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Equations of motion of a relativistic charged particle with curvature dependent actions

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Abstract. We present an introduction to the study of a relativistic particle moving under the influence of its own Frenet-Serret curvatures. With the aim of introducing the notation and conventions used in this paper, we first recall the action of a relativistic particle. We then suggest a mathematical generalization of this action in the sense that now the action may include terms of the curvatures of the world line generated by the particle in Minkowski space-time. We go on to develop a pedagogical introduction to a variational calculus which takes advantage of the Frenet-Serret equations for the relativistic particle. Finally, we consider a relativistic particle coupled to an electromagnetic field that is moving under the influence of its own Frenet-Serret curvatures. Within this frame based on the Frenet-Serret basis, we obtain the equations of motion for several curvature dependent actions of interest in physics. Later, as an illustration of the formalism developed, we consider the simplest case, that of a relativistic particle when no geometrical action is included, in order to show (i) the equivalence of this formalism to the Newton's second law with the Lorentz force and (ii) the integrability in the case of a constant electromagnetic field.

1 Introduction

The dynamics of a mechanical system is governed by *the principle of least action*, which states that the motion of a system, between the times t_i and t_f, is such that the action A, defined by the integral

$$A = \int_{t_i}^{t_f} dt\, L\,, \tag{1.1}$$

takes the least possible value. In general, the Lagrangian L depends only on the positions and velocities of the system, see Ref.[1]. Although sometimes, the motion of the mechanical system can only be described by means of empirical Lagrangians.

For a free relativistic particle, the lagrangian and action are given by

$$A = -mc^2 \int_{t_i}^{t_f} d\,t \sqrt{1 - v^2/c^2} \tag{1.2}$$

where m and v are the mass and velocity of the particle while c is the speed of light in empty space. The dynamics of a relativistic particle can be better studied in *Minkowski space-time*, here denoted by M. The points in M are called *events*, which are generated by the point particle motion. An event has three spatial coordinates x, y, z and the time t, The collection of events forms the particle *world line* in M. The length or *the infinitesimal interval* of the world line on M is given by $ds^2 = -dt^2 + d\vec{x}^2$. Then, following the Wienberg's conventions in Ref.[2], it can be written as $ds^2 = \eta_{\mu\,\nu}\, dx^\mu\, dx^\nu$, where

$$\eta_{\mu\,\nu} = \begin{cases} -1 & \mu = \nu = 0 \\ 1 & \mu, \nu => i, j = 1, 2, 3 \\ 0 & \mu \neq \nu \end{cases} \quad . \tag{1.3}$$

where $\mu = 0, 1, 2, 3$, such that $x^0 = t$ and x^i are x, y, z. Let us now denote the particle world line by X^μ. The X^μ are the embedding functions of the particle motion in M by means of $x^\mu = X^\mu(\xi)$, where ξ is an arbitrary parameter only useful for generating the motion. For example, the *proper time* of the particle is given by

$$ds^2 = -d\tau^2 = \eta_{\mu\nu}\,\frac{dX^\mu}{d\xi}\frac{dX^\nu}{d\xi}d\xi^2. \tag{1.4}$$

It will be convenient to introduce the scalar function γ by

$$\gamma = \eta_{\mu\nu}\,\frac{dX^\mu}{d\xi}\frac{dX^\nu}{d\xi} \tag{1.5}$$

then the particle proper time is given by

$$d\tau = \sqrt{-\gamma}\,d\xi \tag{1.6}$$

and therefore the action of a free relativistic particle can be written as

$$A = \int_{t_i}^{t_f} \frac{d\,t}{\gamma} \equiv \int d\,\tau \tag{1.7}$$

It is possible to consider a mathematical generalization of the action given in Eq. 1.7. The main idea behind this generalization has come from string theory, where the string action included an additional term proportional to the external curvature of the string world-sheet. This additional term was proposed in order to avoid the occurrence of sharp string configurations in the resulting string dynamics, see Ref. [3]. For this reason, it was named the rigid string theory. The natural translation of this idea to the action of a point particle, was to include an additional term to Eq.1.7 with the world-line curvature.

In the purely mathematical sense, the relativistic particle Lagrangian can be generalized by including terms depending on all the curvatures of the particle world-line in the following manner. As the world-line of a particle evolving in a fixed Minkowski space-time of general dimension $N+1$, can have associated up to N curvatures, $\kappa_{1..N}$, a hierarchy of Lagrangians with successively higher curvatures $\kappa_{1..N}$ can be introduced into the action in the form

$$A[X] = \int d\tau\, L_g(\kappa_1, \kappa_2, \kappa_3, ...\kappa_N). \tag{1.8}$$

In fact, in Ref.[4] such mathematical generalization was considered.

The mathematical models pointed out in Eq. 1.8 have also been considered as effective theories for describing the dynamics of an object when its internal structure is not well resolved. For instance, an effective bosonic theory used to describe a super-symmetric particle can be put in terms of some of these models, see Ref. [5]. Besides, the description of spinning particles has incorporated the attractive idea of considering that the spin degrees of freedom are encoded in the world line geometry. In these two examples, the extrinsic curvature in the Lagrangian is expected to supply those extra degrees of freedom. Because the curvature is proportional to the particle's acceleration, such effective actions will then contain terms with derivatives on $X^\mu(\tau)$ equal or higher than two, that is $\frac{d^2X^\mu}{d\,\tau^2}$.

The theoretical study of such curvature dependent actions for the relativistic point particle has a long history. Let us now mention just a few examples of the kind of curvature dependent theories that have been considered so far. It was started in the late 80s, when Plyushchay studied for the first time the physics of the linear theory in the first curvature, that is, $L_g = -\mu + \kappa_1$, see Ref. [6]. Subsequently, in Ref. [7], Plyushchay was able to show that there are three types of explicit solutions for the dynamics of such rigid particle, namely: massive, massless and tachyonic, depending on the value of the particle invariant $M^2 \equiv P^\mu\,P_\mu$, where P^μ is particle's four momentum. Dereli *et al.* in Ref. [8] have considered the theory $L_g = -\mu + \kappa_1^2$ while Nesterenko *et al.* [9] and [10] have studied the integrability of the case $L_g = f(\kappa_1)$ for a general function f. The linear model in the first curvature, $L_g = \kappa_1$, was firstly considered by Plyushchay in Ref. [11] and by Zoller in Ref. [12] who was able to show that the natural coupling of this linear curvature theory to gravity does not yield a consistent theory.

Plyushchay went on to consider the most general curvature dependent lagrangian, $L_g = f(\kappa_1^2)$, with the purpose of studying the quantization of such a particle, see Ref. [13]. The question of quantization of a relativistic particle models with higher derivatives was investigated again by Plyushchay in Ref. [14].

The model of the relativistic particle with an action depending linearly on the curvature and torsion, $L_g = -\mu + cte\,\kappa_1 + cte\,\kappa_2$, was investigated both at the classical and quantum levels by Kuznetsov and Plyushchay, see Refs. [15] , [16], [17] and [18]. A review of the relations between the Majorana equation to the higher derivative particle models was provided in Ref.[19]. Particularly, the model with torsion has been proved to be usefull as a model for relativistic anyons in $(2+1)D$ space-time, see Refs. [20] and [21]. A relation between the relativistic particle with torsion in Minkowski $(2+1)D$ spacetime to the model of the non-relativistic $3D$ Euclidean particle in the field of a monopole has been revealed in Ref. [22]. Plyushchay also demonstrated that the equation of motion of a particle including a linear term on the second curvature (or torsion) of the world-line, coincide with the equation of motion of a relativistic charged particle in an external constant electromagnetic field, see Ref. [23].

Dirac's theory of constrained Hamiltonian dynamics provides a basic tool for understanding gauge symmetry of classical Lagrangian systems and forms the starting point of their quantization. It is notable that this tool is almost confined within the realm of Lagrangian dependent of coordinates and velocities only. The important sector of the higher order Lagrangians remains considerably less explored. A direct connection between the gauge symmetry and the W-algrebra for the rigid relativistic particle was shown by [24]. Later, it was shown by [25] that the quantization in terms of $so(3,2)$ algebra of the Lagrangian linear in the curvature yields massive Dirac equation. A relativistic particle model with curvature as a theory possessing a gauge symmetry was studied by [26], where the constraint analysis of this model and its massless analogue was discussed.

It was shown in Ref.[27] that the particle dynamics with higher derivatives is necessary for the extension of classical field theories to spacetimes of higher dimension, such as the Maxwell-Lorentz electrodynamics; otherwise not all self-energy divergences are eliminated through their absorbing by free parameters in the Lagrangian and redefining them to become finite parameters. Besides, it was noted by [28] that a free relativistic particle with the Lagrangian linear or the quadratic in the curvature may exhibit runaway motions due to instability of zitterbewegung inherent in these models. In contrast, the Lagrangians containing the term k^q, where $0 < q < 1/2$, ensure stable planar zitterbewegungs for a large values of k.

More recently in Ref.[29], Barros has considered a Lagrangian depending linearly on the world line curvature with the purpose of describing the dynamics associated with relativistic particles both massive and massless. He has obtained the whole space of solutions in an space-time with constant curvature. In Ref. [30], Ferrández *et al.* have shown the moduli spaces of solutions in a three dimensional pseudo-Riemannian space for the model of a relativistic particle with curvature and torsion. In Ref. [31], Arroyo *et al.* have considered the model $L_g = m + n\,\kappa_1 + p\,\kappa_2$ and has found all the solutions for the constants m, n, p; or in other words, that the spinning relativistic particles evolve along Lancret curves in a $3D$ space with constant curvature. In Ref. [32] numerical integration of the Cartan equation of motion for a relativistic particle with curvature was carried out in order to explicitly show the particle world-lines.

The authors of Refs.[33] and [34], have explored the correspondence between the geometry of the world lines described by a Frenet-Serret basis and the dynamics of a charged particle as they related the two invariants of the electromagnetic field with the curvatures of the world line.

2 Geometric elements

The mathematical problem of finding the equation of motion associated with an action implies the calculation of the variation of all the terms under the action integral, in such a way that the real motion of the particle will be the one having its first variation identically zero.

A Frenet-Serret vectorial basis can be attached to every point of any smooth curve in $3D$ space generated by the motion of a particle. The basis consists of a set of three vectors: the tangent vector and two ortho-normal vectors, which are called the normal and the bi-normal, respectively. To complete the Frenet-Serret basis, one also needs to introduce the scalar functions κ and τ, known as the curvature and torsion, respectively, see Ref. [35].

Thus, in order to calculate the first variation of all the geometrical terms involved in an action like the one given by Eq. 1.8, we develop in Section 2.2 a variational procedure, based on the Frenet-Serret equation for a relativistic particle. But we first show in Section 2.1 the generalization of a well know procedure in $3D$ space for constructing the Frenet-Serret equations in the four dimensional spacetime $\mathbb{M}$.

The Frenet-Serret basis adapted for the Minkowski spacetime $\mathbb{M}$ has been used to generalize the fundamental theorem of curves in Euclidean $3D$ space, so that the curvature and torsion completely determine the curve up to a rigid motion. In Ref. [34] was shown that in the context of Minkowski spacetime, the three curvatures fixed the world line up to a Poincare transformation.

The authors of [34] also offered a proof for the statement that a world line with a non-vanishing first curvature is plane if and only if the second and third curvatures identically vanish. For the case in which just the third curvature vanishes, the world line lies in a hyperplane.

2.1 The Frenet-Serret basis

Let us now generalize the idea of introducing a vectorial basis at any given point of a curve generated by a relativistic particle evolving in space-time $\mathbf{M}$. Let us start by defining the tangent vector T^μ by $T^\mu = \frac{dX^\mu}{d\tau} \equiv X'^\mu$, where a prime hereafter means a derivative with respect to τ, the proper time defined by Eq.1.6. Making the chance of coordinates $\xi => \tau$ in Eq. 1.4, we get

$$\eta(T,T) \equiv \eta_{\mu\nu} T^\mu T^\nu = -1 \,. \tag{2.1}$$

The set of four-vectors N_i^μ must be orthogonal to the tangent four-vector T^μ, as well as unitary. By taking the derivative of Eq. 2.1 with respect to τ, we find that $\eta(\frac{dT}{d\tau}, T) = 0$, and that these vectors are orthogonal. We then define the four-vector N_1^μ by

$$\frac{dT^\mu}{d\tau} \equiv \kappa_1 \, N_1^\mu \tag{2.2}$$

such that κ_1 is its norm, and it satisfies the relations

$$\eta(T, N_1) = 0 \qquad \eta(N_1, N_1) = 1 \tag{2.3}$$

Let us now take the derivative $\frac{d}{d\tau}$ of the first relation of Eq. 2.3: $\eta(\frac{dT}{d\tau}, N_1) + \eta(T, \frac{dN_1}{d\tau}) = 0$. By using Eq. 2.2 into the left hand side of this last expression, we have that $\kappa_1 + \eta_{\mu\nu} T^\mu \frac{dN_1^\nu}{d\tau}$. It is possible to factorize the tensor $\eta_{\mu\nu}$ from this equation, to obtain $\eta_{\mu\nu} T^\mu \left(-\kappa_1 T^\nu + \frac{dN_1^\nu}{d\tau} \right) = 0$ from which we conclude that the vector in the bracket is orthogonal to the tangent, therefore we can define the second normal vector by

$$\frac{dN_1^\mu}{d\tau} \equiv \kappa_2 \, N_2^\mu + \kappa_1 \, T^\mu \tag{2.4}$$

such that κ_2 is its norm; then the four-vector N_2^μ satisfies the relations

$$\eta(T, N_2) = 0 \qquad \eta(N_2, N_2) = 1 \qquad \eta(N_1, N_2) = 0 \tag{2.5}$$

That N_2 is also orthogonal to N_1 can easily be proved by dotting Eq. 2.4 with N_1.

Let us now continue by taking the derivative of the third relation of Eq. 2.5 with respect to τ. We have $\eta(\frac{dN_1}{d\tau}, N_2) + \eta(N_1, \frac{dN_2}{d\tau}) = 0$. Substituting Eq. 2.4 into the first term of this relation and factorizing the term $\eta(N_1,)$ in its components explicitly, we obtain $\eta_{\mu\nu} N_1^\mu \left(\kappa_2 N_1^\nu + \frac{dN_2^\nu}{d\tau} \right) = 0$, from which we are able to introduce the definition of the third normal four-vector N_3^μ by means of

$$\frac{dN_2^\mu}{d\tau} + \kappa_2 \, N_1^\mu \equiv \kappa_3 \, N_3^\mu \tag{2.6}$$

such that κ_3 is its norm; then the vector N_3 satisfies the relations

$$\eta(T, N_3) = 0 \qquad \eta(N_3, N_3) = 1 \qquad \eta(N_1, N_3) = 0 \qquad \eta(N_2, N_3) = 0 \tag{2.7}$$

By dotting Eq. 2.6 with N_2 we obtain the third relation of Eq. 2.7. Likewise, by dotting Eq. 2.6 with T^μ and commuting the derivative between the terms and substituting Eq. 2.2, we prove the first relation of Eq. 2.7.

Finally, we take the derivative with respect to τ of the third relation of Eq. 2.7: $\eta(\frac{dN_2}{d\tau}, N_3) + \eta(N_2, \frac{dN_3}{d\tau}) = 0$. Substituting Eq. 2.4 into the first term of the precedent expression and factorizing the term $\eta(N_2,)$, we get $\eta_{\mu\nu} N_2^\mu \left(\kappa_3 N_2^\nu + \frac{dN_3^\nu}{d\tau} \right) = 0$. We could introduce a fourth normal vector defining it as the term inside the parenthesis. However, there is no dimension in the space-time $\mathbf{M}$ to associate another normal vector, so we have to close the process and define the term as identically zero, therefore

$$\frac{dN_3^\mu}{d\tau} + \kappa_3 \, N_2^\mu \equiv 0 \tag{2.8}$$

Summing up, so far we have built the Frenet-Serret basis for the relativistic particle in the space $\mathbf{M}$; a basis which is formed by the following vectors

$$
\begin{aligned}
\frac{dT^\mu}{d\tau} &= \kappa_1\, N_1^\mu \\
\frac{dN_1^\mu}{d\tau} &= \kappa_2\, N_2^\mu + \kappa_1\, T^\mu \\
\frac{dN_2^\mu}{d\tau} &= \kappa_3 N_3^\mu - \kappa_2\, N_1^\mu \\
\frac{dN_3^\mu}{d\tau} &= -\kappa_3\, N_2^\mu
\end{aligned}
\tag{2.9}
$$

and with the following properties

$$
\begin{aligned}
&\eta(T, N_i) = 0 \quad i = 1, 2, 3 \\
&\eta(N_i, N_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}
\end{aligned}
\tag{2.10}
$$

The system of equations 2.9 can be easily generalized for a curved space-time by making on all the terms the transformations of derivatives $\frac{dA^\mu}{d\tau}$ to covariant derivatives, that is, $\frac{DA^\mu}{D\tau} = \frac{dA^\mu}{d\tau} + \Lambda^\mu_{\alpha\beta}\frac{dX^\alpha}{d\tau}A^\beta$.

2.2 Relativistic Perturbations.

Let us now consider a small perturbation on a general particle world line embedded in the space-time M by the functions $x^\mu = X^\mu(\tau)$. Let us define δ as a perturbation operator, which action on $X^\mu(\tau)$ can be decomposed along the Frenet-Serret vectors as

$$
\delta X^\mu = \psi_{||} T^\mu + \psi_1 N_1^\mu + \psi_2 N_2^\mu + \psi_3 N_3^\mu \tag{2.11}
$$

where the functions $\psi_{||}, \psi_i$ are numerically small in such a way that we only consider the linear perturbation regime.

By taking the derivative of any vector with respect to τ, we get another vector, which can also be expressed as a linear combination of the Frenet-Serret vectors, for example

$$
\frac{d\,\delta X^\mu}{d\tau} = \left(\psi'_{||} + \kappa_1 \psi_i\right) T^\mu + \left(\psi'_1 - \kappa_2\psi_2 + \kappa_1\psi_{||}\right) N_1^\mu + \left(\psi'_2 - \kappa_3\psi_3 + \kappa_2\psi_1\right) N_2^\mu + \left(\psi'_3 + \kappa_3\psi_2\right) N_3^\mu . \tag{2.12}
$$

Let us now consider the perturbation on the function γ defined in Eq 1.5. We apply to it the variation operator δ, we get

$$
\begin{aligned}
\delta\gamma &= 2\eta_{\mu\nu}\,\frac{dX^\mu}{d\xi}\,\frac{d\delta X^\nu}{d\xi} \\
&= 2\eta_{\mu\nu}\,T^\mu\,\frac{d\delta X^\nu}{d\tau}\,\left(\frac{d\tau}{d\xi}\right)^2 \\
&= 2\left(\frac{d\tau}{d\xi}\right)^2 \eta(T, \frac{d\delta X}{d\tau})
\end{aligned}
\tag{2.13}
$$

where we have used the chain rule in both terms of the dot product shown in the second line, in order to switch to derivatives on τ instead of ξ. Now, according to the third line of Eq. 2.13, we only need to take the tangential component of Eq.2.12 to obtain

$$
\delta\gamma = 2\gamma\left(\psi'_{||} + \kappa_1\,\psi_1\right) , \tag{2.14}
$$

the perturbation of the proper time. Then, by applying the operator δ on Eq. 1.6 and considering that there is a commutation relation with the derivative operator with respect to the arbitrary parameter ξ

$$
\delta\left(\frac{d\tau}{d\xi}\right) = \frac{d\,(\delta\tau)}{d\xi} \tag{2.15}
$$

we get

$$
\delta\,(d\,\tau) = (d\,\tau)\left(\psi'_{||} + \kappa_1\,\psi_1\right) . \tag{2.16}
$$

It is important to point out that the result of Eq.2.16 indicates that the perturbation operator and the derivative operator with respect to the proper time do not commute. For this reason, the chain rule must be used when there are derivatives with respect to τ. As an example, let us consider the perturbation on a derivative of a scalar function $f(\tau)$,

$$\begin{aligned}\delta\left(\frac{df}{d\tau}\right) &= \delta\left(\frac{df}{d\xi}\right)\frac{d\xi}{d\tau}+\frac{df}{d\tau}\delta\left(\frac{d\xi}{d\tau}\right)\\ &= \frac{d\delta f}{d\xi}\frac{d\xi}{d\tau}+\frac{df}{d\tau}\delta\left(\frac{d\xi}{d\tau}\right)\end{aligned} \tag{2.17}$$

and replacing Eqs.1.6 and 2.14, we get

$$\delta\left(\frac{df}{d\tau}\right)=\frac{d\left(\delta f\right)}{d\tau}-\frac{df}{d\tau}\,\left(\psi_{||}'+\kappa_1\,\psi_1\right)\,. \tag{2.18}$$

We can now obtain the perturbation of the tangent four-vector by considering the change $f => X^\mu$ in Eq. 2.18, we get

$$\delta T^\mu\equiv\delta\left(\frac{dX^\mu}{d\tau}\right)=\frac{d\delta X^\mu}{d\tau}-T^\mu\left(\psi_{||}'+\kappa_1\,\psi_1\right) \tag{2.19}$$

and comparing with Eq.2.12 we get that the perturbation of δT^μ

$$\delta T^\mu=\left(\psi_1'-\kappa_2\psi_2+\kappa_1\psi_{||}\right)N_1^\mu+\left(\psi_2'-\kappa_3\psi_3+\kappa_2\psi_1\right)N_2^\mu+\left(\psi_3'+\kappa_3\psi_2\right)N_3^\mu. \tag{2.20}$$

is purely orthogonal to the world-line. For mathematical convenience, we now introduce the α coefficients in the following form

$$\delta T^\mu=\alpha_1\,N_1^\mu+\alpha_2\,N_2^\mu+\alpha_3\,N_3^\mu \tag{2.21}$$

where

$$\begin{aligned}\alpha_1 &= \psi_1'-\kappa_2\psi_2+\kappa_1\psi_{||}\\ \alpha_2 &= \psi_2'-\kappa_3\,\psi_3+\kappa_2\psi_1\\ \alpha_3 &= \psi_3'+\kappa_3\,\psi_2\,.\end{aligned} \tag{2.22}$$

With the purpose of calculating the variation of the first curvature, we solve for κ_1 from the first Frenet-Serret Eq. 2.2, obtaining $\kappa_1=\eta(N_1,\frac{dT}{d\tau})$. Then, applying the δ operator we get,

$$\delta\kappa_1=\eta(\delta N_1,\frac{dT}{d\tau})+\eta(N_1,\delta\frac{dT}{d\tau}). \tag{2.23}$$

The first term vanishes, as it can be probed by making use again of the first Frenet-Serret Eq. 2.2 and factorizing the operator δ, that is $\eta(\delta N_1,\frac{dT}{d\tau})=\kappa_1\eta(\delta N_1,N_1)=\frac{\kappa_1}{2}\delta\eta(N_1,N_1)\equiv 0$. To calculate the second term of Eq. 2.23, we can use Eq.2.18 with $f => T^\mu$, to get

$$\delta\frac{dT^\mu}{d\tau}=\frac{d\delta T^\mu}{d\tau}-\frac{dT^\mu}{d\tau}\left(\psi_{||}'+\kappa_1\psi_1\right) \tag{2.24}$$

Using now Eqs. 2.12 and 2.21 but with α coefficients α defined by Eq. 2.22 instead of the ψ functions, we get the component along the N_1 vector, in such a way that the variation is,

$$\delta\kappa_1=\alpha_1'-\alpha_2\kappa_2-\kappa_1\left(\psi_{||}'+\kappa_1\psi_1\right) \tag{2.25}$$

So, the final result of this variation expressed in terms of the ψ functions is

$$\delta\kappa_1=\psi_1''-\left(\kappa_1^2+\kappa_2^2\right)\psi_1-2\kappa_2\psi_2'-\kappa_2'\psi_2+\kappa_2\kappa_3\psi_3-\kappa_1\psi_{||}''+\kappa_1'\psi_{||} \tag{2.26}$$

Following the same method, we can calculate the variation to first order for all the other curvatures. To get $\delta\kappa_2$ we first calculate δN_1^μ, by solving for the second curvature in the second FS Eq. 2.4 and applying the perturbation operator δ, we obtain

$$\delta\kappa_2=\delta\eta(N_2,\frac{dN_1}{d\tau})=\eta(\delta N_2,\frac{dN_1}{d\tau})+\eta(N_2,\delta\frac{dN_1}{d\tau}) \tag{2.27}$$

Substituting the second FS Eq. 2.4 in the first term and using the commutation relation 2.18 in the second term, we get

$$\delta\kappa_2=\kappa_1\eta(\delta N_2,T)+\kappa_2\eta(\delta N_2,N_2)+\eta(N_2,\frac{d\delta N_1}{d\tau})-\left(\psi_{||}'+\kappa_1\psi_1\right)\eta(N_2,\frac{dN_1}{d\tau}) \tag{2.28}$$

We notice that the second term of Eq. 2.28 vanishes and that we can use again the second FS Eq. 2.4 to simplify the fourth term. The fact that η is constant and that the FS vectors are all orthogonal allow us to interchange the action of the operator δ on the first term $\eta(T, \delta N_2) = -\eta(\delta T, N_2)$; we also notice that the third term can be written in the form

$$\eta(N_2, \frac{d\delta N_1}{d\tau}) = \frac{d}{d\tau}\eta(N_2, \delta N_1) - \eta(\frac{dN_2}{d\tau}, \delta N_1)\,. \tag{2.29}$$

so that Eq. 2.29 becomes

$$\delta\kappa_2 = -\kappa_1\eta(N_2, \delta T) - \kappa_3\eta(N_3, \delta N_1) - \left(\psi_{||}'' + \kappa_1\psi_1\right)\kappa_2 + \frac{d}{d\tau}\eta(N_2, \delta N_1) \tag{2.30}$$

At this point we note that for calculating $\delta\kappa_2$ one only needs to compute the components along the directions N_2 and N_3 of the variation δN_1, as we know that the component along the N_2 direction of δT^μ is α_2.

Let us now calculate the required components of δN_1. Solving for N_1 from the first FS Eq.2.9 and applying the operator δ, we have

$$\delta N_1^\mu = \delta\left(\frac{1}{\kappa_1}\right)\frac{dT^\mu}{d\tau} + \left(\frac{1}{\kappa_1}\right)\delta\frac{dT^\mu}{d\tau} \tag{2.31}$$

and making use of Eq. 2.18, this becomes

$$\delta N_1^\mu = -\left(\frac{\delta\kappa_1}{\kappa_1}\right)N_1^\mu + \frac{1}{\kappa_1}\frac{d\delta T^\mu}{d\tau} - \frac{1}{\kappa_1}\frac{dT^\mu}{d\tau}\left(\psi_{||}' + \kappa_1\psi_1\right) \tag{2.32}$$

Obviously, the needed component is contained in the second term of Eq. 2.32. By once againg using Eq.2.12 with the α coefficients instead of the ψ functions, we get the second term of Eq.2.30, that is,

$$\eta(N_3, \delta N_1) = \frac{1}{\kappa_1}\eta(N_3, \delta\frac{dT^\mu}{d\tau}) = \left(\frac{1}{\kappa_1}\right)(\alpha_3' + \alpha_2\kappa_3) \tag{2.33}$$

we finally get the variation κ_2 in compact notation

$$\delta\kappa_2 = \kappa_1\alpha_2 - \frac{\kappa_3}{\kappa_1}(\alpha_3' + \kappa_3\alpha_2) - \kappa_2\left(\psi_{||}' + \kappa_1\psi_1\right) + \frac{d}{d\tau}\left(\frac{\alpha_2' + \alpha_1\kappa_2 - \alpha_3\kappa_3}{\kappa_1}\right) \tag{2.34}$$

or in terms of the ψ functions, this variation becomes

$$\delta\kappa_2 = -\left(\frac{\kappa_3}{\kappa_1}\right)\psi_3'' + \left(\kappa_1\kappa_3 + \frac{\kappa_3^3}{\kappa_1}\right)\psi_3 - \left(\kappa_1 + 2\frac{\kappa_3^2}{\kappa_1}\right)\psi_2' - \left(\frac{\kappa_3\kappa_3'}{\kappa_1}\right)\psi_2 - \left(2\kappa_1\kappa_2 + \frac{\kappa_2\kappa_3^2}{\kappa_1}\right)\psi_1 - \kappa_2\psi_{||}'\,. \tag{2.35}$$

We proceed in the same way to calculate the variation of κ_3. We start by taking the dot product of the vector N_2^μ with all the terms of the fourth FS Eq. 2.8, that is, $\kappa_3 = -\eta(N_2, \frac{dN_3}{d\tau})$. Then we apply upon it the δ operator

$$\begin{aligned}\delta\kappa_3 &= -\eta(\delta N_2, \tfrac{dN_3}{d\tau}) - \eta(N_2, \delta\tfrac{dN_3}{d\tau})\\ &= \kappa_3\eta(\delta N_2, N_2) - \eta\left(N_2, \tfrac{d\delta N_3}{d\tau} - \tfrac{dN_3}{d\tau}\left(\psi_{||}' + \kappa_1\psi_1\right)\right)\\ &= -\eta(N_2, \tfrac{d}{d\tau}\delta N_3) - \kappa_3\left(\psi_{||}' + \kappa_1\psi_1\right)\end{aligned} \tag{2.36}$$

For calculating the first term of the third line, it is better to make a double transposition of the operator, such that

$$\begin{aligned}\eta(N_2, \tfrac{d}{d\tau}\delta N_3) &= -\eta(\tfrac{dN_2}{d\tau}, \delta N_3) + \tfrac{d}{d\tau}\eta(N_2, \delta N_3)\\ &= \kappa_2\eta(N_1, \delta N_3) + \tfrac{d}{d\tau}\eta(N_2, \delta N_3)\\ &= -\kappa_2\eta(\delta N_1, N_3) + \tfrac{d}{d\tau}\eta(N_2, \delta N_3)\end{aligned} \tag{2.37}$$

then, in compact notation the variation is

$$\delta\kappa_3 = \frac{\kappa_2}{\kappa_1}(\alpha_3' + \kappa_3\alpha_2) - \kappa_3\left(\psi_{||}' + \kappa_1\psi_1\right) - \frac{d}{d\tau}\eta(N_2, \delta N_3) \tag{2.38}$$

Again, it is convenient to express this result in terms of the ψ deformation functions, then

$$\delta\kappa_3 = \frac{\kappa_2}{\kappa_1}\left[(\psi_3'' - \kappa_3^2\psi_3) + (2\kappa_3\psi_2' + \kappa_3'\psi_2)\right] + \left(\frac{\kappa_2^2\kappa_3}{\kappa_1} - \kappa_1\kappa_3\right)\psi_1 - \kappa_3\psi_{||} - \frac{d}{d\tau}\eta(N_2, \delta N_3)\,. \tag{2.39}$$

It should be noted that Eqs. 2.26 , 2.35 and 2.39, with the variation of the curvatures $\delta\kappa_1$, $\delta\kappa_2$ and $\delta\kappa_3$, are the main results of this section. Despite the fact that the total derivative appearing in those equations does not make any contribution to the equation of motion, these terms are important for calculating the conservation laws, as can be seen in Ref. [36].

3 Geometric actions and equations of motion.

The action describing the dynamics of a relativistic charged particle in the presence of an electromagnetic field is

$$A_u[X] = \int d\tau\,\left[-m - qA_\mu T^\mu\right]\,. \tag{3.1}$$

where A^μ are the field potentials. This action describes the motion of an electron in an electromagnetic field. When a term of self-force due to the electron's own charge is considered, the motion can be described by the Dirac-Lorentz equation. It should be noticed that Rohrlich in Ref.[37] has used the FS equations for a $3D$ space time with the purpose of obtaining solutions of the Lorentz-Dirac equations. The term of self-force on the electron includes the particle velocity and certain terms involving the first and second derivatives of the particle velocity. The author of Ref. [38] discovered that mathematical structure of the external force can be obtained by manipulating the Frenet-Serret vectors along the particle world line.

Let us now consider a theory based on the Lagrangian introduced in Eq. 3.1 plus a geometric Lagrangian L_g, which may include world-line curvatures, that is

$$A[X] = \int d\tau\,\left[-m + m_g L_g(\kappa_1,\kappa_2,\kappa_3) - qA_\mu T^\mu\right]\,. \tag{3.2}$$

where m_g is a parameter. The case when L_g is a quadratic function on the first curvature $L_g = \kappa_1^2$ has been studied widely by several authors, see Sect.1.

Let us now consider the general variation of the action given in Eq. 3.2

$$\begin{aligned} \delta A[X] &= \textstyle\int \left[\delta\left(d\tau\right) L + d\tau\,\delta L\right] \\ &= \textstyle\int d\tau \left[\left(\kappa_1\psi_1\right) L + m_g\delta L_g(\kappa_1,\kappa_2,\kappa_3) - q\delta A_\mu - qA_\mu\delta T^\mu\right]\,. \end{aligned} \tag{3.3}$$

By making use of the general variations

$$\begin{aligned} \delta L_g &= \tfrac{\partial L_g}{\partial\kappa_1}\delta\kappa_1 + \tfrac{\partial L_g}{\partial\kappa_2}\delta\kappa_2 + \tfrac{\partial L_g}{\partial\kappa_3}\delta\kappa_3 \\ \delta A_\mu &= \tfrac{\partial A_\mu}{\partial X^\nu}\delta X^\nu = A_{\mu,\nu}\left(\psi_1 N_1^\nu + \psi_2 N_2^\nu + \psi_3 N_3^\nu\right) \end{aligned} \tag{3.4}$$

we are now able to write down some special cases of Lagrangian L_g.

- The model $L_g = \frac{1}{2}\kappa_1^2$, whose dynamical equations take the form

$$\begin{aligned} &-m\kappa_1 - m_g\left[+\kappa_1'' - \tfrac{1}{2}\kappa_1^3 - \kappa_1\kappa_2^2\right] - qF_{\mu\nu}T^\mu N_1^\nu \equiv 0 \\ &-2m_g\kappa_1'\kappa_2 - m_g\kappa_1\kappa_2' - qF_{\mu\nu}T^\mu N_2^\nu \equiv 0 \\ &-m_g\kappa_1\kappa_2\kappa_3 - qF_{\mu\nu}T^\mu N_3^\nu \equiv 0 \end{aligned} \tag{3.5}$$

- The model $L_g = \frac{1}{2}\kappa_2^2$ whose equations of motion are given by

$$\begin{aligned} &-m\kappa_1 - m_g\left[\tfrac{1}{2}\kappa_1\kappa_2^2 + \kappa_2^2\left(2\kappa_1 + \tfrac{\kappa_3^2}{\kappa_1}\right)\right] - qF_{\mu\nu}T^\mu N_1^\nu \equiv 0 \\ &-m_g\tfrac{d}{d\tau}\left(\kappa_1\kappa_2 + \tfrac{2\kappa_2\kappa_3^2}{\kappa_1}\right) + m_g\left(\tfrac{\kappa_2\kappa_3\kappa_3'}{\kappa_1}\right) - qF_{\mu\nu}T^\mu N_2^\nu \equiv 0 \\ &-m_g\kappa_1\kappa_2\kappa_3 - m_g\tfrac{\kappa_2\kappa_3^2}{\kappa_1} - qF_{\mu\nu}T^\mu N_3^\nu \equiv 0 \end{aligned} \tag{3.6}$$

- The model $L_g = \frac{1}{2}\kappa_3^2$ whose equations of motions are

$$
\begin{aligned}
&-m\kappa_1 - m_g\left[\tfrac{1}{2}\kappa_1\kappa_3^2 + \kappa_3^2\left(\kappa_1 - \tfrac{\kappa_2^2}{\kappa_1}\right)\right] - qF_{\mu\nu}T^\mu N_1^\nu \equiv 0\\
&2m_g\tfrac{d}{d\tau}\left(\tfrac{\kappa_2\kappa_3^2}{\kappa_1}\right) - m_g\tfrac{\kappa_2\kappa_3\kappa_3'}{\kappa_1} - qF_{\mu\nu}T^\mu N_2^\nu \equiv 0\\
&-m_g\tfrac{d^2}{d\tau^2}\left(\tfrac{\kappa_2\kappa_3}{\kappa_1}\right) + m_g\tfrac{\kappa_2\kappa_3^3}{\kappa_1} - qF_{\mu\nu}T^\mu N_3^\nu \equiv 0
\end{aligned}
\tag{3.7}
$$

For solving the equations of motion of any of these models, one must get the trajectories of the particles in Minkowski space by integrating all the FS equations coupled with the geometric term of the model.

4 Solution for a free relativistic particle: $m_g = 0$.

Let us first consider the simplest possible case, to test the formalism, when no geometrical Lagrangian is included. Then, the set of Eqs. 3 reduce to

$$
\begin{aligned}
-m\kappa_1 - qF_{\mu\nu}T^\mu N_1^\nu &= 0\\
qF_{\mu\nu}T^\mu N_2^\nu &= 0\\
qF_{\mu\nu}T^\mu N_3^\nu &= 0
\end{aligned}
\tag{4.1}
$$

Let us show now that the first of this equations is the Newton's second law with the Lorentz force in the right side. To see this, substitute κ_1 from the first FS $\kappa_1 = \eta(N_1, \frac{dT}{d\tau})$ we get

$$
-m\eta_{\mu\nu}N_1^\nu \frac{dT^\mu}{d\tau} - qF_{\mu\nu}T^\mu N_1^\nu = 0 \tag{4.2}
$$

factorizing $\eta_{\mu\nu}N_1^\nu$ and re-arranging repeated indices, we obtain

$$
\eta_{\nu\alpha}N_1^\nu\left(-m\frac{dT^\alpha}{d\tau} + qF^\alpha_\beta T^\beta\right) = 0 \tag{4.3}
$$

as N_1^μ is still an arbitrary vector, one must realize that the expression in parentheses is always zero; and this happens because the action Eq. 3.2 reduces to the usual relativistic action if $m_g = 0$, that is

$$
\frac{dT^\mu}{d\tau} = \frac{q}{m}F^\mu_\nu T^\nu \tag{4.4}
$$

We are left with the second and third relations in Eqs. 4.1. It should be noted that for all electromagnetic fields in special relativity, one must have

$$
\begin{aligned}
F_{\mu\nu}T^\mu N_2^\nu &= 0\\
F_{\mu\nu}T^\mu N_3^\nu &= 0
\end{aligned}
\tag{4.5}
$$

4.1 Constant electromagnetic field

Let us consider now the case in which $F_{\mu\nu}$ is constant. We show now from the Frenet-Serret equations 2.9 that the solution curve will have all its curvatures constants. Let us then define $M^\mu = \frac{dT^\mu}{d\tau}$; then according to Eq.4.4, $M^\mu = q/m\, F^\mu_\nu T^\nu$. The derivative is

$$
\frac{dM^\mu}{d\tau} = \frac{q}{m}F^\mu_\nu\frac{dT^\mu}{d\tau} = \frac{q}{m}F^\mu_\nu\, M^\mu. \tag{4.6}
$$

Let us take now the derivative of the magnitude of M, that is

$$
\begin{aligned}
\tfrac{d}{d\tau}\left(\eta_{\mu\nu}M^\mu\, M^\nu\right) &= 2\eta_{\mu\nu}M^\mu\tfrac{dM_\nu}{d\tau}\\
&= 2\eta_{\mu\nu}M^\mu\tfrac{q}{m}F^\nu_\alpha M^\alpha\\
&= 2\tfrac{q}{m}\left(F_{\mu\nu}M^\mu M^\nu\right) \equiv 0
\end{aligned}
\tag{4.7}
$$

where the third line vanishes because F is antisymmetric and constant; we may then conclude that

$$
\eta\left(M, M\right) \equiv \eta\left(\frac{dT}{d\tau}, \frac{dT}{d\tau}\right) = cte \tag{4.8}
$$

In what follows, we will use this result to show that all the curvatures are constant. Let us then substitute the right hand side of Eq. 4.8 with the first Frenet-Serret relation

$$\eta\left(\kappa_1 N_1, \kappa_1 N_1\right) = cte => \kappa_1^2 = cte \ , \tag{4.9}$$

which directly implies that the first curvature is constant. Now, it is possible to rewrite the Newton's second law, Eq. 4.4, in the form

$$\kappa_1 N_1^\mu = \frac{q}{m} F^\mu_\nu T^\nu \ . \tag{4.10}$$

Squarring it we obtain

$$\kappa_1^2 = \left(\frac{q}{m}\right)^2 \eta_{\mu\nu} F^\mu_\alpha F^\nu_\beta T^\alpha T^\beta \equiv cte \ . \tag{4.11}$$

We then recognize κ_1 as an invariant whenever $F_{\mu\nu}$ is constant. Taking the derivative of the Eq. 4.10 with constant κ_1 we learn that the N_1 vector satisfies an equation of the form 4.8,

$$\frac{dN_1^\mu}{d\tau} = \frac{q}{m} F^\mu_\nu N_1^\nu \ . \tag{4.12}$$

Let us now use the second FS Eq. 2.4 and square it

$$\eta\left(\frac{dN_1}{d\tau}, \frac{dN_1}{d\tau}\right) = \eta\left(\kappa_2 N_2 + \kappa_1 T, \kappa_2 N_2 + \kappa_1 T\right) = \kappa_2^2 - \kappa_1^2 = cte \tag{4.13}$$

from which we get that κ_2 must also be a constant. On the other hand, by using Eq. 4.12 into the left hand side of Eq. 4.13, we obtain the relation

$$\kappa_2^2 = \kappa_1^2 + \left(\frac{q}{m}\right)^2 \eta_{\mu\nu} F^\mu_\alpha F^\nu_\beta N_1^\alpha N_1^\beta \equiv cte \tag{4.14}$$

We continue the process by obtaining the term with κ_2 from the second FS relation

$$\kappa_2 N_2^\mu = \frac{dN_1^\mu}{d\tau} - \kappa_1 T^\mu \tag{4.15}$$

Substituting Eq. 4.12 on the right hand side of Eq. 4.15 and taking the τ derivative on the resulting expression, we obtain

$$\frac{dN_2^\mu}{d\tau} = \frac{q}{m} F^\mu_\nu N_2^\nu \ . \tag{4.16}$$

Thus, the square of the derivative is a constant, in such a way that

$$\begin{aligned} &\eta_{\mu\nu}\left(\tfrac{dN_2^\mu}{d\tau}\tfrac{dN_2^\mu}{d\tau}\right) = cte \\ &\eta_{\mu\nu}\left(\kappa_3 N_3^\mu - \kappa_2 N_1^\mu\right)\left(\kappa_3 N_3^\mu - \kappa_2 N_1^\mu\right) = cte \\ &\kappa_3^2 + \kappa_2^2 = cte. \end{aligned} \tag{4.17}$$

from which we conclude that κ_2 is also a constant.

Let us now consider the term with N_3 from the third FS relation 2.6

$$\frac{dN_3^\mu}{d\tau} = \frac{q}{m} F^\mu_\nu N_3^\nu \tag{4.18}$$

by applying a similar procedure we obtain

$$\kappa_3^2 = \left(\frac{q}{m}\right)^2 \eta_{\mu\nu} F^\mu_\alpha F^\nu_\beta N_3^\alpha N_3^\beta \equiv cte \tag{4.19}$$

from which we conclude that κ_3 is a constant.

As we have seen, for an F constant the three curvatures are constants and can be expressed in terms of F itself. In order to show the form of the curvatures in terms of the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, respectively, we use the conventions given by Weinberg[2] for the tensor F, see the Appendix A.

For any vector A^μ, the contraction with the electromagnetic tensor, in the form suggested by the right hand side of the curvatures is given by

$$\eta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} A^\mu A^\nu = -\left(\vec{A}\cdot\vec{E}\right)^2 + \left(A^t \vec{E} + \vec{A}\times\vec{B}\right)^2 \tag{4.20}$$

Then in the coordinate system attached to the laboratory, the tangent vector is

$$T^\mu = (\gamma, \gamma\vec{v}) \tag{4.21}$$

where $\vec{v}$ is the particle velocity and

$$\gamma = \frac{1}{\sqrt{1-\vec{v}^2}}. \tag{4.22}$$

Therefore, the invariant for the fist curvature is

$$\kappa_1^2 = \left(\frac{q\gamma}{m}\right)^2 \left[-\left(\vec{v}\cdot\vec{E}\right)^2 + \left(\vec{E}+\vec{v}\times\vec{B}\right)^2\right] \tag{4.23}$$

It is easy to complete the dot product in the first term of the right hand side, then Eq. 4.23 can be written in the form

$$\kappa_1^2 = \left(\frac{\gamma}{m}\right)^2 \left[\vec{F}^2 - \left(\vec{v}\cdot\vec{F}\right)^2\right] \tag{4.24}$$

where $\vec{F}$ is the Lorentz force $\vec{F} = q\left(\vec{E} - \vec{v}\times\vec{B}\right)$. Equation 4.24 was obtained by [39] and [33].

5 Conclusions

In this paper we have obtained the equations of motion for a charged particle moving in an electromagnetic field when its action includes terms with its own world-line curvatures. This dynamical problem is mathematically difficult even on how to write the equations of motion, particularly when they are in terms of the particle embedding functions $X(\tau)$. An alternative approach useful for handling the mathematics of this problem is based on taking advantage of the Frenet-Serret frame, as we have shown here. We have obtained the Frenet-Serret equations in Minkowski space-time and then we have used them in order to develop a variational calculus well adapted for tackling this kind of problems.

The problem of a relativistic particle moving in an electromagnetic field is interesting both in theoretical and applied physics. For instance, in plasma physics, this problem is concerned with the mechanics of particle acceleration, when heating and radioactive effects are taken into account, see Ref.[40]. This kind of studies are based on the numerical integration of Eq. 4.4 obtained in Section 4, where we also proved the equivalence of the alternative approach. It would be interesting to compare the results obtained by integrating the two formalisms.

Finally, we mention that all the mathematical formalism developed in Sects. 2.1 and 2.2 can easily be translated into an Euclidean space, where the curvature dependent actions and the world-line of the relativistic particle would be replaced by curvature dependent energy functionals and by smoothly continuous curves, respectively, see Refs. [41] and references there in.

The space of solutions in the Euclidean frame is abundant and physically interesting, as can be seen in Ref. [42], where the equilibrium configurations of a $2D$ closed rigid loop were studied. In Ref. [43] the integrability of some curvature dependent energy functionals was established by making use of the constants of integration obtained by applying the Noether's theorem. Subsequently, in Ref. [44], the equilibrium configurations curves in the Euclidean $3D$ space were numerically obtained for the models of Ref. [43].

A Appendix

For the electromagnetic tensor, we follow the notation of Weinberg [2], we then have

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{A.1}$$

and its dual ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\, F^{\alpha\beta}$ and in matrix notation is:

$${}^*F_{\mu\nu} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{pmatrix} \tag{A.2}$$

The field invariant are then the following

$$\begin{aligned} F_{\mu\nu}\, F^{\mu\nu} &= -2\left(\vec{E}^2 - \vec{B}^2\right) \\ {}^*F_{\mu\nu} F^{\mu\nu} &= -2\vec{E}\cdot\vec{B} \end{aligned} \tag{A.3}$$

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