Palestine Journal of Mathematics

Vol. 3(1) (2014) , 94–98

RECURRENCE RELATION OF A UNIFIED GENERALIZED MITTAG-LEFFLER FUNCTION

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Communicated by Jose Luis Lopez-Bonilla

MSC 2010 Classifications: 33E12, 33B15. 11R32.

Keywords and phrases: Generalized Mittag-Leffler function; Recurrence relation; Wiman's function.

The authors are indebted to Dr. B. I. Dave for his kind help in LaTeX type setting.

Abstract. The present work incorporates a recurrence relation and an integral representation of a further generalization of a generalized Mittag-Leffler fuction due to A.K. Shukla and J.C. Prajapati[Surveys in Mathematics and its Applications, Volume 4(2009), 133-138]. At the end, several special cases have also been discussed.

1. Introduction, definitions and Preliminaries

The Mittag-Leffler function has been studied by many researchers either in context with obtaining new properties or by introducing a new generalization and then deriving its properties ([5], [7], [4]). Recently, we have also studied various properties of our newly introduced unification of Generalized Mittag-Leffler function in the form [2]

$$E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^{s}\,(cz)^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)\,[(\lambda)_{\mu n}]^{r}\,(\rho)_{pn}},\tag{1.1}$$

wherein $\alpha,\ \beta,\ \gamma,\ \lambda,\ \rho\ \in\mathbb{C},\ Re(\alpha,\beta,\gamma,\lambda,\rho)>0;\ \delta,\ \mu,\ p,\ c>0$. If $p=1,\ \rho=1,\ r=0,\ s=1,\ \delta=q,\ s=1,\ c=1$, then this yields the generalization due to Shukla and Prajapati [5].
In the next section, we prove the main results.

2. Recurrence Relation

We begin by stating the main theorem.

Theorem 1. For $\alpha,\ \beta,\ \gamma,\ \lambda,\ \rho\ \in\mathbb{C},\ Re((\alpha+a),(\beta+b),\gamma,\lambda,\rho)>0,$ $\delta,\ \mu,\ p,\ c>0,$ we get

$$\begin{aligned}
&E^{\gamma,\delta}_{\alpha+a,\,\beta+b+1,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)\ -\ E^{\gamma,\delta}_{\alpha+a,\,\beta+b+2,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)\\
&\quad=(\alpha+a)^2\,z^2\ddot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)\\
&\qquad+z\,(\alpha+a)\big[\alpha+a+2(\beta+b+1)\big]\dot{E}^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)\\
&\qquad+(\beta+b)\ (\beta+b+2)\ E^{\gamma,\delta}_{\alpha+a,\,\beta+b+3,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r),
\end{aligned}\tag{2.1}$$

where, $\dot{E}^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)\ =\ \dfrac{d}{dz}E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r),$

$$\ddot{E}^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r)\ =\ \frac{d^2}{dz^2}E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\rho,\,p}(cz;s,r).$$

It is easy to obtain the following corollary by letting $\alpha+a\ =\ k$ and $\beta+b\ =\ m$.

Corollary: We have, for $k,\ m\ \in \mathbb{N}$,

$$E^{\gamma,\delta}_{k,\ m+1,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) = E^{\gamma,\delta}_{k,\ m+2,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) + m(m+2)E^{\gamma,\delta}_{k,\ m+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r)\ +\ k^2\, z^2 \ddot{E}^{\gamma,\delta}_{k,\ m+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) + k\,[k+2(m+1)]\ z\dot{E}^{\gamma,\delta}_{k,\ m+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r). \tag{2.2}$$

Proof of Theorem 1. By substituting $\alpha\ =\ \alpha + a\ , \beta\ =\ \beta + b + 1$ in (1.1) and applying the fundamental relation of the Gamma function $\Gamma(z+1)\ =\ z\Gamma(z)$, we have

$$E^{\gamma,\delta}_{\alpha+a,\ \beta+b+1,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b))^{-1}(cz)^{(p\,n+\rho-1)}}{((\alpha+a)(p\,n+\rho-1)+\beta+b)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}} \tag{2.3}$$

and

$$E^{\gamma,\delta}_{\alpha+a,\ \beta+b+2,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}}{((\alpha+a)(p\,n+\rho-1)+\beta+b)[(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}} \cdot \frac{(\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+2))^{-1}}{((\alpha+a)(p\,n+\rho-1)+\beta+b+1)}. \tag{2.4}$$

Equation (2.4) can be written as follows:

$$E^{\gamma,\delta}_{\alpha+a,\ \beta+b+2,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) = \sum_{n=0}^{\infty} \left[\frac{1}{((\alpha+a)(p\,n+\rho-1)+\beta+b)}\ -\ \frac{1}{((\alpha+a)(p\,n+\rho-1)+\beta+b+1)}\right] \cdot \frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}}$$

$$= E^{\gamma,\delta}_{\alpha+a,\ \beta+b+1,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r)\ -\ \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b)} \cdot \frac{((\alpha+a)(p\,n+\rho-1)+\beta+b+1)^{-1}}{[(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}}. \tag{2.5}$$

For the sake of convenience, we denote the last summation in (2.5) by S, then

$$S\ =\ E^{\gamma,\delta}_{\alpha+a,\ \beta+b+1,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) - E^{\gamma,\delta}_{\alpha+a,\ \beta+b+2,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r). \tag{2.6}$$

Applying the following(evident):

$$\frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{(u+1)}$$

and then taking $u = ((\alpha+a)(p\,n+\rho-1)+\beta+b+1)$ to (2.6), we obtain

$$S = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}\ ((\alpha+a)(p\,n+\rho-1)+\beta+b)}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}} + \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}\ ((\alpha+a)(p\,n+\rho-1)+\beta+b)}{[(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}} \cdot \frac{((\alpha+a)(p\,n+\rho-1)+\beta+b+1)}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)}$$

$$= (\alpha+a)\ \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}\ ((\alpha+a)(p\,n+\rho-1)+\beta+b)}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn-1}}$$

$$+\ (\beta+b)\ \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s\ (cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\ [(\lambda)_{\mu\, n}]^r\ (\rho)_{pn}}$$

$$+ (\alpha+a)^2 \sum_{n=1}^{\infty} \frac{(p\,n+\rho-1)\,(cz)^{(p\,n+\rho-1)}\,((\alpha+a)(p\,n+\rho-1)+\beta+b)}{[(\gamma)_{\delta\,n}]^{-s}\,\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$+ x \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,(cz)^{(p\,n+\rho-1)}\,((\alpha+a)(p\,n+\rho-1)+\beta+b)}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$+ y \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,(cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn}}, \tag{2.7}$$

where, $x = (\alpha+a)\,(2\beta+2b+1)$ and $y = (\beta+b)(\beta+b+1)$.
We now express each summation in the right hand side of (2.7) as follows:

$$\frac{d^2}{dz^2}\left[z^2\,E^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r)\right]$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,(p\,n+\rho+1)\,(p\,n+\rho)\,(cz)^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn}}. \tag{2.8}$$

From (2.8) we find that

$$\sum_{n=1}^{\infty} \frac{(p\,n+\rho-1)\,[(\gamma)_{\delta\,n}]^s\,cz^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$= z^2\,\ddot{E}^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) \ + \ 4z\,\dot{E}^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r)$$

$$-\,3 \sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,cz^{(p\,n+\rho-1)}\,(p\,n+\rho-1)\,z^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}. \tag{2.9}$$

Now,

$$\frac{d}{dz}\left[z\,E^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r)\right]$$

$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,(p\,n+\rho)\,cz^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn}},$$

and

$$\sum_{n=1}^{\infty} \frac{[(\gamma)_{\delta\,n}]^s\,cz^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$= z\,\dot{E}^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r). \tag{2.10}$$

Combining (2.9) and (2.10) yields

$$\sum_{n=1}^{\infty} \frac{(p\,n+\rho-1)\,[(\gamma)_{\delta\,n}]^s\,cz^{(p\,n+\rho-1)}}{\Gamma((\alpha+a)(p\,n+\rho-1)+\beta+b+3)\,[(\lambda)_{\mu\,n}]^r\,(\rho)_{pn-1}}$$

$$= z\,\dot{E}^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) + z^2\,\ddot{E}^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) \tag{2.11}$$

Applying (2.10) and (2.11) to (2.7), we find that

$$S = (\alpha+t)^2\,z^2\,\ddot{E}^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) \ + \ z\left[(\alpha+a)^2 + (\alpha+a) + x\right]$$

$$.\dot{E}^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r) \ + \ (\beta+b+y)\,E^{\gamma,\delta}_{\alpha+a,\ \beta+b+3,\ \lambda,\ \mu,\ \rho,\ p}(cz;s,r).$$

Now setting this last identity for S in (2.6), completes the proof of Theorem 1.

3. Integral Representation:

Theorem 2. For $\alpha,\ \beta,\ \gamma,\ \lambda,\ \rho\ \in \mathbb{C},\ Re((\alpha+a),(\beta+b),\gamma,\lambda,\rho) > 0$ $\delta,\ \mu,\ p,\ c > 0,$ we get

$$\int_0^1 (cu)^{\beta+b}\,E^{\gamma,\delta}_{\alpha+a,\ \beta+b,\ \lambda,\ \mu,\ \rho,\ p}((cu)^{\alpha+a};s,r)du$$

$$= \ c^{\beta+b}\left(E^{\gamma,\delta}_{\alpha+a,\ \beta+b+1,\ \lambda,\ \mu,\ \rho,\ p}(c^{\alpha+a};s,r) \ - \ E^{\gamma,\delta}_{\alpha+a,\ \beta+b+2,\ \lambda,\ \mu,\ \rho,\ p}(c^{\alpha+a};s,r)\right). \tag{3.1}$$

Setting $\alpha+a \ = \ k \in \mathbb{N}$ and $\beta+b \ = \ m \in \mathbb{N}$ in (3.1) yields

Corollary:

$$\int_0^1 (cu)^m \, E^{\gamma,\delta}_{k,\, m,\, \lambda,\, \mu,\, \rho,\, p}((cu)^k; s, r)du$$
$$= c^m \Big(E^{\gamma,\delta}_{k,\, m+1,\, \lambda,\, \mu,\, \rho,\, p}(c^k; s, r) \; - \; E^{\gamma,\delta}_{k,\, m+2,\, \lambda,\, \mu,\, \rho,\, p}(c^k; s, r) \Big). \tag{3.2}$$

Proof of the Theorem 2. Putting z=1 in (2.6) gives

$$\sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s \, (c)^{(p\, n+\rho-1)}}{\Gamma((\alpha+a)(p\, n+\rho-1)+\beta+b)} \cdot \frac{((\alpha+a)(p\, n+\rho-1)+\beta+b+1)^{-1}}{[(\lambda)_{\mu\, n}]^r \, (\rho)_{pn}}$$
$$= E^{\gamma,\delta}_{\alpha+a,\, \beta+b+1,\, \lambda,\, \mu,\, \rho,\, p}(c; s, r) - E^{\gamma,\delta}_{\alpha+a,\, \beta+b+2,\, \lambda,\, \mu,\, \rho,\, p}(c; s, r). \tag{3.3}$$

It is easy to find that

$$\int_0^z (cu)^{\beta+b} \, E^{\gamma,\delta}_{\alpha+a,\, \beta+b,\, \lambda,\, \mu,\, \rho,\, p}((cu)^{\alpha+a}; s, r)du$$
$$= \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta\, n}]^s \, c^{(\alpha+a)(p\, n+\rho-1)+\beta+b} \, z^{(\alpha+a)(p\, n+\rho-1)+\beta+b+1}}{\Gamma((\alpha+a)(p\, n+\rho-1)+\beta+s)} \cdot \frac{((\alpha+a)(p\, n+\rho-1)+\beta+b+1)^{-1}}{[(\lambda)_{\mu\, n}]^r \, (\rho)_{pn}}. \tag{3.4}$$

On comparing (3.3) with the identity obtaining by setting z=1 in (3.4) completes the proof of Theorem 2.

4. Special Cases:

1. Setting $r = 0, \rho = p = c = s = 1, \delta = q$ in (2.1), we get recurrence relation of $E^{\gamma,q}_{\alpha,\, \beta}(z)$ [6]:

$$E^{\gamma,\, q}_{\alpha+a,\, \beta+b+1}(z) \; - \; E^{\gamma,\, q}_{\alpha+a,\, \beta+b+2}(z) = (\alpha+a)^2 \, z^2 \ddot{E}^{\gamma,\, q}_{\alpha+a,\, \beta+b+3}(z)$$
$$+ z\, (\alpha+a)\big[\alpha+a+2(\beta+b+1)\big] \dot{E}^{\gamma,\, q}_{\alpha+a,\, \beta+b+3}(z)$$
$$+ \, (\beta+b)\, (\beta+b+2)\, E^{\gamma,\, q}_{\alpha+a,\, \beta+b+3}(z), \tag{4.1}$$

where, $\dot{E}^{\gamma,\, q}_{\alpha,\, \beta}(z) \; = \; \dfrac{d}{dz} E^{\gamma,\, q}_{\alpha,\, \beta}(z)$ and $\ddot{E}^{\gamma,\, q}_{\alpha,\, \beta}(z) \; = \; \dfrac{d^2}{dz^2} E^{\gamma,\, q}_{\alpha,\, \beta}(z)$.

2. Putting $r = a = 0, \gamma = \delta = \rho = p = s = 1; \beta + b = m \in \mathbb{N}$ in (2.1) reduces to a known recurrence relation of $E_{\alpha,\, \beta}(z)$ [1]:

$$E_{\alpha,\, m+1}(z) \; = \; \alpha^2 \, z^2 \ddot{E}_{\alpha,\, m+3}(z) \; + \; \alpha(\alpha+2m+2)\, z\, \dot{E}_{\alpha,\, m+3}(z)$$
$$+ \, m(m+2)\, E_{\alpha,\, m+3}(z) \; + \; E_{\alpha,\, m+2}(z), \tag{4.2}$$

where, $\dot{E}_{\alpha,\, \beta}(z) \; = \; \dfrac{d}{dz} E_{\alpha,\, \beta}(z)$ and $\ddot{E}_{\alpha,\, \beta}(z) \; = \; \dfrac{d^2}{dz^2} E_{\alpha,\, \beta}(z)$.

3. Substituting $r = 0, \rho = p = c = s = 1, \delta = q$ in (3.1), we get integral representation of $E^{\gamma,q}_{\alpha,\, \beta}(z)$ [6]:

$$\int_0^1 u^{\beta+b} \, E^{\gamma,\, q}_{\alpha+a,\, \beta+b}(u^{\alpha+a})du \; = \; E^{\gamma,\, q}_{\alpha+a,\, \beta+b+1}(1) \; - \; E^{\gamma,\, q}_{\alpha+a,\, \beta+b+2}(1) \tag{4.3}$$

4. Substituting $(r = 0, \rho = p = c = \delta = \gamma = s = k = m = 1)$ and $(r = 0,\; \rho = p = c = \delta = s = k = m = 1\,, \gamma \; = \; 2)$ in (3.2) respectively, yields

$$\int_0^1 u\, e^u \, du \; = \; E_{1,\, 2}(1) \; - \; E_{1,\, 3}(1)$$

and

$$\int_0^1 u\, E^{2,1}_{1,\, 1}(1) \; = \; E^{2,1}_{1,\, 2}(1) \; - \; E^{2,1}_{1,\, 3}(1).$$

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Received: November 7, 2012

Accepted: February 12, 2013