contour $\Gamma = \Gamma^+ \cup \Gamma^-$, where

$$\Gamma^+ = \left\{ z = 1 + \eta - i\sqrt{\ln t}; \eta > 0 \right\}$$

$$\Gamma^- = \left\{ z = 1 + \eta - \frac{1}{2}i;\ \eta \leq 0 \right\} \cup \left[1 - \frac{1}{2}i, 1 - i\sqrt{\ln t} \right].$$

For $\xi \in \Gamma^-$, by (2.13), we have for all $s \in [0, 1]$ and for all $\lambda \in [c_1\sqrt{\ln t}, +\infty[$ there exist $\delta > 0$

$$\left\| e^{-is\xi} \frac{v(s)}{(1-i\xi)^k} \cdot \frac{1}{(\xi - B)} \sqrt{\frac{c_0}{2\pi}} e^{-c_0(\lambda - \frac{\xi}{\sqrt{\ln t}})^2} \right\| \leq \frac{C}{(1+|\xi|)^k} e^{-\delta(\lambda^2 + \frac{\xi^2}{\ln t})} \|u_0\|.$$

The contribution de Γ^- to $J(0)$ is bounded in norm by

$$C\sqrt{\ln t} \int_{\lambda \geq c_1\sqrt{\ln t}} e^{-\delta\lambda^2} \|u_0\| = \mathcal{O}\left(e^{-\epsilon \ln t}\right) \|u_0\|. \tag{2.29}$$

For $\xi \in \Gamma^+$ and $s \in [\frac{1}{3}, \frac{2}{3}]$ we have

$$\left\| e^{-is\xi} \frac{V(s)}{(1-i\xi)^k} \cdot \frac{1}{(\xi - B)} \sqrt{\frac{c_0}{2\pi}} e^{-c_0(\lambda - \frac{\xi}{\sqrt{\ln t}})^2} \right\| \leq e^{-\sqrt{\ln t}/3} \frac{C}{(1+|\eta|)^k} e^{-\delta(\lambda^2 + \frac{\xi^2}{\ln t})} \|u_0\|,$$

So, since the contribution of Γ^+ to $J(0)$ is bounded in norm by

$$Ce^{-\sqrt{\ln t}/3} \|u_0\|. \tag{2.30}$$

The contribution to $J(0)$ of the region $\lambda < -c_1\sqrt{\ln t}$ is bounded by the same way.

Finally, it remains to bounding

$$\int_0^1 \|K(u)\|du \leq \left(\int_0^1 \|K(u)\|^2 du \right)^{\frac{1}{2}}. \tag{2.31}$$

By the Plancherel identity,

$$\begin{aligned} \int_{-\infty}^{+\infty} \|K(u)\|^2 du &= C \int_{-\infty}^{+\infty} \left\| \frac{i}{(1-i\xi)^k} \widehat{v\psi'}(\xi) \int_{|\lambda| \geq c_1\sqrt{\ln t}} e^{c_0(\lambda - \xi/\sqrt{\ln t})^2} d\lambda \right\|^2 d\xi \\ &= \int_{-\infty}^{+\infty} \|H(\xi)\|^2 d\xi, \end{aligned} \tag{2.32}$$

for $\xi > \frac{1}{2}c_1 \ln t$,

$$\|H(\xi)\| = \Big\| \int_{|\lambda|>c_1\sqrt{\ln t}} \frac{1}{(1-i\xi)^k} e^{-c_0(\lambda-\xi/\sqrt{\ln t})^2} d\lambda \widehat{v\psi'}(\xi) \Big\| \leq \frac{C}{(\ln t)^k} \|\widehat{v\psi'}(\xi)\| \tag{2.33}$$

and for $\xi \leq \frac{1}{2}c_1 \ln t$,

$$\|H(\xi)\| \leq \int_{|\lambda|>c_1\sqrt{\ln t}} e^{-\delta(\lambda^2+\xi^2/\ln t)} d\lambda \|v\hat{\psi}'(\xi)\| \leq Ce^{-\epsilon \ln t} \|v\hat{\psi}'(\xi)\|. \tag{2.34}$$

Then, by (2.31), (2.32), (2.33), (2.34) and

$$\int_{\infty}^{+\infty} \|\widehat{v\psi'}(\xi)\| = \int \|\psi' v(s)\|^2 ds \leq C \int_0^1 |\psi'(s)|^2 ds \|v_0\|^2$$

(we recall that $v(s) = e^{isS} v_0$ implies $\|v(s)\| \leq \|u_0\|$), we have

$$\int_0^1 \|K(u)\| du \leq C \left(\frac{1}{(\ln t)^k} + e^{-\epsilon \ln t} \right) \|U_0\|. \tag{2.35}$$

By (2.27), (2.28), (2.29), (2.30) and (2.35) we obtain

$$\|I_2\| \leq \frac{C}{(\ln t)^k} \|u_1\|,$$

hence the estimate of I_2.

3 Proof of Theorem 1.3

First, we prove $\varrho \leq 2\min(-D(0), C(\infty))$. Let $\lambda_j \in \mathrm{S}p(A_a^{\alpha,\beta}) \setminus \{0\}$ there exists $\underline{u} = (u_0, u_1) = \big((u_1^0, u_2^0), (u_1^1, u_2^1)\big) \in E_{\lambda_j}$ such that $A_a^{\alpha,\beta} \underline{u} = \lambda_j \underline{u}$ and $u(t,x) = e^{t\lambda_j} u_0$ satisfy (1.1)-(1.6).

As $E(u,t) = e^{2t\,\mathrm{Re}\lambda_j} E(u,0)$ and $E(u,0) = \frac{1}{2}\int_\Omega |\lambda_j|^2 |u_0|^2 + |\nabla_x u_1^0|^2 + \alpha |\nabla_x u_2^0|^2 \neq 0$, we have $\varrho \leq -2\mathrm{Re}\lambda_j$ then $\varrho \leq -2D(0)$. We assume that $\varrho = 2C(\infty) + 4\eta$ with $\eta > 0$ there exists $B > 0$ such that for all $u \in H$ and for all $t \geq 0$ we have the following estimate

$$E(u,t) \leq Be^{-(\varrho-\eta)t} E(u,0). \tag{3.1}$$

Let t fixed such that $Be^{-(\varrho-\eta)t} < e^{-(\varrho-2\eta)t}$, we have $C(t) \leq C(\infty) = \frac{\varrho}{2} - 2\eta$, then there exists $i \in \{1,2\}$ such that $\frac{1}{t}\int_0^t a(x_i(s,\rho_0))ds \leq C(\infty) = \frac{\varrho}{2} - 2\eta$, and there exists $\rho_0 \in T\overline{\Omega}$ with $C(t) < \frac{\varrho}{2} - \eta$ has left a little disturbing ρ_0, we can assume that the outcome of generalized geodesic ρ_0 did as points of intersection with transverse $\partial\Omega$ on $[-2t, +2t]$. by constructing geometric standard optical near γ, we can construct a solution u of (1.1) - (1.6) such that $E(u,0) = 1$ and $E(u,t) > e^{-(\varrho-2\eta)t}$ which contradicts (3.1), so we have $\varrho \leq 2C(\infty)$. To check $\varrho \geq 2\min\{-D(0), C(\infty)\}$, we prove the following lemma :

Lemma 3.1. *For all $T > 0$ and for $\varepsilon > 0$ there exists $C(\varepsilon, T)$ such that for all solution of the evolution equation (1.7) we have*

$$E(u,T) \leq (1+\varepsilon)e^{-2TC(T)} E(u,0) + C(\epsilon,\ T) \|(u_0, u_1)\|_{(L^2(\Omega)\times H^{-1}(\Omega))^2} \tag{3.2}$$

Proof : If (3.2) is false then there exists $T > 0$ and $\varepsilon > 0$ such that for all $k \geq 1$ there exists U_k satisfy

$$E(u_k, T) \geq (1+\varepsilon)e^{-2TC(T)}E(u_k, 0) + k\|(u_0^k, u_1^k)\|_{(L^2(\Omega)\times H^{-1}(\Omega))^2}, \quad E(u_k, 0) = 1. \tag{3.3}$$

Then u_k is bounded in $\left(H^1(I \times \Omega)\right)^2$, $I = [-2T, 2T]$ converges weakly to zero because $\|(u_0^k, u_1^k)\|^2_{(L^2(\Omega)\times H^{-1}(\Omega))^2} \leq \frac{1}{k}E(u_k, T) \leq \frac{1}{k}E(u_k, 0) = \frac{1}{k}$.

Let μ the measure positive onto SZ (see section 4 (4.6)) associated to extract sequence of u_k. Let $\eta \in]0, T[$. As the energy function is decreasing, for all $\varphi \in C_0^\infty(]0, \eta[)$ we have by (3.2)

$$\int_{T-\eta}^{T} \varphi(T-t)E(u_k, t)dt \geq (1+\varepsilon)e^{-2TC(T)} \int_0^\eta \varphi(t)E(u_k, t)dt \tag{3.4}$$

hence

$$\mu((SZ) \cap (t \in]T-\eta, T[)) \geq (1+\varepsilon)e^{-2TC(T)}\mu((SZ) \cap (t \in]0, \eta[)). \tag{3.5}$$

Gold by the propagation Theorem we have

$$\mu((SZ) \cap (t \in]T-\eta, T[)) \leq e^{-2(T-\eta)C(T-\eta)}\mu((SZ) \cap t \in]0, \eta[). \tag{3.6}$$

Since $\mu((SZ) \cap (t \in]0, \eta[)) > 0$ (because if $u_k \to 0$ in $\left(H^1(]0, \eta[\times\Omega)\right)^2$ that implies $u_k \to 0$ in $\left(H^1(J \times \Omega)\right)^2$ for all J this give a contradiction with the fact $E(u_k, 0) = 1$). Since $C(t)$ defined in (1.16) as an infimum over a compact of a continuous function is continuous at $t > 0$, (3.6) contradicts (3.5) to η small, hence the Lemma. □

Let $A_a^{\alpha,\beta,*}$ the adjoint of $A_a^{\alpha,\beta}$, we denote by $E^*_{\lambda_j}$ the characteristic subspace of $A_a^{\alpha,\beta,*}$ associated of the eigenvalue $\overline{\lambda}_j$. Let $H = (H_0^1(\Omega))^2 \oplus (L^2(\Omega))^2$ and for $N \geq 1$

$$H_N = \left\{x \in H \;/\; (x, y)_H = 0, \forall y \in \oplus_{|\lambda_j| \leq N} E^*_{\lambda_j}\right\}. \tag{3.7}$$

Then H_N is invariant under $e^{tA_a^{\alpha,\beta}}$ (indeed let $x \in H_N$, (y_k) a basis of the vectorial space $\oplus_{|\lambda_j| \leq N} E^*_{\lambda_j} \subset D(A_a^{\alpha,\beta,*})$ we have $\frac{d}{dt}\left(e^{tA_a^{\alpha,\beta}}x|A_a^{\alpha,\beta,*}y_k\right) = \sum c_{k,l}\left(e^{tA_a^{\alpha,\beta}}x \mid y_l\right)$ then $\left(e^{tA_a^{\alpha,\beta}}x|y_l\right) \equiv 0$). Let $H^* = (L^2(\Omega))^2 \oplus (H^{-1}(\Omega))^2$ and Φ_N the norm of injection from H_N onto H^*. We have $\lim\limits_{N\to+\infty} \Phi_N = 0$, indeed, we assume that there exists $u_N \in H_N$, $\|u_N\|_H = 1$ and $\|u_N\|_{H^*} \leq \lim\limits_{N\to+\infty} \Phi_N = \rho > 0$. We can assume that u_N converges weakly to u in H, and strongly in H^*. We have $\|u\|_{H^*} \geq \rho$ and $(u, y)_H = 0$, $\forall y \in E^*_{\lambda_j}$, $\forall j$. This is impossible by the fact that $\overline{\oplus E^*_{\lambda_j}} = H$, since $-A_a^{\alpha,\beta,*}$ is a perturbation bounded of self-adjoint A_0.

We can assume $2\min\{-D(0), C(\infty)\} > 0$, let $\eta > 0$ small and $\widetilde{\beta}$ define by $\widetilde{\beta} + \eta = 2\min\{-D(0), C(\infty)\}$. Choosing $T > 0$ such that $4|C(\infty) - C(T)| < \eta$, $2\log 3 < \eta T$ and N such that $C(1,T)\Phi_N^2 \leq e^{-2TC(T)}$. By Lemma 3.1, identifying $u \in H$ to the solution of (1.1)-(1.6) with initial data u

$$\forall u \in H_N, \quad E(u, T) \leq 3e^{-2TC(T)}E(u, 0) \tag{3.8}$$

then H_N is stable by the evolution

$$\forall u \in H_N, \ \forall k \quad E(u, kT) \leq 3e^{-kT[2C(T) - \frac{\log 3}{T}]}E(u, 0) \leq e^{-kT\widetilde{\beta}}E(u, 0) \tag{3.9}$$

as the energy decreases

$$\forall u \in H_N,\ \ \forall t \geq 0,\ \ E(u,t) \leq Be^{-\widetilde{\beta} t}E(u,0),\ \ B = e^{\widetilde{\beta} T}. \tag{3.10}$$

Let $\widetilde{\gamma}$ the contour encircling $\{\lambda_j \mid |\lambda_j| \leq N\}$ in the direct sense and $\Pi = \frac{1}{2i\pi}\int_{\widetilde{\gamma}} \frac{d\lambda}{\lambda - A_a^{\alpha,\beta}}$ the spectral projector on $\oplus_{|\lambda_j|\leq N} E_{\lambda_j} = W_N$; then Π^* is the spectral projector of $A_a^{\alpha,\beta,*}$ on $\oplus_{|\lambda_j|\leq N} E^*_{\lambda_j}$. Then for all $u \in H$, we have

$$u = v + w,\ \ v = \Pi u \in W_N,\ \ w = (\mathrm{i}d - \Pi)u \in H_N. \tag{3.11}$$

As W_N is a finite dimensional and $\widetilde{\beta} < -2D(0)$, We have

$$\exists C,\ \forall u \in W_N,\ \forall t \geq 0,\ E(u,T) \leq Ce^{-\widetilde{\beta} t}E(u,0). \tag{3.12}$$

The decomposition (3.11) is continuous, there exists C_0 such that $E(v,0)+E(w,0) \leq C_0E(u,0)$ and by (3.10), (3.11) and (3.12) implies that $\varrho \geq \widetilde{\beta}$ this achieve the Proof of 1. and 2. of Theorem 1.3 result the fact that $E_{\lambda_j} \subset H_N$ if $|\lambda_j| > N$ (since the projector Π is equal to zero on E_{λ_j} and by (3.10), if $C(\infty) > 0$ and $\widetilde{\beta} < 2C(\infty)$,

for N large enough

$$|\lambda_j| > N \Rightarrow 2\mathrm{R}e\lambda_j \leq -\widetilde{\beta}. \tag{3.13}$$

Then $D(\infty) \leq -C(\infty)$, hence 2. (since $D(\infty) \leq 0$ treats the case $C(\infty) = 0$).

4 Geometric and construction of measure

Near ∂M ($M = \Omega\times\mathbb{R}_+$), we choose the geodesic coordinate system : $(x', x_n) \in \partial M\times[0, r_0] \rightarrow$; $x_n =$ dist$(x, \partial M) =$ dist(x, x') where $r_0 > 0$ small enough. In the system, the principal symbol of $-\Delta$ is $\xi_n^2 + R(x_n, x', \xi')$ and $R_0(x', \xi') = R_{|_{x_n=0}}$ is the metric form on $T^*\partial M$. We denote $\mathcal{G}$ the operator space Q of the form $Q = Q_i + Q_\partial$ where Q_i is a classical pseudo- differential operator onto $\mathbb{R}_t\times\Omega$ with compact support in $\mathbb{R}_t\times$intΩ and Q_∂ is a tangential pseudo differential operator with compact support near $\mathbb{R}\times\partial\Omega$ (i.e $Q_\partial(t, x', x_n) = Q_\partial(x_n)(f)(\cdot, x_n)$ where $Q_\partial(x_n)$ is a C^∞ p.d.o onto $\mathbb{R}_t \times \partial\Omega$ and $Q_\partial = \psi Q_\partial \psi$ with $\psi(t, x_n) \in C_0^\infty(\mathbb{R} \times (-r_0, r_0))$. We denote $\mathcal{G}^{(s)}$ the element of degree s in $\mathcal{G}$ and $\mathcal{G}_{sym}$ the subset of element in $\mathcal{G}$ with self-adjoint principal symbol.

Let $X = \mathbb{R}_t \times \overline{\Omega}$, ${}^bTX$ of the tangent bundle of rung dim$X$, the sections of which are the tangent vector fields to $\mathbb{R} \times \partial\Omega$, ${}^bT * X$ the dual bundle (of the cotangent compressed bundle of Melrose) and $j : T^*X \rightarrow^b T * X$ the canonical maps. Near the ∂X, ${}^bTX$ is generate by the fields ∂_t, $\partial_{x'}$, $x_n\partial_{x_n}$ and

$$j(t, x', x_n; \tau, \xi', \xi_n) = (t, x', x_n; \tau, \xi', v = x_n\xi_n).$$

We denote

$$\mathcal{P}_a^{\alpha,\beta} = \partial_t^2 - D_\alpha + K_a^\beta$$

with principal symbol

$$P^\alpha = \begin{pmatrix} -\tau^2 + |\xi|^2 & 0 \\ 0 & -\tau^2 + \alpha|\xi|^2 \end{pmatrix}$$

we notice that the determinate of the principal symbol is given by [11]:

$$p(t, x; \tau, \xi) = \left(|\xi|^2 - 1\right)\left(\alpha|\xi|^2 - 1\right). \tag{4.1}$$

This leads to two bicharacteristic families in the characteristic set of $\mathcal{P}_a^{\alpha,\beta}$, Char$\mathcal{P}_a^{\alpha,\beta}$, namely those of the symbols

$$p_1(t,x;\tau,\xi) = |\xi|^2 - \tau^2 \quad \text{and} \quad p_\alpha(t,x;\tau,\xi) = \alpha|\xi|^2 - \tau^2,$$

1, $\sqrt{\alpha}$ are respectively the velocity of propagation.

Let $M = \mathbb{R}_+ \times \Omega$. In the interior, i.e. in $T^*(\mathbb{R} \times \Omega)$ wavefront sets propagate independently along the null bicharacteristic of each one of the two families. As the boundary, however, one has to consider the inverse images of the characteristic points, in Char$\mathcal{P}_a^{\alpha,\beta} = p_1^{-1}\{0\} \cup p_\alpha^{-1}\{0\}$ with respect to the projection

$$\Pi : T^*(\overline{M})_{|\partial M} \to T^*(\partial M).$$

We will illustrate what happens at the boundary point $(t,x) \in \partial M$. Let $(\tau,\eta) \neq (0,0)$ be a tangential direction to ∂M at (t,x); i.e. $\eta \cdot \nu(x) = 0$, $\nu(x)$ being the exterior normal to Ω at x. With the assumption $\alpha \neq 1$, we can consider (τ,η) as an element of $T^*_{(t,x)}(\partial M)$, and to look for its inverse image is both characteristic sets means to look for $\lambda \in \mathbb{R}$ such that

$$p_{1,\alpha}\left(t,x;\tau,\eta + \lambda\nu(x)\right) = 0. \tag{4.2}$$

Because of

$$p_{1,\alpha}\left(t,x;\tau,\eta + \lambda\nu(x)\right) = c_{1,\alpha}^2(|\eta|^2 + \lambda^2) - \tau^2,$$

this requires

$$\lambda = \pm\sqrt{\tau^2 - |\eta|^2} \quad \text{or} \quad \lambda = \pm\sqrt{\frac{\tau^2}{\alpha} - \eta^2}. \tag{4.3}$$

Hence, for the existence of such real λ, one of the two relations

$$r_1 = \tau^2 - \eta^2 \geq 0 \quad \text{or} \quad r_\alpha = \tau^2 - \alpha\eta^2 \geq 0$$

must be fulfilled. From the geometrical point of view there are some possibilities for a tangential direction $\xi = (\tau,\eta) \neq (0,0)$, with different number of inverse images with respect to the projection. We can introduce the transversal manifold :

$$\text{Char}\mathcal{T} = \text{Char}\mathcal{T}_\Omega \cup \text{Char}\mathcal{T}_{\partial\Omega},$$

$$\text{Char}\mathcal{T}_\Omega = \left\{(x,t;\xi,\tau)\ ;\ \tau^2 - c_\alpha^2|\xi|^2 = 0,\ t > 0\right\},$$

$$\text{Char}\mathcal{T}_{\partial\Omega} = \left\{(y,t;\xi,\tau)\ ;\ y \in \partial\Omega,\ y \in \partial\Omega, t > 0,\ r_\alpha \geq 0\right\}$$

and the longitudinal manifold of the wave coupled system is

$$\text{Char}\mathcal{L} = \text{Char}\mathcal{L}_\Omega \cup C har\mathcal{L}_{\partial\Omega},$$

$$\text{Char}\mathcal{L}_\Omega = \left\{(x,t;\xi,\tau)\ ;\ \tau^2 - c_1^2|\xi|^2 = 0,\ t > 0\right\},$$

$$\text{Char}\mathcal{L}_{\partial\Omega} = \left\{(y,t;\xi,\tau)\ ;\ y \in \partial\Omega,\ y \in \partial\Omega, t > 0,\ r_1 \geq 0\right\},$$

the characteristic manifold of the system is

$$\text{Char}P = \text{Char}P_\Omega \cup \text{Char}P_{\partial\Omega}$$

and the assumption on the coupled wave $(\alpha \neq 1)$ one obtains

$$\text{Char}P_\Omega = C har\mathcal{T}_\Omega \cup \text{Char}\mathcal{L}_\Omega$$

and

$$\text{Char}P_{\partial\Omega} = \text{Char}\mathcal{T}_{\partial\Omega} \quad \text{if} \ \ \alpha > 1$$

either

$$\text{Char}P_{\partial\Omega} = \text{Char}\mathcal{L}_{\partial\Omega} \quad \text{if} \ \ 0 < \alpha < 1.$$

Finally, we recall that Char$P$ is endowed with a generalized bicharacteristic flow

Definition 4.1. Let $\eta \in T^*\partial\Omega$. We say that

(i) η is a elliptic (or $\eta \in \mathcal{E}$) if and only if $\eta \notin$ (Char$P)_{\partial\Omega}$.

(ii) η is a hyperbolic for the longitudinal wave (or $\eta \in \mathcal{H}_L$) if and only if $r_1 > 0$.

(iii) η is a glancing for the longitudinal wave (or $\eta \in \mathcal{G}_L$) if and only if $r_1 = 0$.

(iv) η is a hyperbolic for the transversal wave (or $\eta \in \mathcal{H}_T$) if and only if $r_\alpha > 0$.

(v) η is a glancing for the transversal wave (or $\eta \in \mathcal{H}_T$) if and only if $r_\alpha = 0$.

We are going now to make a description of a generalized bicharacteristic path and refer to [8] for more details. The generalized bicharacteristic flow lives in Char$P \subset T^*\overline{M}$ and for $\rho \in$ Char$P$, we denote by $G(s,\rho)$ the generalized bicharacteristic path starting from ρ. Since Char$P$ is the disjoint union of CharP_Ω, $\mathcal{H}_T$ and $\mathcal{G}_T$ if $\alpha > 1$ or CharP_Ω, $\mathcal{H}_L$ and $\mathcal{G}_L$ if $\alpha < 1$. We shall consider separately the case where ρ belongs to each one of these sets. Moreover all the description below holds for $|s|$ small.

Case 1. $\rho \in$ CharP_Ω

Here $\rho = (x,t;\xi,\tau)$ where $x \in \Omega$, $t \in (0,T)$, $p(x,t;\xi,\tau) = 0$. Then for $|s|$ small, we have

$$G(s,\rho) = (x(s), t(s), \tau, \xi) \subset T^*(\mathbb{R} \times \Omega)$$

where $(x(s),\xi)$ is the characteristic starting from the point (x,ξ) of

- P_1 if $\rho \in$ Char$\mathcal{L}_\Omega$,
- P_α if $\rho \in$ Char$\mathcal{T}_\Omega$.

Case 2. $\rho \in$ (Char$P)_{\partial\Omega}$ (i.e $0 \leq r_\alpha$) Here $\rho = (x(s), t(s), \eta(s), \tau(s))$ where $x \in \partial\Omega$, $t \in (0,T)$ and the equation $p(x,t,\eta+\xi_n,\tau) = 0$ has roots $\xi_n = \lambda\nu(x)$ described in (4.3).

For $s > 0$ (resp. $s < 0$) let $G^+(s,\rho) = (x^+(s), t(s), \xi^+, \tau(s))$ (resp. $G^+(s,\rho) = (x^-(s), t(s), \xi^-, \tau(s))$ be the outgoing (resp. incoming) bicharacterestic of P. The generalized bicharacteristic path is such that $G(0,\rho) = \rho$ and

$$G(s,\rho) = \begin{cases} G^+(s,\rho) & 0 < s < \epsilon \\ G^-(s,\rho) & -\epsilon < s < 0 \end{cases}$$

Four possibilities may occur

(i)

$$\begin{cases} x^+(s) &= x + 2c_\alpha^2 s\xi^+, \ \ 0 < s < \epsilon, \\ x^-(s) &= x + 2c_\alpha^2 s\xi^-, \ \ -\epsilon < s < 0, \end{cases}$$

where $\xi^+ = \eta - \frac{\sqrt{r_T}}{c_T}\nu(x)$ and $\xi^- = \eta + \frac{\sqrt{r_T}}{c_T}\nu(x)$.

In particular, if $0 < r_\alpha$, one has $x(s) \in \Omega$ for small $|s| \neq 0$.

(ii) If $0 \leq r_1$ (i.e $\eta \in G_L \cup \mathcal{H}_L \subset \mathcal{H}_T$):

i -

$$\begin{cases} x^+(s) &= x + 2c_1^2 s\xi^+, \quad 0 < s < \epsilon, \\ x^-(s) &= x + 2c_1^2 s\xi^-, \quad -\epsilon < s < 0, \end{cases}$$

where $\xi^+ = \eta - \frac{\sqrt{r_1}}{c_1}\nu(x)$ and $\xi^- = \eta + \frac{\sqrt{r_1}}{c_1}\nu(x)$.

ii -

$$\begin{cases} x^+(s) &= x + 2c_1^2 s\xi^+, \quad 0 < s < \epsilon, \\ x^-(s) &= x + 2c_1^2 s\xi^-, \quad -\epsilon < s < 0, \end{cases}$$

where $\xi^+ = \eta - \frac{\sqrt{r_1}}{c_1}\nu(x)$ and $\xi^- = \eta + \frac{\sqrt{r_\alpha}}{c_\alpha}\nu(x)$.

iii -

$$\begin{cases} x^+(s) &= x + 2c_\alpha^2 s\xi^+, \quad 0 < s < \epsilon, \\ x^-(s) &= x + 2c_1^2 s\xi^-, \quad -\epsilon < s < 0, \end{cases}$$

where $\xi^+ = \eta - \frac{\sqrt{r_\alpha}}{c_\alpha}\nu(x)$, $\xi^- = \eta + \frac{\sqrt{r_1}}{c_1}\nu(x)$,

and the manifold characteristic $\text{Char}(\mathcal{P}_a^{\alpha,\beta}) = \{(t, x', x_n; \tau, \xi', \xi_n); \det p = 0\}$. We set

$$Z = j(\text{ Char}(\mathcal{P}_a^{\alpha,\beta})), \quad \hat{Z} = Z \cup j(T^*X_{|x_n=0}). \tag{4.4}$$

We have $Z_{|x_n=0} = \{(t, x', 0; \tau, \xi', 0);\ |\xi'| \leq |\tau| \text{ or } \sqrt{\alpha}|\xi'| \leq |\tau|\}$ and $\hat{Z}_{|x_n=0} = \{(t, x', 0\ ; \tau, \xi', v = 0)\} = T^*(\mathbb{R} \times \partial M) = Z_{|x_n=0} \cup \mathcal{E}$ where $\mathcal{E}$ is the boundary of elliptic region.

As $x_n \in [0, r_0]$ we have $p = \xi_n^2 I_\alpha + R - \tau^2$ id, R is nondegenerate positive matrix we have

$$(t, x', x_n; \tau, \xi', v) \in \hat{Z}, \ \ x_n \in [0, r_0] \Rightarrow \begin{cases} |v| & \leq \quad x_n|\tau| \\ & \text{or} \\ \sqrt{\alpha}|v| & \leq \quad x_n|\tau|. \end{cases} \tag{4.5}$$

We obtain that Z and $\hat{Z}$ are closed conic sets in T^*X. We denote $S\hat{Z}$ and SZ the spherical quotients spaces

$$S\hat{Z} = (\hat{Z} \setminus X)/\mathbb{R}_+^*, \quad SZ = (\hat{Z} \setminus X)/\mathbb{R}_+^* \tag{4.6}$$

which are a locally compact metric spaces. For $Q \in \mathcal{G}^0$ with principal symbol $q = \sigma(Q)$ and we define the function

$$\begin{cases} \kappa(q) \in C^0\left(S\hat{Z}, \text{ end}(\mathbb{C})\right) \\ \rho \in \hat{Z} \setminus X \ \ \kappa(q)(\rho) = q(j^{-1}(\rho)). \end{cases} \tag{4.7}$$

(which is well defined because q is homogeneous and has $\kappa(q)(x', x_n, \xi', \xi_n) = q(x', x_n, \xi', \frac{\xi_n}{x_n})$ for $x \neq 0$ and q is independent of ξ for x sufficiently small.) By (4.7)the set

$$\{\kappa(q),\ q = \sigma(Q),\ Q \in \mathcal{G}^0\}$$

is locally dense in $C^0\left(S\widehat{Z}, \text{end}(\mathbb{C}^2)\right)$ where $C^0\left(S\widehat{Z}, \text{end}(\mathbb{C}^2)\right)$ is provided with the topology of uniform convergence on compact. For $G \in G^0$, and I is an open bounded real interval and $u(x,t) \in \left(H^1(I \times \Omega)\right)^2$ solution of $P_a^{\alpha,\beta}u = 0$ near the boundary, we have $u \in C^k\left(x_n \leq 0; H^{\frac{1}{2}-k}\right)$ with $k \in \mathbb{N}$. If $Q \in \mathcal{G}_I^0$ (i.e, supported in I and zero degree), Q is a bounded operator onto $\left(L^2(I \times \Omega)\right)^2$, $\left(H^1(I \times \Omega)\right)$ and the commutators $[\nabla_x^\alpha, Q]$, $[\partial_t, Q]$ are in $\mathcal{G}_I^0$. We set

$$\varphi(Q,u) = (Qu,u)_{(H^1)^2} = (\nabla_x^\alpha Qu, \nabla_x^\alpha u)_{(L^2)^2} + (\partial_t Qu, \partial_t u)_{(L^2)^2}\,. \tag{4.8}$$

By the integration by parts

$$\begin{aligned} \varphi(Q,u) &= \int_{\mathbb{R}_t\times\partial\Omega} Qu\cdot\partial_\nu^\alpha \overline{u} + 2(\partial_t Qu, \partial_t u)_{(L^2)^2} \\ &- (Qu, K_a^\beta \partial_t u)_{(L^2)^2} + (Qu,u)_{(L^2)^2} \end{aligned} \tag{4.9}$$

where $\partial_\nu^\alpha u = (\partial_\nu u_1, \alpha\partial_\nu u_2)$

According [3], we recall some results useful in this work. We denote $\mathcal{M}^+$ the spaces of Borel measure μ onto $S\hat{Z}$ with $\mathbb{C}$ value Hermitian positive on $\mathbb{C}^2$, a measure μ of $\mathcal{M}^+$ is an element of the dual space $C_0^0\left(S\tilde{Z}\text{end}\right)$ satisfy

$$\langle \mu, q\rangle \geq 0, \forall q \in C^0\left(S\widehat{Z}, \text{end}(\mathbb{C}^2)\right), \tag{4.10}$$

where $\text{end}^+(\mathbb{C}^2)$ denotes the set of positive Hermitian matrices 2×2.

Let (u_k) a bounded sequence in $\left(H^1\left(I\times\Omega\right)\right)^2$, solutions of $Pu^k = 0$ converges weakly to 0. Then $u^k_{|x_n=0}$ (resp. $\partial_\nu u^k{}_{|x_n=0}$) is bounded in $\left(H^{\frac{1}{2}}_{\text{loc}}\left(I\times\partial\Omega\right)\right)^2$ (resp. $\left(H^{-\frac{1}{2}}_{\text{loc}}\left(I\times\partial\Omega\right)\right)^2$) has zero weakly limits.

Proposition 4.2. *There exists a subsequences of* (u_k) *and* $\mu\in\mathcal{M}^+$ *such that*

$$\forall Q\in\mathcal{G}^0, \quad \lim_{k\to\infty}\varphi(Q,u_k) = \langle\mu,\kappa(q)\rangle \tag{4.11}$$

where q *the principal symbol of* Q *and* $\mu = \begin{pmatrix} \mu_1 & \mu_{12} \\ \overline{\mu_{12}} & \mu_2 \end{pmatrix}$.

testing the measure μ on different operators $\mathcal{Q}$, the limit equation (4.11) can be written as

$$\left\{\begin{array}{lcl} \lim_{k\to\infty}(\nabla_x Qu_1^k, \nabla_x u_1^k)_{L^2} + (\partial_t Qu_1, \partial_t u_1) + (Qu_1,u_1) &=& \langle\mu_1,\kappa(q)\rangle \\ \lim_{k\to\infty}\alpha(\nabla_x Qu_2^k, \nabla_x u_2^k)_{L^2} + (\partial_t u_2, \partial_t u_2) + (Qu_2,u_2) &=& \langle\mu_2,\kappa(q)\rangle \\ \lim_{k\to\infty}(\nabla_x Qu_2^k, \nabla_x u_1^k)_{L^2} + (\partial_t Qu_2^k, \partial_t u_1) + (Qu_2^k,u_1) &=& \langle\mu_{12},\kappa(q)\rangle \end{array}\right. \tag{4.12}$$

Proof. According to [3] and we follow the method given by [6]. $u^k_{|x_n=0}$ (resp. $\partial_\nu u^k{}_{|x_n=0}$) has zero weakly limits that implies

$$\forall Q\in\mathcal{G}^{-1}, \quad \lim_{k\to\infty}\varphi(Q,u_k) = 0. \tag{4.13}$$

Let $\chi\in C_0^\infty\left(|x_n|<\epsilon\right)$, $0\leq\chi\leq 1$, $\chi(x)=1$ for $|x|\leq\frac{\epsilon}{2}$ and E is a pseudo-differential operator matrix supported near $\mathrm{C}har(P_a^{\alpha,\beta})$ such that

$$\text{id}-\sigma(E) = \left\{\begin{array}{l} 0 \quad \text{near} \quad \text{neighborhood} \;\; \text{Char}(P)\cap\;\text{supp}(1-\chi), \\ \text{non} \quad \text{negative}, \end{array}\right.$$

for all $\psi\in C_0^\infty(I)$ we have

$$(\text{id}-\chi)(\text{id}-E)\psi u_k \to 0, \quad H^1. \tag{4.14}$$

If $Q = Q_i + Q_\partial \in \mathcal{G}^0_I$ choosing ϵ small we have $\chi Q_i \equiv 0$ and we write

$$Q = \chi Q + (\text{ id} - \chi)Q = \chi Q_\partial + (\text{id} - \chi)QE + (\text{ id} - \chi)Q(\text{id} - E)$$

then Q_∂ is tangential pseudo differential operator, $(\text{ id} - \chi)QE$ is interior pseudo differential operator and $(\text{id} - \chi)Q(\text{ Id} - E)\varphi u_k \to 0$ in $(H^1)^2$ for all $\varphi \in C_0^\infty\left(\overline{X}, \text{End}(\mathbb{C}^2)\right)$

$$\forall Q \in \mathcal{G}^0_{\text{sym}},\ \ \sigma(Q) + M \text{ id positive } \Rightarrow -M \lim_{k\to\infty} \inf \varphi(Q, u_k) \leq -M \lim_{k\to\infty} \sup \|u_k\|^2_{H^1}. \quad (4.15)$$

Indeed, $[\sigma(Q) + M \text{ id}]$ nonnegative matrix implies $[\sigma(\chi Q) + M \text{ id}]$ and $[\sigma(\text{ id}) - \chi)QE) + M \text{ id}]$ are nonnegative matrix and it is sufficient to study independently these cases $Q = Q_\partial$, $Q = Q_i$.

In the first, $Q = Q_\partial$ there exists $\varphi \in C_0^\infty(I)$ such that

$$a_k = (\nabla_x^\alpha Q_\partial u_k, \nabla_x^\alpha u_k)_{(L^2)^2} = (Q_\partial \nabla_x^\alpha \varphi u_k, \nabla_x^\alpha \varphi u_k) + b_k$$

with $b_k = ([\nabla_x^\alpha, Q_\partial]u_k, \nabla_x^\alpha u_k)_{L^2} \to 0$. For all $\varepsilon > 0$ there exists B_∂ of zero degree, C_∂ of -1 degree tangential d.p.o such that $Q_\partial + (M + \varepsilon) \text{ id} = B^* B_\partial + C_\partial$. As $C_\partial \nabla_x^\alpha \varphi u_k \to 0$ in $(L^2)^2$ (because (φu_k) is a bounded sequence near the boundary in $C^1\left(x_n \geq 0, \left(H^{-\frac{1}{2}}_{t,x'}\right)^2\right)$), we have $\lim \inf a_k \leq -(M + \epsilon) \lim\sup \|\nabla_x^\alpha \varphi u_k\|$, the same method to $(\partial_t Q_\partial u_k, \partial_t u_k)$ because $\lim\sup \|\partial_t \varphi u_k\| \leq \lim \|u_k\|_H$.

So we have

$$\begin{aligned} &Q \in \mathcal{G}^0_I, \\ &\sigma(Q)_{|CharP} = 0 \ \text{ and } \ \sigma(Q)_{|x_n \leq \varepsilon} = 0 \Rightarrow \lim_k \varphi(Q, u_k) = 0. \end{aligned} \quad (4.16)$$

Let $\sigma(\mathcal{G}) = \{q = \sigma(Q); Q \in \mathcal{G}\}$, that is a vectorial subspace of functions space C^0 homogenies of zero degree onto $T^*X \setminus X$ with value in $\text{End}(\mathbb{C}^2)$ endowed with the L^∞ and there exists a subset dense of $\sigma(\mathcal{G})$. By (4.15) and (4.16), there exists a subsequence of (u_k) and a linear map $\tilde{\varphi}$ from $\sigma(\mathcal{G})$ onto $\mathbb{C}$ such that

$$\forall Q \in \mathcal{G}^0,\ \ \lim_{k\to\infty} \varphi(Q, u^k) = \tilde{\varphi}\left(\sigma(Q)\right), \quad (4.17)$$

$$|\tilde{\varphi}(q)| \leq \|q\|_{L^\infty} \lim\sup |u_k|^2_{H^1}. \quad (4.18)$$

Moreover, we have

$$q \in \sigma(\mathcal{G}^0) \ \text{ and } \ \kappa(q) = 0 \Rightarrow \tilde{\varphi}(q) = 0 \quad (4.19)$$

because if $\kappa(q) = 0$, for all $\varepsilon > 0$, there exists $\chi \in C_0^\infty(\mathbb{R}, \text{ end}(\mathbb{C}^2))$ supported near $x = 0$ such that $|\chi q|_{L^\infty} \leq \varepsilon$ and $(\text{ id} - \chi)q = \sigma(Q)$ where $Q \in \mathcal{G}^0$ satisfies (4.16). By Riesz Theorem there exists a Radon measure μ in the dual of $C_0^0\left(S\tilde{Z}, \text{End}(\mathbb{C}^2)\right)$ such that

$$\forall Q \in \mathcal{G}^0,\ \ \lim_k \varphi(Q, u_k) = \langle \mu, \kappa(\sigma(Q)) \rangle \quad (4.20)$$

with μ is positive Hermitian by (4.15) and a measure μ_∂ on $S(T^*\partial X)$ such that

$$\forall Q \in \mathcal{G}^0_I,\ \lim_k \in t_{\partial X} Q u_k \partial_\nu u_k = \int \sigma(Q)_{|x_n=0} d\nu_\partial \quad (4.21)$$

and by (4.9) we have

$$\mu = \mu_\partial + \mu_{\text{cin}} \tag{4.22}$$

where μ_∂ is considered to measure on $S\tilde{Z}$ through the injection $S(T^*\partial X) \hookrightarrow S\tilde{Z}$. If the sequence u_k satisfies the Dirichelet condition $u_{k|\partial X} \equiv 0$ then $\mu_\partial \equiv 0$ and if $Q = Q_\partial \in \mathcal{G}_I^0$ with compact support near $x_n = 0$, $(t, x', \tau, \xi') \in T^*\partial X$ we have Qu_k bounded in $C^\infty(\bar{X})$

We have $\tilde{Z}_{x=0} = T^*Y$, since the sequence u^k satisfies the Dirichlet $u_{k|\partial X} \equiv 0$ then $\mu_\partial \equiv 0$.

4.1 Propagation Theorem to boundary

We assume that there is no contact of infinity order between the geodesics of $\overline{\Omega}$ and the boundary $\partial\Omega$. In this section we recall some concepts and properties to the boundary value problem of coupled waves system. Let $u_k(t, x)$ a sequence of solution of the following problem

$$\begin{cases} \left(\partial_t^2 - D_\alpha + K_a^\beta \partial_t\right) u_k = 0, \ u_k|_{\mathbb{R}_{\times\partial\Omega}} = 0 \\ \left(u_{k|_{t=0}}, \partial_t u_{k|_{t=0}}\right) \text{ bounded in } \left(H_0^1(\Omega)\right)^2 \times \left(L^2(\Omega)\right)^2 \end{cases} \tag{4.23}$$

has null weak limits, $\mu = 2\mu_{cin}$ associated measures on (SZ), $\mu^+ = 2\mu_{cin}^+$ their restrictions to $(SZ)^+$.

Theorem 4.3. *For all $s \in \mathbb{R}$ we have*

$$G(s)^*(\mu) = \langle \exp\left(-\int_0^s K^\beta_{a(G(\sigma)(\rho))} d\sigma\right), \mu\rangle. \tag{4.24}$$

Precisely, for all B a Boral set of SZ, we have

$$\mu(G(s)(B)) = \int_B H(s,\rho) d\mu = \sum_{i,j} \int_B H_{ij} d\mu_{ji}$$

with $H(s,\rho) = \exp\left(-\int_0^s K^\beta_{a(G(\sigma)(\rho))} d\sigma\right)$.

Proof. We set $\mu_s = H(s,\rho)\mu$. As $\{G(s)\}$ is a C^0-homeomorphic group of SZ and change t to $-t$ returns change a to $-a$. Then it is sufficient to prove that

$$G(s)^*(\mu^+) \leq \mu_s^+ \quad \text{for all } s > 0. \tag{4.25}$$

If K is a compact of $(SZ)^+ \cap (t = 0)$ and J a compact of $\mathbb{R}$. We denote

$$K_J = \{G(\sigma)(\rho); \rho \in K, \ \sigma \in J\}.$$

The fact that $G(s)(t, x, \xi) = (t + s, G(s)(x, \xi))$, the map $\Theta : ((SZ)^+ \cap (t = 0)) \times \mathbb{R} \to (SZ)$; $\Theta(\rho, \sigma) = G(\sigma)\rho$ is a homomorphic that redress the flow ($G(\rho, \sigma + s) = G(s)\Theta(\rho, \sigma)$). To prove (4.25) it is sufficient to verify the following properties

$$\begin{cases} \forall \alpha_1 > 0, \exists \beta_1 > 0 \text{ such that} \\ \text{for all } K' \subset\subset \text{int}(K) \subset K \subset ((SZ)^+ \cap (t = 0)), \ \text{diam} K \leq \beta_1 \\ \text{and for all } b_0 < b_0' < b_1' < b_1, \ b_1 - b_0 \leq \beta_1 \\ \text{with } J = [b_0, b_1], \ J' = [b_0', b_1'] \\ \text{we have } G(s)^*(\mu)(K'_{J'}) \leq (1 + \alpha_1)\mu_s(K_J). \end{cases} \tag{4.26}$$

Indeed, by the redress flow, we can consider the measures μ^+ and μ_s^+ onto product $((SZ)^+\cap(t=0))\times\mathbb{R}$, we denote by $\tilde\mu^+,\ \tilde\mu_s^+$ and $\tilde\nu_s^+=G(s)^*(\tilde\mu^+)$. By (4.26) we deduce that

$$\tilde\nu_s^+(E')\le(1+\alpha_1)\mu_s^+(E) \tag{4.27}$$

for $E'=K'\times I$, $E=K\times I$, $K'\Subset K$, $\operatorname{diam}(K)\le\beta_1$, $I=]b_0,b_1[$, $b_1-b_0\le\beta_1$ by increasing limits, and for $E=O\times I$, O open set with $\operatorname{diam}(O)\le\beta_1$ with $\operatorname{diam}(O)\le\beta_1$ and by decreasing limits for $E=E'=O\times L$ for any interval L with $\operatorname{diam}(L)\le\beta_1$ then we have by additivity of measure and increasing limits we have

$$\tilde\nu_s^+(V)\le(1+\alpha_1)\mu_s^+(V),\ \ \forall V\ open$$

then $\tilde\nu_s^+\le(1+\alpha_1)\mu_s^+$, for all $\alpha_1>0$, hence (4.25).

Now we prove (4.26), we have $G(s)^*(\mu^+)(K'_{J'})=\mu(K'_{J'+s})$ and we can assume $0<\beta_1\ll s$. We set $u^k=u$ and we identify $u(x,t)$ to

$$\underline{u}(x,t)=(u(x,t),\partial_t u(x,t))\in\big(C^0(\mathbb{R},(H_0^1(\Omega))^2)\cap C^1(\mathbb{R},(L^2(\Omega))^2)\big)\oplus C^0(\mathbb{R},(L^2(\Omega))^2).$$

We set

$$H=(H_0^1(\Omega))^2\oplus(L^2(\Omega))^2,\quad H'=(L^2(\Omega))^2\oplus(H^{-1}(\Omega))^2,\quad \mathcal{H}_1=L^2(\mathbb{R},H)\ \text{ and }\ \mathcal{H}_0=L^2(\mathbb{R},H')$$

and for $\underline{v}=((u_0,v_0),(u_1,v_1))\in\mathcal{H}_i$

$$\left|\underline{v}\right|=\left\|\underline{v}\right\|_{\mathcal{H}_i}. \tag{4.28}$$

We recall that the operator $\mathcal{A}_a^{\alpha,\beta}$ with boundary Dirichlet and that $e^{t\mathcal{A}_a^{\alpha,\beta}}$ is bounded on H and H', we denote by C some independent constants of k index concerning the sequence u^k and by C_0 some independent constants of $k,\ K',\ K,\ J,\ J'$ and $b_0,\ b_1$ given in a fixed compact of $\mathbb{R}$.

Let $\varphi\in C_0^\infty(\mathbb{R})$, equal to 1 on $[b_0-1,b_1'+s+1]$, $\psi(t)\in C^\infty(\mathbb{R})$, $0\le\psi\le1$, in a neighborhood of $[b_1,+\infty[$, $\psi\equiv1$ in a neighborhood of $]-\infty,b_0]$, $\Psi(t)\in C_0^\infty(]b_0,b_1[)$, $0\le\Psi\le1$ and $\mathrm{id}-\Psi$, $\Psi\equiv1$ in a neighborhood of suppψ'. If $Q\in\mathcal{G}^0$ and $\rho\in Z\setminus X$, we write $\rho\notin\mathrm{ES}(Q)$ if $j^{-1}(\rho)\cap\mathrm{Car}P_a^{\alpha,\beta}$ not meet the essential supported of Q that is define because if ρ is an interior point, Q is a d.p.o. near the point $\rho'=J^{-1}(\rho)\in\mathcal{P}_a^{\alpha,\beta}$.

So we write for K compact of $Z\setminus X$, $Q=\mathrm{Id}$ near of K if $K\cap\mathrm{ES}(Q-\mathrm{Id})=\emptyset$. Let $Q_0\in\mathcal{C}^0$ with its principal symbol $q_0=\sigma(Q_0)$, id$-q_0$ positive, such that $\mathrm{Q}_0\subset\{G(\sigma)(\rho)\ ;\ \rho\in\mathrm{int}(K),\ b_0-\epsilon<\sigma<b_1'+s+\epsilon\}$ with $\epsilon>0$ small and $Q_0=\mathrm{Id}$ near of $K'_{[b_0,b_1'+s]}$, and let $Q_1\in\mathcal{G}^0$ with q_1 its principal symbol with q_1 and $\mathrm{id}-q_1$ are nonnegative and such that $Q_1=\mathrm{id}$ near of $K'_{J'+s}$, $\mathrm{ES}(Q_1)$ include in a neighborhood of $K'_{J'+s}$ and $Q_0=\mathrm{id}$ near of $\mathrm{ES}(Q_1)$.

Let $Q\in\mathcal{G}^0$ and $\underline{v}=((u_0,v_0),(u_1,v_1))\in\mathcal{H}_i$ we set $Q\underline{v}=(Q(u_0,v_0),Q(u_1,v_1))$.

We have $\left(\partial_t-\mathcal{A}_a^{\alpha,\beta}\right)\underline{u}=0$, then $\left(\partial_t-\mathcal{A}_a^{\alpha,\beta}\right)\psi\underline{u}=\psi'(t)\underline{u}$.

Let

$$\underline{w}=-\int_{-\infty}^{t}e^{(t-\sigma)\mathcal{A}_a^{\alpha,\beta}}\psi'(\sigma)\underline{u}(\sigma)d\sigma,$$

we have $(\partial_t-A_a^{\alpha,\beta})\underline{w}=-\psi'(t)\underline{u}$, then $(\partial_t-A_a^{\alpha,\beta})[\underline{u}-\psi(t)\underline{u}-\underline{w}]=0$, since $\underline{u}-\psi(t)\underline{u}-\underline{w}=0$ for $t<b_0$ that result

$$\underline{u}=\psi(t)\underline{u}+\underline{w}. \tag{4.29}$$

We have $\left(\partial_t-\mathcal{A}_a^{\alpha,\beta}\right)Q_0\underline{w}=-Q_0\psi'\underline{u}-[\partial_t-\mathcal{A}_a^{\alpha,\beta},Q_0]\underline{w}$ and we let

$$\underline{h}=-\int_{-\infty}^{t}e^{(t-\sigma)\mathcal{A}_a^{\alpha,\beta}}Q_0\psi'(\sigma)\underline{u}(\sigma)d\sigma,$$

hence $(\partial_t - \mathcal{A}_a^{\alpha,\beta})\underline{h} = -Q_0\psi'(t)\underline{u}$

$$(\partial_t - \mathcal{A}_a^{\alpha,\beta})[Q_0\underline{w} - \underline{h}] = -[\partial_t - \mathcal{A}_a^{\alpha,\beta}, Q_0]\underline{w}. \tag{4.30}$$

The key point is the following estimation

$$\Big|Q_1(Q_0\underline{w} - \underline{h})\Big| \leq C\Big|\varphi\underline{u}\Big|_0. \tag{4.31}$$

that result by the propagation Theorem of Melrose-Sjöstrand.

Indeed, let $F = \{\ u \in L^2_{\mathrm{loc}(t)}(X) \mid \mathcal{P}_a^{\alpha,\beta}u = 0,\ u_{|\partial X} = 0\Big\}$ inner the norm $|\varphi\underline{u}|_0$ and WF_b the wavefront at the boundary. Let $\underline{w}$, $\underline{h}$ associate to u as given below, we have $WF_b(\underline{u}) \subset Z$ that implies $WF_b(\underline{w}) \subset Z$, $WF_b(\underline{h}) \subset Z$ and $WF_b([\partial_t - A_a^{\alpha,\beta}]\underline{w}) \subset Z \setminus \{\rho,\ Q_0 = \text{id near } \rho\}$. As $WF_b(Q_0\underline{w}) \subset (b_0, +\infty)$ by the propagation theorem (see [11]), we have $WF_b(Q_0\underline{w} - h) \cap \mathrm{ES}(Q_1) = \emptyset$ then $Q_1(Q_0\underline{w} - \underline{h}) \in C^\infty(\overline{X})$. As $\underline{u} \mapsto Q_1(Q_0\underline{w} - \underline{h})$ is continuous from F onto $\mathcal{H}_0$ and (4.31) result of closed graph theorem.

We have

$$\underline{h} = -\int_{-\infty}^{t} e^{(t-\sigma)\mathcal{A}_a^{\alpha,\beta}}\psi'(\sigma)\Psi(\sigma)Q_0\underline{u}d\sigma - \int_{-\infty}^{t} e^{(t-\sigma)\mathcal{A}_a^{\alpha,\beta}}[Q_0, \psi'\Psi]\underline{u}d\sigma, \tag{4.32}$$

then $\underline{h} \in C^0(\mathbb{R}, H)$ and for $t \in [b_0 - 1, b_1' + s + 1]$,

$$\|\underline{h}\|_H \leq C_0\|\psi'\|_{L^2}|\Psi Q_0\underline{u}|_1 + C|\varphi\underline{u}|_0. \tag{4.33}$$

because $[Q_0, \psi'\Psi]\underline{u} = (Q_{-1}u(t,x),\ Q_{-1}\partial u(x,t))$ with $Q_{-1} \in \mathcal{G}^{-1}$ then

$$\begin{aligned} \Big|[Q_0,\ \psi'\Psi]u\Big|_1 &\leq \Big\|\nabla_x Q_{-1}u\Big\|_{\left(L^2(\mathbb{R}\times\Omega)\right)} + \Big\|\partial_t u\Big\|_{\left(L^2(\mathbb{R}\times\Omega)\right)} \\ &\leq C\Big\|\varphi u\Big\|_{\left(L^2(\mathbb{R}\times\Omega)\right)} \end{aligned} \tag{4.34}$$

Let d a real constant, $\mathcal{A}_d^{\alpha,\beta} = \begin{pmatrix} 0 & \text{id} \\ D_\alpha & -K_d^\beta \end{pmatrix}$. We have $(\partial_t - \mathcal{A}_d^{\alpha,\beta})\underline{h} = -Q_0\psi'\underline{u} + (\mathcal{A}_a^{\alpha,\beta} - \mathcal{A}_d^{\alpha,\beta})\underline{h}$ then

$$\begin{aligned} \underline{h} \;=\; & -\textstyle\int_{-\infty}^t e^{-(t-\sigma)A_d^{\alpha,\beta}}\psi'(\sigma)\Psi(\sigma)Q_0\underline{u}, \\ & -\textstyle\int_{-\infty}^t e^{-(t-\sigma)A_d^{\alpha,\beta}}[Q_0, \psi'\Psi]\underline{u}, \\ & +\textstyle\int_{-\infty}^t e^{-(t-\sigma)A_d^{\alpha,\beta}}(A_a^{\alpha,\beta} - A_d^{\alpha,\beta})\underline{h}. \end{aligned} \tag{4.35}$$

There results for all $t \in [b_0, b_1' + s + \epsilon'],\ \epsilon' > 0,\ \epsilon' \ll \epsilon$

$$\begin{aligned} \Big|\Big|\underline{u}(t)\Big|\Big|_H \leq \big(e^{-d(t-b_1)}e^{|d|\beta} &+ C_0\|a(x) - d\|_{L^\infty(T_\epsilon)}(t - b_0)\big) \\ &\cdot\|\psi'\|_{L^2}|\Psi Q_0\underline{u}|_1 + C|\varphi\underline{u}|_0 \end{aligned} \tag{4.36}$$

where $T_\epsilon = K_{[b_0-\epsilon, b_1+s+\epsilon]}$. Indeed, we write by (4.36)

$$\underline{h} = (1) + (2) + (3)$$

We have $WF_b(\underline{h}) \subset \{t > t_0\}$ and $WF_b((\partial_t - A_a)\underline{h}) = WF_b(Q_0\psi'(t)\underline{u}) \subset (SE(Q_0) \cap (t > b_0))$. By Cauchy Schwartz we obtain

$$\begin{aligned}\Big\| \int_{-\infty}^{t} & e^{(t-\sigma)A_d^{\alpha,\beta}}(A_a^{\alpha,\beta}-A_d^{\alpha,\beta})h(\sigma)d\sigma\Big\| \\ \leq & \quad C_0(t-b_0)\Big\|a(x)-d\Big\|_{L^\infty(T_\varepsilon}\Big\|\Psi'\Big\|_{L^2}\Big|\psi Q\underline{u}\Big|_1+C\Big|\varphi\underline{u}\Big|_0\end{aligned} \tag{4.37}$$

that give the term (3). We can see the term (2) by (4.34).

Finally for the term (1), we see that if (e_j,w_j) is the orthonormal basis of eigenfunctions of $H_0^1(\Omega)$, $-\Delta e_j=\omega_j^2 e_j$, $\omega_j\geq 0$, we denote by $\lambda_{ji}^{\pm}$, $i=1,2$ roots of $\lambda^4+2d\lambda^3+(\beta^2+\alpha\omega_j^2+\omega_j^2)\lambda^2+2d\alpha\omega_j^2\lambda+\alpha\omega_j^4=0$. The family $\big((e_j,\ \alpha e_j),\ \lambda_{ji}^{\pm}(e_j,\ \alpha e_j)\big)$, $i=1,2$ constitute an orthonormal basis in H of eigenfunctions of $\mathcal{A}_a^{\alpha,\beta}$. For j large, we have $\mathrm{Re}(\lambda_{ji}^{\pm})=-\frac{d}{2}$, we obtain

$$\begin{aligned}(1) \quad \leq & \quad \int_{b_0}^{b_1} e^{-(t-\sigma)\frac{d}{2}}\Big|\psi'(\sigma)\Big|\Big\|\Psi Q_0\underline{u}\Big\|_H d\sigma+C\Big|\varphi\underline{u}\Big|_0 \\ \leq & \quad e^{-(t-b_1)\frac{d}{2}+\frac{d}{2}\beta_1}\Big\|\psi'\Big\|_{L^2}\Big|\Psi Q_0\underline{u}\Big|_1+C\Big|\varphi\underline{u}\Big|_0.\end{aligned} \tag{4.38}$$

this give (4.36). We have $\lim_k\Big|\varphi\underline{u}^k\Big|_0=0$, and since $\sigma\left(Q_1^*Q_1\right)=\ \mathrm{Id}$ on $K'_{J'+s}$

$$\mu^+(K'_{J'+s})\leq \limsup_k\Big|Q_1\underline{u}^k\Big|_1^2. \tag{4.39}$$

Let $\chi\in C_0^\infty\left(]b_0'+s-\epsilon,b_1'+s+\epsilon[\right)$, with $\chi\equiv 1$ on $SE(Q_1)$. By (4.29), (4.31) and $q_1=\sigma(Q_1)\in[0,1]$, we have

$$\limsup_k\Big|Q_1\underline{u}^k\Big|_1^2\leq\limsup_k\Big|\chi\underline{h}^k\Big|_1^2 \tag{4.40}$$

and by (4.36)

$$\begin{aligned}\limsup_k|\chi\underline{h}^k|_1^2 \quad \leq & \quad (b_1'-b_0'+2\epsilon)\|\psi'\|^2_{(L^2)^2}\left(e^{-\frac{d}{2}(b_0'+s-b_1-\epsilon)}e^{\frac{d}{2}[\beta+\epsilon]}\right. \\ + & \quad C_0\|a(x)-d\|_{L^\infty(T_\epsilon)}(b_1'+s+\epsilon-b_0)^2\big)\limsup_k|\Psi Q_0U^k|^2.\end{aligned} \tag{4.41}$$

As $b_1'-b_0<b_1-b_0$, we can assume $(b_1'-b_0'+2\epsilon)\ \|\psi'\|_{L^2}\leq 1$. Moreover, $\mathrm{Id}-\sigma(\Psi Q_0)$ non negative and supported in K_J, then we deduce from (4.39), (4.40), (4.40) with $T=T_{\epsilon=0}$

$$\mu^+(K'_{J'+s})\leq\mu^+(K_J)\left[e^{-ds}e^{\frac{3}{2}\beta_1 d}+C_0\|a(x)-d\|_{L^\infty(T)}(b_1'+s-b_0)\right]^2. \tag{4.42}$$

This estimation is valid for $K'\Subset K$, $J'\Subset J$, $b_1-b_0\leq\beta$ and $s>\beta$ where $b_0,\ b_1,\ d$ are bounded constant. For all $\rho\in K_J$, we have

$$\Big|e^{sd}G(s,\rho)-\mathrm{Id}\Big|\leq C_0 s\|a(x)-d\|_{L^\infty(T)}.$$

Let $\delta>0$ small enough, there exist $s_\delta>0$ and β_δ such that ρ_0 is in K_J we have $\mathrm{d}iam(K)<\beta_\delta$, $0<s\leq s_\delta$ and $b_1-b+0\leq\delta_s$ and choosing $d=a(\rho_0)$, $\|a(x)-d\|_{L^\infty(T)}\leq C_0\delta$ and by (4.42)

$$\mu^+(K'_{J'+s})\leq\mu^+(K_J)H(s,,\rho_0)(1+C_0\delta s). \tag{4.43}$$

Proving (4.26), let $s>0$ and $\beta<\inf(\beta_\delta,\delta s_\delta)$. By iterating at most $N=\frac{s}{s_\delta}$ times the inequality with $s=s_\delta$ and a sequence $J'=J_1\Subset J_2\Subset...\Subset J_N=J$ of intervals and a sequence of compacts $K'=K_1\Subset K_2\Subset K_3\Subset ....\Subset K_N=K$, we obtain for any $\rho_0\in \mathrm{i}nt(K)$

$$\mu^+(K'_{J'})\leq\mu^+(K_J)H(s,\rho)(1+C_0\delta). \tag{4.44}$$

Since $(1+C_0\delta s_\delta)^{\frac{s}{s_\delta}} \leq 1+C_0\delta$. As we have $\mu_s^+(KJ) = \int_{K_J} H(s,\rho)$ and for β small $|H(s,\rho) - H(s,\rho_0)| \leq C_0\delta$ for $\rho \in K_J$, hence the function H the function H remaining in a compact $(0. + \infty)$

$$\left|\mu_s^+(K_j) - \mu^+(K_J)H(s,\rho\right| \leq C_0\delta\mu_s^+(K_J) \tag{4.45}$$

and (4.26) deduced from (4.44) and (4.45).

References

[1] Bardos, C., Lebeau, G. and Rauch, G. Sharp sufficient conditions for the observation, control and stabilization of weaves from the boundary, *SIAM J. Control Optimization,* **30** (1992), 1024-1065.

[2] Burq, N. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonnance au voisinage du réel, *Acta Math.,* **180** (1998), 1-29.

[3] Burq, N. et Lebeau, G. Mesures de défaut de compacité, applications au système de lamé, *Ann. Sci. Ec. Norm. Sup.,* **34** (2001), 817-870.

[4] Daoulatli, M., Dehman, B. and Khenissi, M., Local energy decay for the elastic system with nonlinear damping in an exterior domain, *SIAM J. Control Optimization,* **48** (2010), 5254-5275.

[5] Batty, C. J. K. and Duyckaerts, T. Non-uniform stability forbounded semi-groups on Banach spaces, *J. Evol. Equ.*, **8** (2008), 765-780.

[6] Gérard, P. Microlocal defect measures, Comm. P. D. E., **16** (1991), 1761-1794.

[7] Gohberg, I.C and Krein, M.G. Introduction to the therory of linear non self adjoint operators, translations of mathematical monograph, *Amer. Math. Soc* **18** (1969).

[8] Lebeau.G, Equation des ondes amorties, *in Algebraic and geometric methods in mathematical physics, Kluwer Academic Publishers, Nertherlands,* (1996), 73-109.

[9] Robbiano, L. Fonctions de coût et contrôle des solutions des équations hyperboliques, *Asymptotic Analysis* , **10** (1995), 95-115.

[10] Rauch, J. and Taylor, M., Decay of solutions to nondissipative hyperbolic systems on compact manifolds, Comm.Pure Appl. Math. **28** (1975) 501-523.

[11] Taylor, M.E. Pseudodifferential operators, *Princeton University Press, NEW JERSSEY.*

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