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On the convergence of SP-iterative scheme for three multivalued nonexpansive mappings in $CAT(\kappa)$ spaces

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Abstract In this paper strong convergence and Δ-convergence theorem is established for the SP-iterative scheme for three multivalued nonexpansive mappings in $CAT(\kappa)$ spaces for any $\kappa > 0$. Our results extend and improve the recent ones announced by [6, 20].

1 Introduction

The study of fixed points for multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [15]. The Banach contraction principle has been extended in different directions either by using generalized contractions for multivalued mappings and hybrid pairs of single and multivalued mappings, or by using more general spaces. The terminology $CAT(\kappa)$ spaces was introduced by M. Gromov to denote a distinguished class of geodesic metric spaces with curvature bounded above by $\kappa \in \mathbb{R}$. In recent years, $CAT(\kappa)$ spaces have attracted the attention of many authors as they have played a very important role in different aspects of geometry. A very thorough discussion on these spaces and the role they play in geometry can be found in the book by M.R. Bridson and A. Haefliger [1]. As it was noted by W.A. Kirk in his fundamental works [8, 9], the geometry of $CAT(\kappa)$ spaces is rich enough to develop a very consistent theory on fixed point under metric conditions. These works were followed by a series of new works by different authors (see for instance [2, 4, 10, 11, 17, 18, 21].

Also, since any $CAT(\kappa)$ space is a $CAT(\kappa')$ space for $\kappa' > \kappa$, all results for $CAT(0)$ spaces immediately apply to any $CAT(\kappa)$ with $\kappa > 0$.

Recently B. Piątek in [20] proved that an iterative sequence generated by the Halpern algorithm converges to a fixed point in the complete $CAT(\kappa)$ spaces.

Very recently, J.S. He et al. in [6] proved that the sequence defined by Mann's algorithm Δ-converges to a fixed point in complete $CAT(\kappa)$ spaces.

2 Preliminaries

Let (E, d) be a bounded metric space, then for $D, K \in E$ nonempty, set
$r_x(D) = \sup\{d(x, y) : y \in D\}, x \in E$,
$rad_K(D) = \inf\{r_x(D) : x \in K\}$,
$diam(D) = \sup\{d(x, y) : x, y \in D\}$.

Let $x, y \in E$. A geodesic path joining x to y (or, more briefly, a geodesic from x to y) is a map $c : [0, l] \to E$ such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry between $[0, l]$ and $c([0, l])$, and $d(x, y) = l$. Usually, the image $c([0, l])$ of c is called a geodesic segment (or metric segment) joining x and y. A metric segment joining x and y is not necessarily unique in general. In particular, in the case when the geodesic segment joining x and y is unique, we use $[x, y]$ to denote the unique geodesic segment joining x and y, this means that $z \in [x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(z, x) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$. In this case, we will write $z = tx \oplus (1 - t)y$ for simplicity. For fixed $D \in (0, +\infty)$, the space (E, d) is called a D-geodesic space if any two points of E with their distance smaller than D are joined by a geodesic segment. An Ą∞-geodesic space is simply called a geodesic space. Recall that a geodesic triangle $\triangle := \triangle(x, y, z)$ in the metric space (E, d) consists of three points in E (the vertices of $\triangle$) and three geodesic segments between each pair of vertices (the edges of $\triangle$). For the sake of saving printing space, we write $p \in \triangle$ when a point $p \in E$ lies in the union of $[x, y], [x, z]$ and $[y, z]$.

The triangle $\overline{\triangle}$ is called a comparison triangle for $\triangle$ if $d(x, y) = d(\overline{x}, \overline{y})$, $d(x, z) = d(\overline{x}, \overline{z})$ and $d(z, y) = d(\overline{z}, \overline{y})$. By [[1], Lemma 2.14, page 24], a comparison triangle for $\triangle$ always exists provided that the perimeter $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ (where $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ and ∞ otherwise). A point $P \in [\overline{x}, \overline{y}] \subset \overline{\triangle}$ is called a comparison point for $p \in [x, y] \subset \triangle$ if $d(\overline{p}, \overline{x}) = d(p, x)$. Recall that a geodesic triangle $\triangle$ in E with perimeter less than $2D_\kappa$ is said to satisfy the $CAT(\kappa)$ inequality if, given $\overline{\triangle}$ a comparison triangle in M_k^2 for $\triangle$, one has that

$$d(p, q) \leq d(\overline{p}, \overline{q}), \quad \forall p, q \in \triangle,$$

where $\overline{p}$ and A$\overline{q}$ are respectively the comparison points of p and q.

The model spaces M_κ^2 are defined as follows.

Definition 2.1. Given a real number κ, we denote by M_κ^2 the following metric spaces:

(i) if $\kappa = 0$ then M_κ^2 is Euclidean space $\mathbb{E}^n$;

(ii) if $\kappa > 0$ then M_κ^2 is obtained from the sphere $\mathbb{S}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{\kappa}}$.

(iii) if $\kappa < 0$ then M_κ^2 is obtained from hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by $\frac{1}{\sqrt{-\kappa}}$..

The metric space (E, d) is called a $CAT(\kappa)$ space if it is D_κ-geodesic and any geodesic triangle in E of perimeter less than $2D_\kappa$ satisfies the $CAT(\kappa)$ inequality.

Proposition 2.2. *M_κ^2 is a geodesic metric space. If $\kappa \leq 0$, then M_κ^2 is uniquely geodesic and all balls in M_κ^2 are convex. If $\kappa > 0$, then there is a unique geodesic segment joining $x, y \in M_\kappa^2$ if and only if $d(x, y) < \frac{\pi}{\sqrt{\kappa}}$. If $\kappa > 0$, closed balls in M_κ^2 of radius smaller than $\frac{\pi}{2\sqrt{\kappa}}$ are convex.*

Proposition 2.3. *Let E be a $CAT(\kappa)$ space. Then any balls in E of radius smaller than $\frac{\pi}{2\sqrt{\kappa}}$ are convex.*

In a geodesic space E with unique metric segments, the metric d is said to be convex if for $p, x, y \in E$, $t \in (0, 1)$ and a point $m \in [x, y]$ such that $d(x, m) = td(x, y)$ and $d(y, m) = (1-t)d(x, y)$, then $d(p, m) \leq (1-t)d(p, x) + td(p, y)$. And for all $x, y, z \in E$ and $t \in [0, 1]$ then,

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z) \tag{2.1}$$

$\mathbb{R}$-trees are a particular class of $CAT(\kappa)$ spaces for any real κ which will be named at certain points of our exposition (see [1], pg. 167] for more details).

Definition 2.4. An $\mathbb{R}$-tree is a metric space E such that:

(i) it is a uniquely geodesic metric space,

(ii) if x, y and $z \in T$ are such that $[y, x] \bigcap [x, z] = \{x\}$, then $[y, x] \bigcup [x, z] = [y, z]$.

Therefore the family of all closed convex subsets of a $CAT(\kappa)$ space has uniform normal structure in the usual meteric (or Banach space) sense.

A subset K of E is said to be convex if K includes every geodesic segment joining any two of its points. We will denote by $P(E)$ the family of nonempty proximinal subsets of E, by $CC(E)$ the family of nonempty closed convex subsets of E, and by $KC(E)$ the family of nonempty compact convex subsets of E. Let H be the Hausdorff distance on $CC(E)$, that is

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

for every $A, B \in CC(E)$, where $d(x, B) = \inf\{d(x, y) : y \in B\}$ is the distance from the point x to the subset B.

A multivalued mapping $T : E \to CC(E)$, is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad \forall x, y \in E.$$

A point $x \in E$ is said to be a fixed point for a multivalued mapping T if $x \in Tx$. We use the notation $F(T)$ standing for the set of fixed points of a mapping T.

Let us recall the following definitions.

Definition 2.5. Three of multivalued nonexpansive mappings $T, S, R : K \to CC(K)$, where K a subset of E, are said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F)$ or $d(x, Sx) \geq f(d(x, F)$ or $d(x, Rx) \geq f(d(x, F)$ for all $x \in K$, where $F = F(T) \bigcap F(S) \bigcap F(R)$, the set of all common fixed points of the mappings T, S and R

Definition 2.6. The mapping $T : E \to CC(E)$, is called hemicompact if, for any sequence $\{x_n\}$ in E such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $x_{n_r} \to p \in E$.

Proposition 2.7. *The modulus of convexity for $CAT(\kappa)$ space E (of dimension $\geq$2) and number $r < \frac{\pi}{\sqrt{\kappa}}$ and let m denote the midpoint of the segment $[x, y]$ joining x and y define by the modulus δ_r by sitting*

$$\delta(r, \epsilon) = \inf\{1 - \frac{1}{r} d(a, m)\},$$

where the infimum is taken over all points $a, x, y \in E$ satisfying $d(a, x) \leq r$, $d(a, y) \leq r$ and $\epsilon \leq d(x, y) < \frac{\pi}{\sqrt{\kappa}}$

Next we state the following useful lemmas.

Lemma 2.8. *[see [13], Lemma 7]. Let (E, d, W) be a uniformly convex hyperbolic with modulus of uniform convexity δ. For any $r > 0$, $\epsilon \in (0, 2)$, $\lambda \in [0, 1]$ and $a, x, y \in X$, if $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \leq \epsilon r$ then $d((1 - \lambda)x \oplus \lambda y, a) \leq (1 - 2\lambda(1 - \lambda)\delta(r, \epsilon)r$.*

Lemma 2.9. *Let E be a complete $CAT(\kappa)$ space with modulus of convexity $\delta(r, \epsilon)$ and let $x \in E$. Suppose that $\delta(r, \epsilon)$ increases with r (for a fixed ϵ) and suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in E such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$ for some $r = 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Proof. The proof similar as in lemma 9 in [12]. □

For scaler valued case, we state the following iterative processes as the following:
In 1953, W. R. Mann defined the Mann iteration [14] as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \tag{2.2}$$

where $\{\alpha_n\}$ is a sequences of positive numbers in $[0, 1]$.

In 1974, S. Ishikawa defined the Ishikawa iteration [5] as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{2.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$.

In 2008, S. Thianwan defined the new two step iteration [22] as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{2.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$.

In 2001, M. A .Noor defined the three step Noor iteration [16] as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T z_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n, \end{aligned} \tag{2.5}$$

where where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0, 1]$.

Recently, Phuengrattana and Suantai defined the SP-iteration [19] as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)z_n + \beta_n T z_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n, \end{aligned} \tag{2.6}$$

where where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$. clearly the Mann and Ishikawa iteration are special cases of the Noor iteration.

In the following definition, we extend the SP-iteration process (2.6) to the case of three multivalued nonexpansive mappings on closed and convex subset of E modifying the above ones.

Definition 2.10. Let E be a $CAT(\kappa)$ space, K be a nonempty closed and convex subset of E and $T, S, R : K \to CC(K)$, be three multivalued nonexpansive mappings. The sequence $\{x_n\}$ of the modified SP-iteration is defined by:

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)y_n \oplus \alpha_n u_n, \quad n \in \mathbb{N}, \\ y_n &= (1-\beta_n)z_n \oplus \beta_n v_n, \\ z_n &= (1-\gamma_n)x_n \oplus \gamma_n w_n, \end{aligned} \tag{2.7}$$

where $u_n \in Ty_n$, $v_n \in Sz_n$, $w_n \in Rx_n$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [0,1]$.

Remark 2.11. **(i)** If $\gamma_n = 0$ then we have

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)y_n \oplus \alpha_n u_n, \quad n \in \mathbb{N}, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n v_n, \end{aligned} \tag{2.8}$$

where $u_n \in Ty_n$, $v_n \in Sx_n$ and $\{\alpha_n\}, \{\beta_n\} \in [0,1]$.

(ii) If $\gamma_n = \beta_n = 0$ then we obtain

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n u_n, \quad n \in \mathbb{N}, \end{aligned} \tag{2.9}$$

$$\tag{2.10}$$

where $u_n \in Tx_n$,and $\{\alpha_n\} \in [0,1]$.

In this paper, we study the strong convergence and Δ-convergence of SP-iterative scheme for three multivalued nonexpansive mappings in $CAT(\kappa)$ spaces for any $\kappa > 0$ under some conditions. Our results extend and improve the some results in [6, 20].

3 Strong convergence theorems

First of all, we prove the following lemmas, which play very important rule in the latter.

Lemma 3.1. *Let K be a nonempty closed and convex subset of a complete $CAT(\kappa)$ space E with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$, let $T, S, R : K \to CC(K)$ be three multivalued nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (2.7). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$ then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.*

Proof. Assume that $F \neq \emptyset$. Let $p \in F$. Then from (2.7) we have,

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)y_n \oplus \alpha_n u_n, p) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n d(u_n, p) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n H(Ty_n, Tp) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n d(y_n, p) \\ &= d(y_n, p), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)z_n \oplus \beta_n v_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(v_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n H(Sz_n, Sp) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\ &= d(z_n, p), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} d(z_n,p) &= d((1-\beta_n)x_n \oplus \beta_n w_n, p) \\ &\le (1-\beta_n)d(x_n,p) + \beta_n d(w_n,p) \\ &\le (1-\beta_n)d(x_n,p) + \beta_n H(Rx_n,Rp) \\ &\le (1-\beta_n)d(x_n,p) + \beta_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we obtain,

$$d(y_n,p) \le d(x_n,p) \tag{3.4}$$

From (3.3) and (3.4) we have,

$$d(x_{n+1},p) \le d(x_n,p).$$

Thus $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F$, hence $\{x_n\}$ is bounded. □

Lemma 3.2. *Let K be a nonempty closed and convex subset of a complete $CAT(\kappa)$ space E with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$ and let $T,S,R : K \to CC(K)$, be three multivalued nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (2.7). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$ then $\lim_{n\to\infty} d(x_n,Ty_n) = \lim_{n\to\infty} d(x_n,Sz_n) = \lim_{n\to\infty} d(x_n,Rx_n) = 0$.*

Proof. Let $p \in F \neq \emptyset$. By lemma (3.1), $\lim_{n\to\infty} d(x_n,p)$ exists, $\{x_n\}$ is bounded. Put

$$\limsup_{n\to\infty} d(x_n,p) = c. \tag{3.5}$$

From (3.4) we have

$$\limsup_{n\to\infty} d(y_n,p) \le c, \tag{3.6}$$

also

$$d(u_n,p) \le d(y_n,p),$$

for all $n \ge 1$. so

$$\limsup_{n\to\infty} d(u_n,p) \le c. \tag{3.7}$$

From (3.3) we have

$$\limsup_{n\to\infty} d(z_n,p) \le c, \tag{3.8}$$

also

$$d(v_n,p) \le d(z_n,p),$$

thus

$$\limsup_{n\to\infty} d(v_n,p) \le c. \tag{3.9}$$

Further,

$$\begin{aligned} c = \lim_{n\to\infty} d(x_{n+1},p) &= \lim_{n\to\infty} d((1-\alpha_n)y_n \oplus \alpha_n u_n, p) \\ &\le \lim_{n\to\infty} [(1-\alpha_n)d(y_n,p) + \alpha_n d(u_n,p)] \\ &\le \lim_{n\to\infty} [(1-\alpha_n)\limsup_{n\to\infty} d(y_n,p) + \alpha_n \limsup_{n\to\infty} d(u_n,p)] \\ &\le \lim_{n\to\infty} [(1-\alpha_n)c + \alpha_n c] = c, \end{aligned}$$

this gives

$$\lim_{n\to\infty} ((1-\alpha_n)d(y_n,p) + \alpha_n d(u_n,p)) = c. \tag{3.10}$$

Applying lemma (2.9) we obtain

$$\lim_{n\to\infty} d(y_n, u_n) = 0. \tag{3.11}$$

Noting that

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)y_n \oplus \alpha_n u_n, p) \\ &\leq d((1-\beta_n)y_n + \beta_n(u_n, p) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n d(u_n, p) \\ &\leq (1-\alpha_n)d(y_n, u_n) + d(u_n, p), \end{aligned}$$

which yields that

$$c \leq \liminf_{n\to\infty} d(u_n, p),$$

then from (3.7) we have,

$$c = \lim_{n\to\infty} d(u_n, p).$$

In turn

$$d(v_n, p) \leq d(y_n, p),$$

this implies that

$$c \leq \liminf_{n\to\infty} d(y_n, p). \tag{3.12}$$

By (3.6) and (3.14), we obtain

$$c = \lim_{n\to\infty} d(y_n, p).$$

Further,

$$d(v_n, p) \leq d(z_n, p),$$

this gives

$$\limsup_{n\to\infty} d(v_n, p) \leq c$$

Moreover,

$$\begin{aligned} c = \lim_{n\to\infty} d(y_n, p) &= \lim_{n\to\infty} d((1-\beta_n)z_n \oplus \beta_n v_n, p) \\ &\leq \lim_{n\to\infty} [(1-\beta_n) \limsup_{n\to\infty} d(z_n, p) + \beta_n \limsup_{n\to\infty} d(v_n, p)] \\ &\leq \lim_{n\to\infty} [(1-\beta_n)c + \beta_n c] = c \\ &\leq \lim_{n\to\infty} d(x_n, p) = c, \end{aligned}$$

hence

$$\lim_{n\to\infty} d((1-\beta_n)z_n \oplus \beta_n v_n, p) = c.$$

From lemma (2.9) we have

$$\lim_{n\to\infty} d(z_n, v_n) = 0, \tag{3.13}$$

and

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)z_n \oplus \beta_n v_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(v_n, p) \\ &\leq (1-\beta_n)[d(z_n, v_n) + d(v_n, p)] + \beta_n d(v_n, p) \\ &\leq (1-\beta_n)d(z_n, v_n) + \beta_n d(v_n, p), \end{aligned}$$

this yields that

$$c \leq \liminf_{n\to\infty} d(v_n, p),$$

so (3.9) gives that

$$c = \lim_{n\to\infty} d(v_n, p).$$

In turn

$$d(v_n, p) \leq d(z_n, p).$$

this yields

$$c \leq \liminf_{n\to\infty} d(z_n, p). \tag{3.14}$$

By (3.6) and (3.14), we obtain

$$c = \lim_{n\to\infty} d(z_n, p).$$

Moreover,

$$\begin{aligned} c = \lim_{n\to\infty} d(z_n, p) &= \lim_{n\to\infty} d((1-\gamma_n)x_n \oplus \gamma_n w_n, p) \\ &\leq \lim_{n\to\infty} [(1-\gamma_n) \limsup_{n\to\infty} d(x_n, p) + \gamma_n \limsup_{n\to\infty} d(w_n, p)] \\ &\leq \lim_{n\to\infty} [(1-\gamma_n)c + \gamma_n c] = c. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} d((1-\gamma_n)x_n \oplus \gamma_n w_n, p) = c.$$

By lemma (2.9) we have

$$\lim_{n\to\infty} d(x_n, w_n) = 0, \tag{3.15}$$

$$\begin{aligned} \lim_{n\to\infty} d(z_n, x_n) &= \lim_{n\to\infty} d((1-\gamma_n)x_n \oplus \gamma_n w_n, x_n) \\ &\leq \lim_{n\to\infty} [(1-\gamma_n) \limsup_{n\to\infty} d(x_n, x_n) + \gamma_n \limsup_{n\to\infty} d(w_n, x_n)], \end{aligned}$$

that is,

$$\lim_{n\to\infty} d(z_n, x_n) = 0$$

and

$$\begin{aligned} \lim_{n\to\infty} d(y_n, z_n) &= \lim_{n\to\infty} d((1-\gamma_n)z_n \oplus \gamma_n v_n, z_n) \\ &\leq \lim_{n\to\infty} [(1-\gamma_n) \limsup_{n\to\infty} d(z_n, z_n) + \gamma_n \limsup_{n\to\infty} d(v_n, z_n)], \end{aligned}$$

then

$$\lim_{n\to\infty} d(y_n, z_n) = 0. \tag{3.16}$$

Also

$$d(u_n, x_n) \leq d(u_n, y_n) + d(y_n, x_n), \tag{3.17}$$

then,

$$\lim_{n\to\infty} d(u_n, x_n) = 0. \tag{3.18}$$

Also

$$d(v_n, x_n) \le d(v_n, z_n) + d(z_n, x_n), \tag{3.19}$$

that is,

$$\lim_{n\to\infty} d(v_n, x_n) = 0. \tag{3.20}$$

Then we have

$$d(x_n, Ty_n) \quad \le \quad d(x_n, u_n) \to 0, as \;\; n \to \infty \;\; .$$

and

$$d(x_n, Sz_n) \quad \le \quad d(x_n, v_n) \to 0, as \;\; n \to \infty \;\; .$$

and

$$d(x_n, Sx_n) \quad \le \quad d(x_n, w_n) \to 0, as \;\; n \to \infty \;\; .$$

□

The following theorems gives the strong convergence of three multivalued nonexpansive mappings, with value in closed and convex space,

Theorem 3.3. *Let K be a nonempty closed and convex subset of a complete $CAT(\kappa)$ space E with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$, and let $T, S, R : K \to CC(K)$, be three multivalued nonexpansive mappings satisfying condition (I), $\{x_n\}$ be the sequence as defined in (2.7). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, then $\{x_n\}$ converges strongly to a common fixed point of T, S, R.*

Proof. Since T, S, R, satisfies condition (I), we have $\lim\limits_{n\to\infty} f(d(x_n, F)) = 0$. Thus there is a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ and a sequence $\{p_r\} \subset F$ such that

$$d(x_{n_r}, p_r) < \frac{1}{2^r},$$

for all $r > 0$. By lemma (3.1) we obtain that

$$d(x_{n_{r+1}}, p_r) \le d(x_{n_r}, p_r) < \frac{1}{2^r}.$$

We now show that $\{p_r\}$ is a Cauchy sequence in K. Observe that

$$\begin{aligned} d(p_{r+1}, p_r) &\le d(p_{r+1}, x_{n_{r+1}}) + d(x_{n_{r+1}}, p_r) \\ &< \frac{1}{2^{r+1}} + \frac{1}{2^r} \\ &< \frac{1}{2^{r-1}}. \end{aligned}$$

This shows that $\{p_r\}$ is a Cauchy sequence in K and thus converges to $p \in K$. Since

$$\begin{aligned} d(p_r, Tp) &\le H(Tp, Tp_r) \\ &\le d(p, p_r), \end{aligned}$$

and $p_r \to p$ as $r \to \infty$, it follows that $d(p, Tp) = 0$, which implies that $p \in Tp$.

Similarity

$$\begin{aligned} d(p_r, Sp) &\le H(Sp, Sp_r) \\ &\le d(p, p_r), \end{aligned}$$

and $p_r \to p$ as $r \to \infty$, it follows that $d(p, Sp) = 0$, which implies that $p \in Sp$.

Similarity

$$\begin{aligned} d(p_r, Rp) &\le H(Rp, Rp_r) \\ &\le d(p, p_r), \end{aligned}$$

and $p_r \to p$ as $r \to \infty$, it follows that $d(p, Rp) = 0$, which implies that $p \in Rp$. Consequently, $p \in F \neq \infty$. $\lim\limits_{n\to\infty} d(x_n, p)$ exists, we conclude that $\{x_n\}$ converges strongly to a common fixed point p. □

Theorem 3.4. *Let K be a nonempty closed and convex subset of a complete $CAT(\kappa)$ space E with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$, and let $T, S, R : K \to CC(K)$, be three hemicompact and continuous multivalued nonexpansive mappings. If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, then $\{x_n\}$ be the sequence as defined in (2.7) converges strongly to a common fixed point of T, S, R.*

Proof. From lemma (3.2) we obtain $\lim_{n\to\infty} d(x_n, Ty_n) = \lim_{n\to\infty} d(x_n, Sz_n) = \lim_{n\to\infty} d(x_n, Rx_n) = 0$ and T, S and R are hemicompact, there is a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $x_{n_r} \to p$ as $r \to \infty$ for some $p \in K$. Since T, S and R are continuous, we have

$$\begin{aligned} d(p,Tp) &\leq d(p,x_{n_r}) + d(x_{n_r},Ty_{n_r}) + H(Ty_{n_r},Tp) \\ &\leq 2d(p,x_{n_r}) + d(x_{n_r},Ty_{n_r}) \to 0 \ \ as \ \ r \to \infty, \end{aligned}$$

and

$$\begin{aligned} d(p,Sp) &\leq d(p,x_{n_r}) + d(x_{n_r},Sx_{n_r}) + H(Sx_{n_r},Sp) \\ &\leq 2d(p,x_{n_r}) + d(x_{n_r},Sx_{n_r}) \to 0 \ \ as \ \ r \to \infty. \end{aligned}$$

and

$$\begin{aligned} d(p,Rp) &\leq d(p,x_{n_r}) + d(x_{n_r},Rx_{n_r}) + H(Rx_{n_r},Rp) \\ &\leq 2d(p,x_{n_r}) + d(x_{n_r},Sx_{n_r}) \to 0 \ \ as \ \ r \to \infty. \end{aligned}$$

This implies that $p \in Tp$, $p \in Sp$ and $p \in Rp$. by lemma (3.1) $\lim_{n\to\infty} d(x_n, p)$ exists, thus $p \in F$ is the strong limit of the sequence $\{x_n\}$ itself. □

Corollary 3.5. *Let E be a uniformly convex Banach space and K a nonempty closed and convex subset of E. Let T, S be two multivalued nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (2.8) and T and S are hemicompact and continuous. If $\boldsymbol{F} \neq \emptyset$ and $Tp = Sp = \{p\}$ for any $p \in \boldsymbol{F}$, then $\{x_n\}$ converges strongly to a common fixed point of T and S.*

4 Δ-convergence of the SP-multivalued iteration

In this section, we will study the Δ-convergence of SP-iterative process (2.7) for three multivalued nonexpansive mappings in $CAT(\kappa)$ spaces satisfying conditions (I).

Let E be a complete $CAT(\kappa)$ space and $\{x_n\}$ a bounded sequence in E. For $x \in E$, we set

$$r(x,(x_n)) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in E\},$$

the asymptotic radius $r_C(\{x_n\})$ with respect to $C \subseteq E$ of $\{x_n\}$ is given by

$$r_C(x_n) = \inf\{r(x,\{x_n\}) : x \in C,$$

the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is given by the set

$$A(\{x_n\}) = \{x \in E : r(x,\{x_n\}) = r(\{x_n\})\}.$$

Therefore, the following equivalence holds for any point $u \in E$:

$$u \in A(\{x_n\}) \iff \limsup_{n\to\infty} d(u,x_n) \leq \limsup_{n\to\infty} d(x,x_n), \quad \forall x \in E. \tag{4.1}$$

A sequence $\{x_n\}$ in E is said to Δ-converge to $x \in E$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$. we denote $\omega_w(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

Proposition 4.1. *Let E be a complete $CAT(\kappa)$ space and let $\{x_n\}$ be a sequence in E with $r(\{x_n\}) < D_\kappa/2$. Then the following assertions hold.*

- *$A(\{x_n\})$ consists of exactly one point.*

- *$\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 4.2. *([3], Lemma 2.7).*

(i) *Every bounded sequence in E has a Δ-convergent subsequence.*

(ii) *If K is a closed convex subset of E and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

(iii) *If K is a closed convex subset of E and if $T : K \to E$ is a nonexpansive mapping, then the conditions, $\{x_n\}$ Δ-converges to x and $d(x_n, f(x_n)) \to 0$, imply $x \in K$ and $f(x) = x$.*

Lemma 4.3. *[3] If $\{x_n\}$ is a bounded sequence in E with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = u$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 4.4. *Let K be a closed and convex subset of E, and let $T : K \to CC(K)$ be a multivalued nonexpansive mapping. Suppose $\{x_n\}$ is a bounded sequence defined by (2.9) in K such that $\lim\limits_{n\to\infty} d(x, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F$, then $w(x_n) \in F$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$, then there exists a subsequence u_n of x_n such that $A(u_n) = \{u\}$. By Lemma (4.2)(i) and (ii) there exists a subsequence v_n of $\{u_n\}$ such that $\triangle \lim\limits_{n\to\infty} v_n = v \in K$. By Lemma (4.2)(iii), $v \in F(T)$. By Lemma (4.3), $u = v$. This shows that $\omega_w(x_n) \subset F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let u_n be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$, $\{d(x_n, u)\}$ converges. By Lemma (4.3), $x = u$. □

Theorem 4.5. *Let K be a nonempty closed convex subset of a complete $CAT(\kappa)$ space E with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$, and let $T : K \to CC(K)$, satisfying condition (I). If $\{x_n\}$ be the sequence in K defined by (2.9) such that $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim\limits_{n\to\infty} x_n = v$, then $v \in Tv$.*

Proof. Let $\Delta - \lim\limits_{n\to\infty} x_n = v$. We note that by Lemma (4.2), $v \in K$. For each $n \geq 1$, we choose $z_n \in T(v)$ such that $d(x_n, z_n) = dist(x_n, T(v))$. By the compactness of $T(v)$ there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim\limits_{n\to\infty} z_{n_k} = w \in T(v)$. Since T satisfies the condition (I) we have,

$$dist(x_{n_k}, T(v)) \leq dist(x_{n_k}, T(x_{n_k})) + d(x_{n_k}, v), \tag{4.2}$$

Note that

$$d(x_{n_k}, w) \leq d(x_{n_k}, z_{n_k}) + d(z_{n_k}, w) \leq dist(x_{n_k}, T(x_{n_k})) + d(x_{n_k}, v) + d(z_{n_k}, w). \tag{4.3}$$

Thus

$$\limsup_{n\to\infty} d(x_{n_k}, w) \leq \limsup_{n\to\infty} d(x_{n_k}, v). \tag{4.4}$$

By the uniqueness of asymptotic centers, we have $v = w \in T(v)$. □

Theorem 4.6. *Let K be a nonempty closed and convex subset of a complete $CAT(\kappa)$ space E with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$, and let $T, S, R : K \to CC(K)$, be three multivalued nonexpansive mappings satisfying condition (I), $\{x_n\}$ be the sequence as defined in (2.7). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, then $\{x_n\}$ is Δ-converges to a common fixed point of T, S, R.*

Proof. It follows from Lemma (3.2), $\lim\limits_{n\to\infty} d(x_n, Ty_n) = \lim\limits_{n\to\infty} d(x_n, Sz_n) = \lim\limits_{n\to\infty} d(x_n, Rx_n) = 0$. Now we let $\omega_w(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences u_n of x_n. We claim that $\omega_w(x_n) \subseteq F$. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas (4.2) and (4.3), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim\limits_{n\to\infty} v_n = v$. Since $\lim\limits_{n\to\infty} d(x_n, Ty_n) = \lim\limits_{n\to\infty} d(x_n, Sz_n) = \lim\limits_{n\to\infty} d(x_n, Rx_n) = 0$, by Theorem (4.5) we have $v \in F$, and the $\lim\limits_{n\to\infty} d(x_n, v)$ exists by Lemma (3.1). Hence $u = v \in F$ by Lemma (4.3). This shows that $\omega_w(x_n) \subset F$. Next we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F$ and $d(x_n, v)$ converges, by Lemma (4.3) we have $x = u$. □

Corollary 4.7. *Let K be a nonempty closed convex subset of a complete $CAT(\kappa)$ space E with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$, and let $T, S : K \to CC(K)$, be two multivalued nonexpansive mappings satisfying condition (I), $\{x_n\}$ be the sequence as defined in (2.8). If $F \neq \emptyset$ and $Tp = Sp = \{p\}$ for any $p \in F$, then $\{x_n\}$ is Δ-converges to a common fixed point of T, S.*

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