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On analogues involving zero-divisors of a domain-theoretic result of Ayache

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Dedicated to Patrick Smith and John Clark, in recognition of their many contributions to ring theory

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Abstract. Ayache has recently proved that if R is an integrally closed domain such that each overring of R is treed, then R is a locally pseudo-valuation domain. We investigate the extent to which the analogue (in which one concludes that R is a locally pseudo-valuation ring) holds if R is generalized to a commutative ring. A positive result is obtained if R is an idealization $D(+)K$ where D is an integrally closed LPVD each of whose overrings is treed (for instance, a Prüfer domain) with quotient field K. However, the analogue fails in general for (quasi-local) idealizations of the form $R = A(+)A$ where $A = \mathbb{Z}/p^n\mathbb{Z}$ with p a prime number and $n \geq 3$. A positive result for the analogue is obtained for certain reduced rings R (namely, weak Baer rings that are integrally closed), but an example using the $A + B$ construction shows that the analogue fails in general for reduced rings that are total quotient rings.

1 Introduction

All rings considered below are commutative, unital and nonzero. Our starting point is the following recent result of A. Ayache [5, Theorem 3.10]: if R is an integrally closed (commutative integral) domain such that each overring of R is treed, then R is an LPVD, that is, a locally pseudo-valuation domain, in the sense of [17]. (As usual, by an *overring* of a ring A, we mean an A-subalgebra of the total quotient ring $\text{tq}(A)$ of A, that is, a ring B such that $A \subseteq B \subseteq \text{tq}(A)$; and a ring is said to be *treed* if its prime spectrum, as a poset under inclusion, is a tree.) Our main interest here is to study the extent to which the analogue of the above results holds if R is replaced by an arbitrary commutative ring which is integrally closed (in $\text{tq}(R)$) and one seeks to determine whether such R must be an LPVR, that is, a locally pseudo-valuation ring, in the sense of [8]. For motivation, note that a domain is an LPVR if and only if it is an LPVD. It is also important to record that a quasi-local domain is a PVR (pseudo-valuation ring, in the sense of [1], [2]) if and only if it is a PVD (pseudo-valuation domain, in the sense of [22]).

This paragraph and the next paragraph may be skipped or skimmed quickly on a first reading, as their sole purpose is to provide further motivation for the question of whether a domain-theoretic result extends to a ring-theoretic setting. We begin by recalling the result of McAdam [26, Corollary 11] that any integrally closed quasi-local going-down domain must be a divided domain. It follows easily that any integrally closed going-down domain must be a locally divided domain. One was led to ask whether, more generally, any integrally closed going-down ring must be a locally divided ring. (For the purpose of this motivation, one need not be concerned with the definitions of these kinds of domains or rings. It is important, however, to know the following facts: the concept of "going-down domain" from [13] and [19] was generalized to "going-down ring" in [16]; the concept of "locally divided domain" from [14] was generalized to "locally divided ring" in [7] (cf. also [6], where the concept of "divided domain" from [14] was generalized to "divided ring"); each LPVD is a locally divided domain [17, Corollary 2.3]; each locally divided domain is a going-down domain, but not conversely [14, Proposition 2.1, Example 2.9]; each going-down domain is treed [13, Theorem 2.2]; and each locally divided ring is a treed going-down ring [7, Proposition 3.1].) Although any integrally closed going-down domain must be a locally divided ring (thanks to McAdam's result and the fact that a domain is a locally divided ring if and only if it is a locally divided domain), Badawi and the author were able to answer the above question in the negative, by constructing, for each $n, 1 \leq n \leq \infty$, an example of a quasi-local integrally closed treed going-down ring that has Krull dimension n and is not a (locally) divided ring [7, Example 3.10].

Despite counter-examples of the kind in [7, Example 3.10], one can also find in [7] generalizations of several results on locally divided domains and going-down domains to the context of locally divided rings and going-down rings (often for rings R such that either $Z(R) = \text{Nil}(R)$ or R is reduced.) Our work here is meant in that spirit of [7], where, to echo a phrase from the first paragraph of [7], we seek “connections, some with the flavor of domain-theoretic studies and others differing from such phenomena in the presence of zero-divisors."

Recall that if A is a ring and E is an A-module, the *idealization* $R := A(+)E$ is the ring whose additive structure is that of $A \oplus E$ and whose multiplication is given by $(a_1, e_1)(a_2, e_2) := (a_1a_2, a_1e_2 + a_2e_1)$ for all $a_1, a_2 \in A$ and all $e_1, e_2 \in E$. One views A as a subring of this idealization R via the canonical injective (unital) ring homomorphism $A \to R$, $a \mapsto (a, 0)$. The above-mentioned construction in [7, Example 3.10] made use of a particular idealization. For this reason, we begin the next section by asking whether the idealization construction sustains a generalization of the motivating result of Ayache [5, Theorem 3.10]. In the spirit of the above quotation from [7], we find both positive answers and negative answers. In detail, Proposition 2.1 gives a positive answer in case $R = D(+)K$ where D is a particular kind of integrally closed locally divided domain (for instance, a Prüfer domain) with quotient field K. On the other hand, Proposition 2.3 gives a negative answer in case $R = A(+)A$ where $A := \mathbb{Z}/p^n\mathbb{Z}$, with p any prime number and n any integer such that $n \geq 3$.

Recall that a ring R is said to be *reduced* if R has no nonzero nilpotent elements. Perhaps the most obvious example of a reduced ring is given by any domain. However idealizations $R = A(+)E$, which are the context for the first three results in Section 2, are never reduced rings (when $E \neq 0$). So, as we continue to consider the question whether a ring R must be an LPVR, given that R is integrally closed and each overring of R is treed, we sharpen the focus by restricting to reduced rings. As was the case for idealizations, we find both positive answers and negative answers. Indeed, Proposition 2.4 gives a positive answer in case R is also assumed to be a weak Baer ring. (Background on weak Baer rings is recalled as needed in Section 2.) A family of rings that illustrates Proposition 2.4 is given in Corollary 2.5. However, the context of reduced rings also supports a negative answer to our basic question. Indeed, in what is probably the most complicated work in this note, Proposition 2.6 gives an example of an integrally closed reduced treed ring R such that $\text{tq}(R) = R$ (so, trivially, each overring of R is treed) and R is not an LPVR. This result is achieved with the help of the $A + B$ construction (in the sense of [25, Section 8]; cf. also [23]). To make Proposition 2.6 more self-contained, we summarize some of the required background about the $A + B$ construction prior to Proposition 2.6.

As usual, if R is a ring, then $Z(R)$ denotes the set of zero-divisors of R, $\text{Nil}(R)$ the set of nilpotent elements of R, $\text{Spec}(R)$ the set of prime ideals of R, and $\text{Max}(R)$ the set of maximal ideals of R. Any unexplained material is standard, as in [21], [23], [24].

2 Results

We begin with a positive result by showing that certain idealizations satisfy an analogue of the motivating domain-theoretic result of Ayache.

Proposition 2.1. *Let D be an integrally closed LPVD each of whose overrings is treed (for instance, let D be a Prüfer domain) with quotient field K. Then the idealization $R := D(+)K$ is an integrally closed ring, but not a domain, such that each overring of R is treed and R is an LPVR.*

Proof. First, for the parenthetical assertion, it is well known that any Prüfer domain is an LPVD [17, page 149] and each of its overrings is also a Prüfer domain [21, Theorem 26.1 (1)] and, hence, treed. Next, it is easy to check that the set of non-zero-divisors of R is $R \setminus Z(R) = \{(d, b) \in R \mid d \neq 0,\, b \in K\}$ (cf. [23, Theorem 25.3]). It follows (cf. [23, Corollary 25.5 (3)]) that $\text{tq}(R)$ $(= R_{R \setminus Z(R)})$ can be identified, as an R-algebra, with $K(+)K$. (In detail, the R-algebra homomorphism $\text{tq}(R) \to K(+)K$, $(\delta, \beta)/(d, b) \mapsto (\delta/d, -\delta b/d^2 + \beta/d)$ for all $\delta \in D$, $0 \neq d \in D$ and $\beta, b \in K$, is both injective and surjective.) In particular, if $0 \neq d \in D$ and $b \in K$, the multiplicative inverse of (d, b) is identified with $(1/d, -b/d^2)$. Consequently, the overrings of R are the rings of the form $E(+)K$, where E ranges over the set of overrings of D. In view of the standard order-isomorphism of posets under inclusion $\text{Spec}(E) \to \text{Spec}(E(+)K)$ (given by $P \mapsto P(+)E$) [23, Theorem 25.1], we see that $E(+)K$ is treed since (in fact, if and only if) E is treed. Thus, each overring of R is treed. Moreover, since D is an integrally closed domain with quotient field K, it follows from [23, Theorem 25.6] (or from [23, Corollary 25.8]) that R is integrally closed (in $\text{tq}(R)$). Of course, R is not a domain (since, for instance, $0 \neq (1, 0) \in$

Nil(R)). Finally, to see that R is an LPVR, we need only check the conditions in [8, Proposition 3.4 (g)]: $Z(D)$ $(= \{0\})$ =Nil(D); D is an LPVR (since it is an LPVD); $K = \text{tq}(D)$; and D is a domain. □

In the next remark, it will be useful to recall that a ring (resp., domain) is a quasi-local LPVR (resp., a quasi-local LPVD) if and only if it is a PVR (resp., a PVD).

Remark 2.2. (a) It is natural to ask if an example of the kind constructed in Proposition 2.1 can be quasi-local. In fact, that ring R $(:= D(+)K)$ in Proposition 2.1 is quasi-local if and only if D is quasi-local. (This follows from the fact [23, Theorem 25.1] that for any idealization $A(+)E$, the assignment $P \mapsto P(+)E$ gives an order-isomorphism $\text{Spec}(A) \to \text{Spec}(A(+)E)$.) In other words, the ring R in Proposition 2.1 is quasi-local if and only if D is an integrally closed PVD each of whose overrings is treed (for instance, a valuation domain) with quotient field K.

(b) We next give an example showing that a pseudo-valuation domain D of the kind in (a) need not be a valuation domain. (The following example also shows that an LPVD satisfying the hypotheses of Proposition 2.1 need not be a Prüfer domain.) Indeed, consider $D := k + Yk(X)[[Y]]$, where k is a field, X is an indeterminate over k, and Y is an analytic indeterminate over $k(X)$. By [22, Example 2.1], D is a PVD but not a valuation domain. In fact, the canonically associated valuation overring of D is $V := k(X)[[Y]]$ (cf. [22, Theorem 2.10], [3, Proposition 2.6]). Of course, D is integrally closed (since k is algebraically closed in $k(X)$), by the lore of the classical $D+M$ construction [21, Exercise 11 (2), page 202]. Moreover, each ring contained between D and V is treed, by [12, Theorem 2.5]. (To apply the cited result, one needs to know that each PVD is a going-down domain, in the sense of [13], [19]; this, in turn, follows by combining [15, page 560] and [14, Proposition 2.1].) Since each overring of D is comparable under inclusion with V (by the first part of the proof of [9, Theorem 3.1]) and each overring of a valuation domain is treed, it follows that each overring of D is treed.

Despite hopes that may have been raised by Proposition 2.1, we show next that some idealizations, such as $R = \mathbb{Z}/8\mathbb{Z}(+)\mathbb{Z}/8\mathbb{Z}$, do not satisfy an analogue of Ayache's result.

Proposition 2.3. *Let p be a prime number and let n be an integer such that $n \geq 3$. Put $A := \mathbb{Z}/p^n\mathbb{Z}$. Then $R := A(+)A$ is an integrally closed ring (but not a domain) such that each overring of R is treed, but R is not an LPVR. Moreover, R is quasi-local and* $\text{tq}(R) = R$.

Proof. As $n \geq 3$, it follows from [8, Corollary 3.3] that R is not a PVR. Also, by [23, Theorem 25.1], R inherits the "quasi-local" property from the first coordinate A in $A(+)A$ $(= R)$. Thus, R is not an LPVR. The order-isomorphism $\text{Spec}(A) \to \text{Spec}(R)$ also shows that R inherits from A the property of having only one prime ideal. As R then has Krull dimension 0, it follows that $\text{tq}(R) = R$ (cf. [24, Theorem 84]). Then, *a fortiori*, R is integrally closed; and each overring of R (namely, R itself) is treed. Finally, R is not a domain, since $0 \neq (0, 1) \in \text{Nil}(R)$. □

Proposition 2.3 is best possible in the following sense. If n is 1 or 2, then $\mathbb{Z}/p^n\mathbb{Z}(+)\mathbb{Z}/p^n\mathbb{Z}$ is a PVR [8, Corollary 3.3] and, hence, an LPVR.

Note that idealizations are intuitively very "far" from being domains, since $E^2 = 0$ in $A(+)E$. This leads to the question whether an analogue of Ayache's result holds for rings in which one rules out the existence of non-zero nilpotent elements, that is, for reduced rings. In the spirit of Proposition 2.1, we begin to answer this question by giving a positive result: see Proposition 2.4. First, recall that a ring R is said to be a *weak Baer ring* if, for each $r \in R$, the annihilator of r in R is generated by an idempotent element of R. Since any domain is a weak Baer ring, Proposition 2.4 generalizes the motivating result of Ayache beyond the context of domains (but one should note that the proof of Proposition 2.4 uses that result of Ayache).

Proposition 2.4. *Let R be an integrally closed weak Baer ring such that each overring of R is treed. Then R is an LPVR.*

Proof. Let S be any multiplicatively closed subset of R. Then, since R is a weak Baer ring, there is a canonical isomorphism $\text{tq}(R_S) \cong \text{tq}(R)_S$, so that the overrings of R_S may be identified with the rings B_S as B ranges over the set of overrings of R [27, Lemme 2.5 and Corollaire 2.6]. Let $M \in \text{Max}(R)$. It follows that each overring of R is of the form $B_{R \setminus M}$ for some (uniquely determined) overring B of R. As each such B is assumed to be treed, each overring of R_M must be treed. Also, R_M inherits the "integrally closed" property from R, since relative integral closure commutes with the formation of rings of fractions [10, Proposition 16, page 314]. Therefore, if R_M is a weak Baer ring for each $M \in \text{Max}(R)$, we can (replace R with some R_M and thus) assume also that R is quasi-local. In fact, since R is a weak Baer ring, each R_M

is a domain (and hence a weak Baer ring). Thus, we have reduced to proving the assertion in case R is a (quasi-local) domain. Therefore, an appeal to the result of Ayache [5, Theorem 3.10] completes the proof. □

The next result gives an explicit family of rings R that satisfy the hypotheses of Proposition 2.4 (and are not domains).

Corollary 2.5. *For some $n \geq 3$, let $R_1, \ldots, R_n$ be a finite list of integrally closed PVDs, each of which has the property that each overring is treed. Then $R := R_1 \times \ldots \times R_n$ is an integrally closed weak Baer ring (but not a domain) such that each overring of R is treed (and R is an LPVR).*

Proof. Since $n > 1$, R is not a domain. It is well known that

$$\text{tq}(R) \cong \text{tq}(R_1) \times \cdots \times \text{tq}(R_n)$$

(cf. the proof of [20, Proposition 4.4]). Since each R_i is assumed to be integrally closed, it follows easily that R is integrally closed. It also follows (cf. [18, Lemma III.3 (d)]) that each overring of R is of the form $S_1 \times \cdots \times S_n$ for some suitable overrings S_i of R_i ($i = 1, \ldots, n$). As each S_i is assumed to be treed, it follows that each overring of R must be treed.

There are several ways to show that R is a weak Baer ring. We will do so by using the criterion that (i) tq(R) is a von Neumann regular ring and (ii) each prime ideal of R contains only one minimal prime ideal of R. Of course, (i) holds since tq(R) is a direct product of fields. As for (ii), write any $P \in \text{Spec}(R)$ as

$$P = R_1 \times \cdots \times R_{i-1} \times Q_i \times R_{i+1} \times \cdots \times R_n$$

for some uniquely determined i and $Q_i \in \text{Spec}(R_i)$, and note that (since R_i is a domain) the only minimal prime ideal of R that P can contain is

$$R_1 \times \cdots \times R_{i-1} \times \{0\} \times R_{i+1} \times \cdots \times R_n.$$

We next give three proofs of the parenthetical assertion. In view of the information assembled above, the first of these is by an application of Proposition 2.4. For a second, and more direct, proof of the parenthetical assertion, combine [8, Proposition 3.4 (e)] with the fact that each PVD is an LPVR. For the third (and most basic) proof, consider any $P \in \text{Spec}(R)$, as above write

$$P = R_1 \times \cdots \times R_{i-1} \times Q_i \times R_{i+1} \times \cdots \times R_n$$

for some uniquely determined i and $Q_i \in \text{Spec}(R_i)$, and note that R_P is ring-isomorphic to $(R_i)_{Q_i}$, which is a PVR (by [22, Proposition 2.6] or [1, Theorem 12, Corollary 4]). □

Next, we show that even reduced rings can fail to exhibit an analogue of Ayache's result. The construction of a suitable culprit in Proposition 2.6 below depends on the $A + B$ construction, in the sense of [25, Section 8]. For the sake of completeness, we pause to review that construction. (In the proof of Proposition 2.6, we will cite results on $A+B$ rings as needed from [25, Theorems 8.3 and 8.4], for convenience as formulated in [11, Theorem 6.1]; the reader is cautioned that the notation and results given below for the $A + B$ rings vary somewhat from those for the $A + B$ construction that was introduced in [23].)

Let D be a domain and let $\mathcal{P}$ be a nonempty subset of Spec(D). (In our application below, we will take $\mathcal{P} := \text{Max}(D)$.) Let $\mathcal{A}$ be an index set for $\mathcal{P}$, and let $\mathcal{I} = \mathcal{A} \times \mathbb{N}$ (where $\mathbb{N}$ denotes the set of natural numbers). For each $i = (\alpha, n)$ in $\mathcal{I}$, let K_i be the quotient field of D/P_α. Next, let $B := \sum K_i$ and form the ring $R := D + B$ from the direct sum of D and B by defining multiplication by $(r, b)(s, c) = (rs, rc+sb+bc)$. One refers to R as the *$A+B$ ring corresponding to* D and $\mathcal{P}$. It is easy to see that R has no nonzero elements whose square is 0, and so R is a reduced ring. Moreover, B is an ideal of R and $R/B \cong D$. Thus, $B \in \text{Spec}(R)$.

Proposition 2.6. *There exists an $A + B$ ring R such that $R = \text{tq}(R)$, R is an integrally closed reduced ring but not a domain, and (each overring of) R is treed, but R is not an LPVR.*

Proof. It was pointed out in [4, Remark 3.8 (b)] that the ring A^0 constructed by Ribenboim in [28, page 165] is a quasi-local integrally closed domain of Krull dimension 1 which is not a PVD. (The "quasi-local integrally closed' aspect of A^0 is interesting but will not be needed below.) Take this ring (or any other ring with similar properties) to be an ambient domain D, take $\mathcal{P} := \text{Max}(D)$, and let R be the $A+B$ ring corresponding to D and $\mathcal{P}$. We proceed to verify that this ring R has all the asserted properties.

We noted above that any $A+B$ ring is reduced. Moreover, no $A+B$ ring can be a domain (since $K_iK_j = 0$ whenever $i \neq j$). Next, because we have used $\mathcal{P} := \text{Max}(D)$, it follows from [25, Theorem 6.1 (3)] that $\text{tq}(R) = D + B$ $(= R)$. Thus, the only overring of R is R itself, and so R is integrally closed. Next, if M denotes the unique maximal ideal of D, it follows from the description of $\text{Spec}(R)$ in [25, Theorem 6.1 (4), (5)] that the only proper containment of prime ideals of R is $B \subset M + B$. It follows easily that R (and hence each overring of R) is treed. It remains only to prove that R is not an LPVR. This, in turn, follows from the fact that the class of LPVRs is stable under formation of homomorphic images [8, Proposition 3.4 (d)], since $R/B \cong D$ is a quasi-local domain which is not a PVD (and hence is not an LPVR). The proof is complete. □

In closing, we point out one reason that the domain A^0 of Ribenboim [28] was used in the proof of Proposition 2.6. Recall that the following are some of the properties that were required of such a domain, namely, that it be a domain of Krull dimension 1 which is not a PVD. Any such domain needs to be somewhat esoteric. For instance, one can see via [21, Exercise 12 (4), pages 202-203] and [15, Proposition 4.9 (i)] that the classical $D+M$ construction (with $M \neq 0$) cannot produce such a domain.

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