RECURRENT RELATION OF A UNIFIED GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. The present work incorporates a recurrence relation and an integral representation of a further generalization of a generalized Mittag-Leffler function due to A.K. Shukla and J.C. Prajapati [Surveys in Mathematics and its Applications, Volume 4(2009), 133-138]. At the end, several special cases have also been discussed.

1. Introduction, definitions and Preliminaries

The Mittag-Leffler function has been studied by many researchers either in obtaining new properties or by introducing a new generalization and then deriving its properties ([5], [7], [4]). Recently, we have also studied various properties of our newly introduced unification of Generalized Mittag-Leffler function in the form [2]

\[ E_{\gamma, \delta}^{\alpha, \beta, \lambda, \mu, \rho, p}(cz; s, r) = \sum_{n=0}^{\infty} \frac{[\gamma]^n}{\Gamma(\alpha(pn + \rho - 1) + \beta)} \frac{(cz)^n}{(\rho)^n}, \]

where, \( \alpha, \beta, \gamma, \lambda, \rho \in \mathbb{C}, \text{Re}(\alpha, \beta, \gamma, \lambda, \rho) > 0; \delta, \mu, p, c > 0. \) If \( p = 1, \rho = 1, r = 0, s = 1, \delta = q, s = 1, c = 1, \) then this yields the generalization due to Shukla and Prajapati [5]. In the next section, we prove the main results.

2. Recurrence Relation

We begin by stating the main theorem.

Theorem 1. For \( \alpha, \beta, \gamma, \lambda, \rho \in \mathbb{C}, \text{Re}((\alpha + a), (\beta + b), \gamma, \lambda, \rho) > 0, \delta, \mu, p, c > 0, \) we get

\[ E_{\alpha+a, \beta+b+1, \lambda, \mu, \rho}^{\gamma, \delta}(cz; s, r) = E_{\alpha+a, \beta+b, \lambda, \mu, \rho}^{\gamma, \delta}(cz; s, r) = (\alpha + a)^2 z^2 E_{\alpha+a, \beta+b+3, \lambda, \mu, \rho}^{\gamma, \delta}(cz; s, r) \\
+ z (\alpha + a)(\alpha + a + 2(\beta + b + 1)) E_{\alpha+a, \beta+b+3, \lambda, \mu, \rho}^{\gamma, \delta}(cz; s, r) \\
+ (\beta + b)(\beta + b + 2) E_{\alpha+a, \beta+b+3, \lambda, \mu, \rho}^{\gamma, \delta}(cz; s, r), \]

where, \( \dot{E}_{\alpha, \beta, \lambda, \mu, \rho}^{\gamma, \delta}(cz; s, r) = \frac{d}{dz} E_{\alpha, \beta, \lambda, \mu, \rho}^{\gamma, \delta}(cz; s, r), \)

\[ \ddot{E}_{\alpha, \beta, \lambda, \mu, \rho}^{\gamma, \delta}(cz; s, r) = \frac{d^2}{dz^2} E_{\alpha, \beta, \lambda, \mu, \rho}^{\gamma, \delta}(cz; s, r), \]

It is easy to obtain the following corollary by letting \( \alpha + a = k \) and \( \beta + b = m. \)
Corollary: We have, for \( k, m \in \mathbb{N} \),
\[
E_{k, m+1, \lambda, \mu, \rho, p}(cz; s, r) = E_{k, m+2, \lambda, \mu, \rho, p}(cz; s, r) + m(m + 2)E_{k, m+3, \lambda, \mu, \rho, p}(cz; s, r) + k^2 z^2 E_{k, m+3, \lambda, \mu, \rho, p}(cz; s, r).
\]
(2.2)

Proof of Theorem 1. By substituting \( \alpha = \alpha + \beta = \beta + b + 1 \) in (1.1) and applying the fundamental relation of the Gamma function \( \Gamma(z + 1) = z\Gamma(z) \), we have
\[
E_{\alpha+1, \beta+b+1, \lambda, \mu, \rho, p}(cz; s, r) = \sum_{n=0}^{\infty} \frac{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b + 1)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}^r (n)\mu_n^r (\rho)pn
\]
(2.3)
and
\[
E_{\alpha+1, \beta+b+2, \lambda, \mu, \rho, p}(cz; s, r) = \sum_{n=0}^{\infty} \frac{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b + 1)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}^r (n)\mu_n^r (\rho)pn
\]
(2.4)
Equation (2.4) can be written as follows:
\[
E_{\alpha+1, \beta+b+1, \lambda, \mu, \rho, p}(cz; s, r) = \sum_{n=0}^{\infty} \frac{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b + 1)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}^r (n)\mu_n^r (\rho)pn
\]
(2.5)
For the sake of convenience, we denote the last summation in (2.5) by \( S \), then
\[
S = E_{\alpha+1, \beta+b+1, \lambda, \mu, \rho, p}(cz; s, r) - E_{\alpha+1, \beta+b+2, \lambda, \mu, \rho, p}(cz; s, r).
\]
(2.6)
Applying the following (evident):
\[
\frac{1}{u} = \frac{1}{u(u + 1)} + \frac{1}{(u + 1)}
\]
and then taking \( u = (\alpha + a)(p n + \rho - 1) + \beta + b + 1 \) to (2.6), we obtain
\[
S = \sum_{n=0}^{\infty} \frac{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b + 1)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}^r (n)\mu_n^r (\rho)pn
\]
(2.6)
\[
= (\alpha + a) \sum_{n=1}^{\infty} \frac{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b + 1)}{(\alpha + a)(p n + \rho - 1) + \beta + b + 1}\right)}^r (n)\mu_n^r (\rho)pn
\]
(2.6)
\[
+ (\beta + b) \sum_{n=0}^{\infty} \frac{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b + 3)}{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b + 3)}\right)}{\left(\frac{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b + 3)}{\Gamma(\alpha + a)(p n + \rho - 1) + \beta + b + 3)}\right)}^r (n)\mu_n^r (\rho)pn
\]
\[\begin{align*}
&(\alpha + a)^2 \sum_{n=1}^{\infty} \frac{(p n + \rho - 1) (cz)^{(p n + \rho - 1)}}{(\gamma \delta n)^{a}} \Gamma((\alpha + a)(p n + \rho - 1) + \beta + b) \\
+ x \sum_{n=1}^{\infty} \frac{([\gamma \delta n])^{a} (cz)^{(p n + \rho - 1)}}{\Gamma((\alpha + a)(p n + \rho - 1) + \beta + b)} + y \sum_{n=0}^{\infty} \frac{([\gamma \delta n])^{a} (cz)^{(p n + \rho - 1)}}{\Gamma((\alpha + a)(p n + \rho - 1) + \beta + b + 3)} \right)
\end{align*}\]

where, \(x = (\alpha + a)(2 \beta + 2 b + 1)\) and \(y = (\beta + b)(\beta + b + 1)\).

We now express each summation in the right hand side of (2.7) as follows:

\[\begin{align*}
\frac{d^2}{dz^2} \left[ x^2 E_{\alpha+a, \beta+b+3, \lambda, \mu, \rho}^{\gamma, \delta} (cz; s, r) \right] &= \sum_{n=0}^{\infty} \frac{([\gamma \delta n])^{a} (p n + \rho - 1) (cz)^{(p n + \rho - 1)}}{\Gamma((\alpha + a)(p n + \rho - 1) + \beta + b + 3)} \right)
\end{align*}\]

(2.8)

From (2.8) we find that

\[\begin{align*}
\sum_{n=1}^{\infty} \frac{1}{\Gamma((\alpha + a)(p n + \rho - 1) + \beta + b + 3)} \right)
\end{align*}\]

(2.9)

Now,

\[\begin{align*}
\frac{d}{dz} \left[ x^2 E_{\alpha+a, \beta+b+3, \lambda, \mu, \rho}^{\gamma, \delta} (cz; s, r) \right] &= \sum_{n=0}^{\infty} \frac{([\gamma \delta n])^{a} (p n + \rho - 1) (cz)^{(p n + \rho - 1)}}{\Gamma((\alpha + a)(p n + \rho - 1) + \beta + b + 3)} \right)
\end{align*}\]

(2.10)

Combining (2.9) and (2.10) yields

\[\begin{align*}
\sum_{n=1}^{\infty} \frac{1}{\Gamma((\alpha + a)(p n + \rho - 1) + \beta + b + 3)} \right)
\end{align*}\]

(2.11)

Applying (2.10) and (2.11) to (2.7), we find that

\[S = (a + t)^2 x^2 E_{\alpha+a, \beta+b+3, \lambda, \mu, \rho}^{\gamma, \delta} (cz; s, r) + z \left[ (a + a) + (a + a) + x \right] E_{\alpha+a, \beta+b+3, \lambda, \mu, \rho}^{\gamma, \delta} (cz; s, r) + (\beta + b + y) E_{\alpha+a, \beta+b+3, \lambda, \mu, \rho}^{\gamma, \delta} (cz; s, r).
\]

Now setting this last identity for \(S\) in (2.6), completes the proof of Theorem 1.

### 3. Integral Representation:

**Theorem 2.** For \(\alpha, \beta, \gamma, \lambda, \rho \in \mathbb{C}, Re((\alpha + a), (\beta + b), \gamma, \lambda, \rho) > 0\)

\(\delta, \mu, \rho, c > 0\), we get

\[\int_{0}^{1} (cu)^{\beta+b} E_{\alpha+a, \beta+b}^{\gamma, \delta} (cu)^{\alpha+a}; s, r) du = e^{\beta+b} \left( E_{\alpha+a, \beta+b+1}^{\gamma, \delta} (cu)^{\alpha+a}; s, r) - E_{\alpha+a, \beta+b+2}^{\gamma, \delta} (cu)^{\alpha+a}; s, r) \right).
\]

(3.1)

Setting \(\alpha + a = k \in \mathbb{N}\) and \(\beta + b = m \in \mathbb{N}\) in (3.1) yields
Corollary:
\[
\int_0^1 (cu)^m E_{\gamma, \delta}^\gamma k, m, \lambda, \mu, \rho, \gamma (cu)^k; s, r) du = \sum_{n=0}^{\infty} \frac{E_{\gamma, \delta}^\gamma k_{m+1, \lambda, \mu, \rho, \gamma}(cu)^{k_n}; s, r) - E_{\gamma, \delta}^\gamma k_{m+2, \lambda, \mu, \rho, \gamma}(cu)^{k_n}; s, r)}{(\gamma + n + 1)\Gamma(\gamma + n + 1 + \beta + b)}.
\] (3.2)

Proof of the Theorem 2. Putting \( z = 1 \) in (2.6) gives
\[
\int_0^1 (\gamma + n + 1 + \beta + b) \frac{E_{\gamma, \delta}^\gamma k_{m+1, \lambda, \mu, \rho, \gamma}(cu)^{k_n}; s, r) - E_{\gamma, \delta}^\gamma k_{m+2, \lambda, \mu, \rho, \gamma}(cu)^{k_n}; s, r)}{(\gamma + n + 1)\Gamma(\gamma + n + 1 + \beta + b)}.
\] (3.3)

It is easy to find that
\[
\int_0^1 (\gamma + n + 1 + \beta + b) \frac{E_{\gamma, \delta}^\gamma k_{m+1, \lambda, \mu, \rho, \gamma}(cu)^{k_n}; s, r) - E_{\gamma, \delta}^\gamma k_{m+2, \lambda, \mu, \rho, \gamma}(cu)^{k_n}; s, r)}{(\gamma + n + 1)\Gamma(\gamma + n + 1 + \beta + b)}.
\] (3.4)

On comparing (3.3) with the identity obtained by setting \( z = 1 \) in (3.4) completes the proof of Theorem 2.

4. Special Cases:

1. Setting \( r = 0, \rho = p = c = s = 1, \delta = q \) in (2.1), we get recurrence relation of \( E_{\alpha, \beta}^{\gamma, q}(z) \) [6]:
\[
E_{\alpha + a, \beta + b + 1}(z) = E_{\alpha + a, \beta + b + 2}(z) = (\alpha + a)^2 E_{\alpha + a, \beta + b + 3}(z)
\]
\[
+ z(\alpha + a)(\alpha + a + 2(\beta + b + 1)) E_{\alpha + a, \beta + b + 3}(z)
\]
\[
+ (\beta + b)(\beta + b + 2) E_{\alpha + a, \beta + b + 3}(z),
\] (4.1)

where, \( \dot{E}_{\alpha, \beta}^{\gamma, q}(z) = \frac{d}{dz} E_{\alpha, \beta}^{\gamma, q}(z) \) and \( \ddot{E}_{\alpha, \beta}^{\gamma, q}(z) = \frac{d^2}{dz^2} E_{\alpha, \beta}^{\gamma, q}(z) \).

2. Putting \( r = a = 0, \gamma = \delta = \rho = p = s = 1; \beta + b = m \in \mathbb{N} \) in (2.1) reduces to a known recurrence relation of \( E_{\alpha, \beta}^{\gamma, q}(z) \) [1]:
\[
E_{\alpha, m+1}(z) = \alpha^2 z^2 E_{\alpha, m+1}(z) + \alpha(\alpha + 2m + 2) z E_{\alpha, m+1}(z)
\]
\[
+ m(m+2) E_{\alpha, m+1}(z) + E_{\alpha, m+1}(z),
\] (4.2)

where, \( \dot{E}_{\alpha, \beta}^{\gamma, q}(z) = \frac{d}{dz} E_{\alpha, \beta}^{\gamma, q}(z) \) and \( \ddot{E}_{\alpha, \beta}^{\gamma, q}(z) = \frac{d^2}{dz^2} E_{\alpha, \beta}^{\gamma, q}(z) \).

3. Substituting \( r = 0, \rho = p = c = s = 1, \delta = q \) in (3.1), we get integral representation of \( E_{\alpha, \beta}^{\gamma, q}(z) \) [6]:
\[
\int_0^1 u^{\beta + b} E_{\alpha + a, \beta + b + 1}(u^{\alpha + a}) du = E_{\alpha + a, \beta + b + 1}(1) - E_{\alpha + a, \beta + b + 2}(1)
\] (4.3)

4. Substituting \( r = 0, \rho = p = c = \delta = \gamma = s = k = m = 1 \) and \( r = 0, \rho = p = c = \delta = s = k = m = 1, \gamma = 2 \) in (3.2) respectively, yields
\[
\int_0^1 u e^u du = E_{1, 2}(1) - E_{1, 3}(1)
\]
and
\[
\int_0^1 u E_{1, 1}(1) = E_{1, 2}(1) - E_{1, 3}(1).
\]
References


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