Palestine Journal of Mathematics

Vol. 7(2)(2018) , 385–396

Wilf's conjecture for numerical semigroups

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Communicated by Ayman Badawi

2000 Mathematical Subject Classification: 11D07, 20M14

Keywords and phrases: Numerical semigroup, Wilf conjecture, Frobenius number.

Abstract Let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity m, embedding dimension ν and conductor $c = qm - \rho$ for some $q, \rho \in \mathbb{N}$ with $\rho < m$. Let n be the cardinality of the set of elements $x \in S; x < c$. Wilf conjecture says that $c \leq \nu n$. Despite a lot of activities around this conjecture, it is still open. The aim of this paper is first to prove that Wilf's conjecture holds for S if $(2 + \frac{1}{q})\nu \geq m$. This generalizes the case when $2\nu \geq m$, proved by Sammartano in [9]. We also prove the conjecture for $m - \nu \leq 5$, and also for $m = 9$. These cases result from the following: let Ap$(S, m) = \{w_0 < w_1 < \ldots < w_{m-1}\}$ be the Apéry set of S. The conjecture holds if $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m - 1$ (Theorem 4.1).

1 Introduction and notations

Let $\mathbb{N}$ denotes the set of natural numbers, including 0. A *numerical semigroup* S is an additive submonoid of $(\mathbb{N}, +)$ of finite complement in $\mathbb{N}$, that is $0 \in S$, if $a, b \in S$ then $a + b \in S$, and $\mathbb{N} \setminus S$ is a finite set. The elements of $\mathbb{N} \setminus S$ are called the *gaps* of S and their cardinality is denoted by $g(S)$ and is called the genus of S. The largest gap is denoted by $f = f(S)$ =max$(\mathbb{N} \setminus S)$ and is called the *Frobenius number* of S. The smallest non zero element $m = m(S) = \min(S^*)$ is called the *multiplicity* of S ($S^* = S \setminus \{0\}$) and $n = |\{s \in S;\ s < f(S)\}|$ is also denoted by $n(S)$. Every numerical semigroup S is minimally generated, i.e.

$$S =< g_1, \ldots, g_\nu >= \mathbb{N}g_1 + \ldots + \mathbb{N}g_\nu$$

for suitable unique coprime integers $g_1, \ldots, g_\nu$. The cardinality of the minimal set of generators of S is denoted by $\nu = \nu(S)$ and is called the *embedding dimension* of S. An integer $x \in \mathbb{N} \setminus S$ is called a *pseudo-Frobenius number* if $x + S^* \subseteq S$. The *type* of the semigroup, denoted by $t(S)$ is the cardinality of the set of pseudo-frobenius numbers. The *Apéry set* of S with respect to $a \in S$ is defined as Ap$(S, a) = \{s \in S; s - a \notin S\}$.

The invariants associated with a numerical semigroup S are connected with equalities and inequalities. For example, $f(S) + 1 = g(S) + n(S)$, $\nu(S) \leq m(S)$.... In [10], H. S. Wilf proposed the following conjecture:

$$f(S) + 1 \leq \nu(S)n(S).$$

Suggesting a regularity in the set $\mathbb{N} \setminus S$. Although the problem has been considered by several authors (cf. [1], [2], [4], [5], [6], [7], [9]), only special cases have been solved and it remains wide open. In [4], D. Dobbs and G. Matthews proved Wilf's conjecture for $\nu \leq 3$. In [7] N. Kaplan proved it for $f + 1 \leq 2m$ and in [5] S. Eliahou extended Kaplan's work for $f + 1 \leq 3m$.

In this paper, we prove Wilf's conjecture in some relevant cases. More precisely, we prove that the conjecture holds for numerical semigroups S when $(2 + \frac{1}{q})\nu \geq m$ (where $f + 1 = qm - \rho, \rho < m$). This generalizes the case proved by A. Sammartano ([9]), who showed that Wilf's conjecture holds for $2\nu \geq m$. We also prove the conjecture when $m - \nu = 5$, and also for $m = 9$. Our main idea is based on counting the elements of S in some intervals of length m. This gives us an equivalent form of Wilf's conjecture, and allows us to prove the conjecture in the cases cited above.

The paper is organized as follows. In section 2 we use some notations and prove some results in order to give an equivalent form of Wilf's conjecture. In section 3 we give some technical results needed in the paper. Section 4 is the heart of the paper. Let Ap$(S, m) = \{0 = w_0 < w_1 <$

$\dots < w_{m-1}\}$. First, we show that Wilf's conjecture holds for numerical semigroups that satisfy $w_{m-1} \geq w_1 + w_\alpha$ and $(2+\frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m-1$ (see Theorem 4.1). Then we prove Wilf's conjecture for numerical semigroups with $m - \nu \leq 4$. This implies the case where $2\nu \geq m$. We also prove that numerical semigroups with $m - \nu = 5$ satisfy Wilf's conjecture. This allows us to prove the conjecture for $m = 9$. Finally we prove, using the previous cases, that Wilf's conjecture holds for numerical semigroups with $(2+\frac{1}{q})\nu \geq m$.

A good reference on numerical semigroups is [8].

2 Equivalent form of Wilf's conjecture

Let S be a numerical semigroup and the notations be as in the introduction. For the sake of clarity, we shall use the notations $\nu, f, n, c, m...$ for $\nu(S), f(S), n(S), c(S), m(S)....$ In this section, we will introduce some notations and prove some results in order to give an equivalent form of Wilf's conjecture. Let $q, \rho \in \mathbb{N}, 0 \leq \rho < m$ such that $c = f + 1 = qm - \rho$. Given a nonnegative integer k, we define the kth interval I_k of length m as

$$I_k = [km - \rho, (k+1)m - \rho[= \{km - \rho, km - \rho + 1, \dots, (k+1)m - \rho - 1\}.$$

We denote by

$$n_k = |S \cap I_k|.$$

For all $j \in \{1, \dots, m-1\}$, we define η_j to be the number of intervals I_k with $n_k = j$.

$$\eta_j = \big|\{k \in \mathbb{N};\ |S \cap I_k| = j\}\big|.$$

Proposition 2.1. Under the previous notations, we have the following:

$i)$ $1 \leq n_k \leq m-1$ for all $0 \leq k \leq q-1$ and $n_k = m$ for all $k \geq q$.

$ii)$ $\sum_{j=1}^{m-1} \eta_j = q$.

$iii)$ $\sum_{j=1}^{m-1} j\eta_j = \sum_{k=0}^{q-1} n_k = n(S) = n$.

Proof. $i)$ obvious. We will prove $ii)$ and $iii)$.

$ii)$ $\sum_{j=1}^{m-1} \eta_j = \sum_{j=1}^{m-1} \big|\{k \in \mathbb{N};\ |I_k \cap S| = j\}\big| = \sum_{j=1}^{m-1} |\{k \in \mathbb{N};\ n_k = j;\ 0 \leq k \leq q-1\}| = q$.

$iii)$ $\sum_{j=1}^{m-1} j\eta_j = \sum_{j=1}^{m-1} j\big|\{k \in \mathbb{N};\ |I_k \cap S| = j\}\big| = \sum_{j=1}^{m-1} j|\{k \in \mathbb{N};\ n_k = j;\ 0 \leq k \leq q-1\}| = \sum_{k=0}^{q-1} n_k = n$. ■

Remark: We shall use the notation $\lfloor x \rfloor$ for the largest integer smaller than or equal to x.

Next, we will express η_j in terms of the Apéry set.

Proposition 2.2. Let $\mathrm{Ap}(S, m) = \{w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}\}$. Under the previous notations, for all $1 \leq j \leq m-1$ we have

$$\eta_j = \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor.$$

Proof. Fix $0 \leq k \leq q-1$ and let $1 \leq j \leq m-1$. We will show that the interval I_k contains exactly j elements of S if and only if $w_{j-1} < (k+1)m - \rho \leq w_j$. Recall to this end that for all $s \in S$, there exist $0 \leq i \leq m-1$ and $a \in \mathbb{N}$ such that $s = w_i + am$.

Suppose that I_k contains exactly j elements of S. Suppose, by contradiction, that $w_{j-1} \geq (k+1)m - \rho$. We have $w_{m-1} > \dots > w_{j-1} \geq (k+1)m - \rho$, thus $w_{m-1}, \dots, w_{j-1} \in \cup_{t=k+1}^{q} I_t$. Hence, I_k contains at most $j-1$ elements of S (namely $w_0 + km = km, w_1 + k_1 m, w_2 + k_2 m, \dots, w_{j-2} + k_{j-2} m$ for some $k_1, \dots, k_{j-2} \in \{0, \dots, k-1\}$). This contradicts the fact that I_k contains exactly j elements of S.
Let us prove that $(k+1)m - \rho \leq w_j$. If $w_j < (k+1)m - \rho$, then $w_0 < \dots < w_j < (k+1)m - \rho$, thus $w_0, \dots, w_j \in \cup_{t=0}^{k} I_t$. Hence, I_k contains at least $j+1$ elements of S which are : $w_0 + km =$

$km, w_1 + k_1 m, w_2 + k_2 m, \ldots, w_j + k_j m$ for some $k_1, \ldots, k_j \in \{0, \ldots, k-1\}$. This is again a contradiction.

Conversely, suppose that $w_{j-1} < (k+1)m - \rho \leq w_j$. Since $w_{j-1} < (k+1)m - \rho$ then $w_0 < \ldots < w_{j-1} < (k+1)m - \rho$, whence $w_0, \ldots, w_{j-1} \in \cup_{t=0}^{k} I_t$. In particular I_k contains at least j elements of S, namely $w_0 + km = km, w_1 + k_1 m, w_2 + k_2 m, \ldots, w_{j-1} + k_{j-1} m$ for some $k_1, \ldots, k_{j-1} \in \{0, \ldots, k-1\}$. On the other hand $w_j \geq (k+1)m - \rho$ implies that $w_{m-1} > \ldots > w_j \geq (k+1)m - \rho$, so $w_{m-1}, \ldots, w_j \in \cup_{t=k+1}^{q} I_t$. Thus, I_k contains at most j elements of S which are: $w_0 + km = km, w_1 + k_1 m, w_2 + k_2 m, \ldots, w_{j-1} + k_{j-1} m$ for some $k_1, \ldots, k_{j-1} \in \{0, \ldots, k-1\}$. Hence, if $w_{j-1} < (k+1)m - \rho \leq w_j$, then I_k contains exactly j elements of S and this proves our assertion.

We finally have the following:

$$\begin{aligned}
\eta_j &= |\{k \in \mathbb{N} \text{ such that } |I_k \cap S| = j\}| \\
&= |\{k \in \mathbb{N} \text{ such that } w_{j-1} < (k+1)m - \rho \leq w_j\}| \\
&= |\{k \in \mathbb{N} \text{ such that } \tfrac{w_{j-1}+\rho}{m} < (k+1) \leq \tfrac{w_j+\rho}{m}\}| \\
&= |\{k \in \mathbb{N} \text{ such that } \tfrac{w_{j-1}+\rho}{m} - 1 < k \leq \tfrac{w_j+\rho}{m} - 1\}| \\
&= |\{k \in \mathbb{N} \text{ such that } \lfloor \tfrac{w_{j-1}+\rho}{m} \rfloor \leq k \leq \lfloor \tfrac{w_j+\rho}{m} \rfloor - 1\}| \\
&= \lfloor \tfrac{w_j+\rho}{m} \rfloor - \lfloor \tfrac{w_{j-1}+\rho}{m} \rfloor.
\end{aligned}$$

■

Proposition 2.3 gives an equivalent form of Wilf's conjecture using Proposition 2.1 and Proposition 2.2.

Proposition 2.3. Let the notations be as above. We have S satisfies Wilf's conjecture *if and only if*

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 0.$$

Proof. By Proposition 2.1, we have

$$f + 1 \leq n\nu \Leftrightarrow qm - \rho \leq \nu \sum_{k=0}^{q-1} n_k \Leftrightarrow \sum_{k=0}^{q-1} m - \rho \leq \sum_{k=0}^{q-1} n_k \nu \Leftrightarrow \sum_{k=0}^{q-1} (n_k \nu - m) + \rho \geq 0 \Leftrightarrow$$

$$\sum_{j=1}^{m-1} \eta_j (j\nu - m) + \rho \geq 0.$$

And by Proposition 2.2, we get

$$\sum_{j=1}^{m-1} \eta_j (j\nu - m) + \rho \geq 0 \Leftrightarrow \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 0.$$

■

3 Technical results

Let S be a numerical semigroup and let the notations be as in sections 1 and 2. In this section, we give some technical results used through the paper. Recall that $\text{Ap}(S, m) = \{w_0 = 0 < w_1 < \ldots < w_{m-1}\}$.

Remark 3.1. With the notations above, we have the following:

$i)$ $\lfloor \frac{w_0+\rho}{m} \rfloor = 0$.

$ii)$ For all $1 \leq i \leq m-1$, we have $\lfloor \frac{w_i+\rho}{m} \rfloor \geq 1$ (as $w_i > m$).

$iii)$ For all $1 \leq i \leq m-1$, we have $\lfloor \frac{w_i+\rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor$ or $\lfloor \frac{w_i+\rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor + 1$.

$iv)$ If $\lfloor \frac{w_i+\rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor + 1$, then $\lfloor \frac{w_i+\rho}{m} \rfloor \geq 2$ and $\rho \geq 1$.

$v)$ For all $0 \leq i < j \leq m-1$, we have $\lfloor \frac{w_i+\rho}{m} \rfloor \leq \lfloor \frac{w_j+\rho}{m} \rfloor$.

$vi)$ $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor = \lfloor \frac{qm-\rho-1+m+\rho}{m} \rfloor = q$ (as $w_{m-1} = f+m$). ■

Let $1 < \alpha < m-1$. Using Remark 3.1 we get the following inequalities which will be used later in the paper:

$$\begin{aligned}
\sum_{j=1}^{\alpha}(\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) &= \sum_{j=1}^{\alpha}\lfloor \frac{w_j+\rho}{m} \rfloor (j\nu - m) - \sum_{j=1}^{\alpha}\lfloor \frac{w_{j-1}+\rho}{m} \rfloor (j\nu - m) \\
&= \sum_{j=1}^{\alpha}\lfloor \frac{w_j+\rho}{m} \rfloor (j\nu - m) - \sum_{j=0}^{\alpha-1}\lfloor \frac{w_j+\rho}{m} \rfloor ((j+1)\nu - m) \\
&= \lfloor \frac{w_\alpha+\rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_0+\rho}{m} \rfloor (\nu - m) - \sum_{j=1}^{\alpha-1}\lfloor \frac{w_j+\rho}{m} \rfloor \nu \\
&= \lfloor \frac{w_\alpha+\rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_1+\rho}{m} \rfloor \nu - \sum_{j=2}^{\alpha-1}\lfloor \frac{w_j+\rho}{m} \rfloor \nu \\
&\geq \lfloor \frac{w_\alpha+\rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_1+\rho}{m} \rfloor \nu - \sum_{j=2}^{\alpha-1}\lfloor \frac{w_\alpha+\rho}{m} \rfloor \nu \\
&= \lfloor \frac{w_\alpha+\rho}{m} \rfloor (\alpha\nu - m) - \lfloor \frac{w_1+\rho}{m} \rfloor \nu - \lfloor \frac{w_\alpha+\rho}{m} \rfloor (\alpha - 2)\nu \\
&= -\lfloor \frac{w_1+\rho}{m} \rfloor \nu + \lfloor \frac{w_\alpha+\rho}{m} \rfloor (2\nu - m).
\end{aligned}$$

Consequently,

$$\sum_{j=1}^{\alpha}(\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) \geq -\lfloor \frac{w_1+\rho}{m} \rfloor \nu + \lfloor \frac{w_\alpha+\rho}{m} \rfloor (2\nu - m). \tag{3.1}$$

On the other hand,

$$\begin{aligned}
\sum_{j=\alpha+1}^{m-1}(\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) &\geq \sum_{j=\alpha+1}^{m-1}(\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)((\alpha+1)\nu - m) \\
&= ((\alpha+1)\nu - m)(\sum_{j=\alpha+1}^{m-1}\lfloor \frac{w_j+\rho}{m} \rfloor - \sum_{j=\alpha+1}^{m-1}\lfloor \frac{w_{j-1}+\rho}{m} \rfloor) \\
&= ((\alpha+1)\nu - m)(\sum_{j=\alpha+1}^{m-1}\lfloor \frac{w_j+\rho}{m} \rfloor - \sum_{j=\alpha}^{m-2}\lfloor \frac{w_j+\rho}{m} \rfloor) \\
&= (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor)((\alpha+1)\nu - m).
\end{aligned}$$

Hence,

$$\sum_{j=\alpha+1}^{m-1}(\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) \geq \big(\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor\big)\big((\alpha+1)\nu - m\big). \tag{3.2}$$

Lemma 3.2. Suppose that $w_i \geq w_j + w_k$. We have the following:

$i)$ $\lfloor \frac{w_i+\rho}{m} \rfloor \geq \lfloor \frac{w_j+\rho}{m} \rfloor + \lfloor \frac{w_k+\rho}{m} \rfloor - 1$.

$ii)$ If $\lfloor \frac{w_i+\rho}{m} \rfloor = \lfloor \frac{w_j+\rho}{m} \rfloor + \lfloor \frac{w_k+\rho}{m} \rfloor - 1$, then

$$\lfloor \frac{w_j+\rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor + 1,\ \lfloor \frac{w_k+\rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor + 1 \text{ and } \rho \geq 1.$$

In particular, $\lfloor \frac{w_j+\rho}{m} \rfloor \geq 2$, $\lfloor \frac{w_k+\rho}{m} \rfloor \geq 2$ and $\rho \geq 1$.

Proof. $i)$ Since $w_i \geq w_j + w_k$, then $\frac{w_i+\rho}{m} \geq \frac{w_j+w_k+\rho}{m}$. Consequently, $\lfloor \frac{w_i+\rho}{m} \rfloor \geq \lfloor \frac{w_j+\rho}{m} \rfloor + \lfloor \frac{w_k}{m} \rfloor$. By Remark 3.1 (iii), $\lfloor \frac{w_k}{m} \rfloor \geq \lfloor \frac{w_k+\rho}{m} \rfloor - 1$. Hence, $\lfloor \frac{w_i+\rho}{m} \rfloor \geq \lfloor \frac{w_j+\rho}{m} \rfloor + \lfloor \frac{w_k+\rho}{m} \rfloor - 1$.

$ii)$ Suppose by the way of contradiction that $\lfloor \frac{w_j+\rho}{m} \rfloor \neq \lfloor \frac{w_j}{m} \rfloor + 1$ or $\lfloor \frac{w_k+\rho}{m} \rfloor \neq \lfloor \frac{w_k}{m} \rfloor + 1$ or $\rho < 1$. By Remark 3.1 (iii) and that $\rho \geq 0$, it follows that $\lfloor \frac{w_j+\rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor$ or $\lfloor \frac{w_k+\rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor$ or $\rho = 0$. Since $w_i \geq w_j + w_k$, we have

$$\lfloor \frac{w_i+\rho}{m} \rfloor \geq \lfloor \frac{w_j+w_k+\rho}{m} \rfloor.$$

Since $\lfloor \frac{w_j+\rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor$ or $\lfloor \frac{w_k+\rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor$ or $\rho = 0$, it follows that $\lfloor \frac{w_i+\rho}{m} \rfloor \geq \lfloor \frac{w_j+\rho}{m} \rfloor + \lfloor \frac{w_k+\rho}{m} \rfloor$, which contradicts the hypothesis. Hence,

$$\lfloor \frac{w_j+\rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor + 1,\ \lfloor \frac{w_k+\rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor + 1 \text{ and } \rho \geq 1.$$

Using Remark 3.1 (ii), it follows that $\lfloor \frac{w_j+\rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor + 1 \geq 2$, $\lfloor \frac{w_k+\rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor + 1 \geq 2$ and $\rho \geq 1$. ■

4 Main Results

Let S be a numerical semigroup and let the notations be as in sections 1, 2 and 3. The aim of this section is to prove that Wilf's conjecture holds for S in the following cases:

(i) $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m-1$.

(ii) $m - \nu \leq 5$. (Note that the case $m - \nu \leq 4$ results from the fact that Wilf's conjecture holds for $2\nu \geq m$. This case has been proved in [9]), however we shall give a proof in order to cover it through our techniques).

We shall then deduce the conjecture when $(2 + \frac{1}{q})\nu \geq m$, and also when $m = 9$.

Next, we will show that Wilf's conjecture holds if $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$.

Theorem 4.1. Let the notations be as above. In particular S is a numerical semigroup with multiplicity m, embedding dimension ν and conductor $f + 1 = qm - \rho$ for some $q, \rho \in \mathbb{N}$; $0 \leq \rho \leq m-1$, and $\text{Ap}(S, m) = \{w_0 = 0 < w_1 < w_2 < \ldots < w_{m-1}\}$. Suppose that $w_{m-1} \geq w_1 + w_\alpha$ for some $1 < \alpha < m-1$. If $(2 + \frac{\alpha-3}{q})\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. We are going to use the equivalent form of Wilf's conjecture given in Proposition 2.3. Since $w_{m-1} \geq w_1 + w_\alpha$, Lemma 3.2 (i) implies that $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor \geq \lfloor \frac{w_1+\rho}{m} \rfloor + \lfloor \frac{w_\alpha+\rho}{m} \rfloor - 1$. Let $x = \lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor$. Then, $x \geq -1$ and $\lfloor \frac{w_1+\rho}{m} \rfloor + \lfloor \frac{w_\alpha+\rho}{m} \rfloor = q - x$. Now using (3.1) and (3.2), we have

$$\begin{aligned}
&\sum_{j=1}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho \\
&\geq -\lfloor \frac{w_1+\rho}{m} \rfloor \nu + \lfloor \frac{w_\alpha+\rho}{m} \rfloor (2\nu - m) + (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor)\big((\alpha+1)\nu - m\big) + \rho \\
&= \lfloor \frac{w_1+\rho}{m} \rfloor \Big(-\nu + \big((\alpha+1)\nu - m\big) - \big((\alpha+1)\nu - m\big) \Big) + \lfloor \frac{w_\alpha+\rho}{m} \rfloor (2\nu - m) \\
&\quad + (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor)\big((\alpha+1)\nu - m\big) + \rho \\
&= \lfloor \frac{w_1+\rho}{m} \rfloor (\alpha\nu - m) + \lfloor \frac{w_\alpha+\rho}{m} \rfloor (2\nu - m) + (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor)\big((\alpha+1)\nu - m\big) + \rho \\
&= (\lfloor \frac{w_1+\rho}{m} \rfloor + \lfloor \frac{w_\alpha+\rho}{m} \rfloor)(2\nu - m) + \lfloor \frac{w_1+\rho}{m} \rfloor (\alpha - 2)\nu \\
&\quad + (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor)\big((\alpha+1)\nu - m\big) + \rho \\
&= (q - x)(2\nu - m) + \lfloor \frac{w_1+\rho}{m} \rfloor (\alpha - 2)\nu + x\big((\alpha+1)\nu - m\big) + \rho.
\end{aligned}$$

Consequently,

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho \geq (q - x)(2\nu - m) + \lfloor \frac{w_1+\rho}{m} \rfloor (\alpha - 2)\nu + x\big((\alpha+1)\nu - m\big) + \rho. \tag{4.1}$$

Since $x = \lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor \geq -1$, then we have two cases:

- If $x = -1$, then by Lemma 3.2 (ii), we have $\lfloor \frac{w_1+\rho}{m} \rfloor \geq 2$. From (4.1), it follows that

$$\begin{aligned}\sum_{j=1}^{m-1}(\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho &\geq (q+1)(2\nu - m) + 2(\alpha-2)\nu - \big((\alpha+1)\nu - m\big) + \rho \\ &= \nu(2q + \alpha - 3) - qm + \rho \\ &= q\big(\nu(2 + \frac{\alpha-3}{q}) - m\big) + \rho \geq 0.\end{aligned}$$

- If $x \geq 0$, then by Remark 3.1 (ii), we have $\lfloor \frac{w_1+\rho}{m} \rfloor \geq 1$. From (4.1), it follows that

$$\begin{aligned}\sum_{j=1}^{m-1}(\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho &\geq (q-x)(2\nu - m) + (\alpha-2)\nu + x\big((\alpha+1)\nu - m\big) + \rho \\ &= \nu\big(2q + (\alpha-2)(x+1) + x\big) - qm + \rho \\ &> \nu(2q + \alpha - 3) - qm + \rho \\ &= q\big(\nu(2 + \frac{\alpha-3}{q}) - m\big) + \rho \geq 0.\end{aligned}$$

Using Proposition 2.3, we get that S satisfies Wilf's conjecture. ■

Theorem 4.1 will give us some cases where Wilf's conjecture holds. We shall need the following notations. Let $\leq_S$ be the partial order defined by $a \leq_S b$ if and only if $b - a \in S$. Then define the following sets:

$$\min(\text{Ap}(S,m)) = \{w \in \text{Ap}(S,m)^* \text{ such that } w \text{ is minimal with respect to } \leq_S\}.$$

$$\max(\text{Ap}(S,m)) = \{w \in \text{Ap}(S,m)^* \text{ such that } w \text{ is maximal with respect to } \leq_S\}.$$

If S is minimally generated by $m, g_2, \ldots, g_\nu$ then, by [3] Lemma 3.2

(i) min(Ap(S,m))=$\{g_2, ..., g_\nu\}$.

(ii) max(Ap(S,m))=$\{w$ such that $w - m$ is a pseudo-frobenius number of $S\}$.

In particular

$i)$ $|\text{Ap}(S,m)^* \backslash \min(\text{Ap}(S,m))| = m - \nu$.

$ii)$ $|\max(\text{Ap}(S,m))| = t(S)$ (where $t(S)$ denotes the type of S).

Note that (see [6], Lemma 6, for example), if $w \in \text{Ap}(S,m)$ and $u \leq_S w$ with $u \in S$, then $u \in \text{Ap}(S,m)$. This implies the following:

Corollary 4.2. Let $x \in \text{Ap}(S,m)^*$. We have the following:

$i)$ $x \in \min(\text{Ap}(S,m))$ if and only if $x \neq w_i + w_j$ for all $w_i, w_j \in \text{Ap}(S,m)^*$.

$ii)$ $x \in \max(\text{Ap}(S,m))$ if and only if $w_i \neq x + w_j$ for all $w_i, w_j \in \text{Ap}(S,m)^*$.

The results above imply also the following:

Lemma 4.3. Let the notations be as in Theorem 4.1. If $m - \nu > \frac{\alpha(\alpha-1)}{2}$ for some $\alpha \in \mathbb{N}^*$, then $w_{m-1} \geq w_1 + w_\alpha$.

Proof. Suppose by the way of contradiction that $w_{m-1} < w_1 + w_\alpha$ and let w be such that $w \in \text{Ap}(S,m)^* \backslash \min(\text{Ap}(S,m))$ (such an element exists because $m > \nu$). Hence, $w \leq w_{m-1} < w_1 + w_\alpha$ and from Corollary 4.2 (i), it follows that $w = w_i + w_j$ for some $w_i, w_j \in \text{Ap}(S,m)^*$. Thus the only possible values for w are included in $\{w_i + w_j; 1 \leq i \leq j \leq \alpha - 1\}$. It follows that $|\text{Ap}(S,m)^* \backslash \min(\text{Ap}(S,m))| = m - \nu \leq \frac{\alpha(\alpha-1)}{2}$, which contradicts the hypothesis. ■

Next, we will deduce Wilf's conjecture for numerical Semigroups with $m - \nu > \frac{\alpha(\alpha-1)}{2}$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $\alpha > 1$ in $\mathbb{N}$. This will be used later in order to show that the conjecture holds for numerical semigroups with $(2 + \frac{1}{q})\nu \geq m$, and also to cover the result in [9] saying that the conjecture is true for $2\nu \geq m$.

Corollary 4.4. Let the notations be as above. Suppose that $m-\nu>\frac{\alpha(\alpha-1)}{2}$ for some $1<\alpha<m-1$. If $(2+\frac{\alpha-3}{q})\nu\geq m$, then S satisfies Wilf's conjecture.

Proof. If $m-\nu>\frac{\alpha(\alpha-1)}{2}$, then, by Lemma 4.3, $w_{m-1}\geq w_1+w_\alpha$. Now use Theorem 4.1. ∎

In the following Lemma, we will show that Wilf's conjecture holds for numerical semigroups with $m-\nu\leq 3$. This will enable us later to prove the conjecture for numerical semigroups with $(2+\frac{1}{q})\nu\geq m$ and to cover the result in [9] saying that the conjecture is true for $2\nu\geq m$.

Lemma 4.5. Let the notations be as above. If $m-\nu\leq 3$, then S satisfies Wilf's conjecture.

Proof. We shall assume that $\nu\geq 4$ (the case $\nu\leq 3$ is solved in [4]).

i) If $m-\nu=1$, then $m=\nu+1\geq 5$ ($\nu\geq 4$). We are going to show Wilf's conjecture holds by using Proposition 2.3. By taking $\alpha=1$ in (3.2), we get

$$\sum_{j=2}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)\geq(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor)(2\nu-m).\quad\text{Hence,}$$

$$\begin{aligned}
&\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho\\
&=(\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_0+\rho}{m}\rfloor)(\nu-m)+\sum_{j=2}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho\\
&\geq\lfloor\frac{w_1+\rho}{m}\rfloor(\nu-m)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor)(2\nu-m)+\rho\\
&=\lfloor\frac{w_1+\rho}{m}\rfloor\big(\nu-m+(2\nu-m)-(2\nu-m)\big)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor)(2\nu-m)+\rho\\
&=\lfloor\frac{w_1+\rho}{m}\rfloor(3\nu-2m)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor)(2\nu-m)+\rho\\
&=\lfloor\frac{w_1+\rho}{m}\rfloor(m-3)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor)(m-2)+\rho.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho\geq\quad&(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor)(m-2)+\rho\\
&+\lfloor\frac{w_1+\rho}{m}\rfloor(m-3).
\end{aligned}\tag{4.2}$$

Since $m-\nu=1>0=\frac{1.0}{2}$, then by Lemma 4.3, it follows that $w_{m-1}\geq w_1+w_1$. Consequently, by Lemma 3.2 (i), we have $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor\geq\lfloor\frac{w_1+\rho}{m}\rfloor+\lfloor\frac{w_1+\rho}{m}\rfloor-1$.

- If $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor=-1$. Then by Lemma 3.2, we have $\lfloor\frac{w_1+\rho}{m}\rfloor\geq 2$. By using (4.2) and $m\geq 5$, then
$$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho\geq 2(m-3)-(m-2)+\rho\geq 0.$$
- If $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor\geq 0$. By using (4.2) and $m\geq 5$, then
$$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho\geq(m-3)+\rho\geq 0.$$

Now the assertion results from Proposition 2.3.

ii) If $m-\nu\in\{2,3\}$. We have $m-\nu>1=\frac{2(1)}{2}$. If $(2-\frac{1}{q})\nu\geq m$, then by Corollary 4.4 S satisfies Wilf's conjecture. Now suppose that $(2-\frac{1}{q})\nu<m$. Since Wilf's conjecture holds for $q\leq 3$ (see [7], [5]), we may assume that $q\geq 4$.

- If $m-\nu=2$. Then $(2-\frac{1}{q})\nu<\nu+2$. Hence, $\nu<2(\frac{q}{q-1})\leq\frac{8}{3}$. By [4], S satisfies Wilf's conjecture.
- If $m-\nu=3$. Then $(2-\frac{1}{q})\nu<\nu+3$. Hence, $\nu<3(\frac{q}{q-1})\leq 4$. By [4], S satisfies Wilf's conjecture.

Thus Wilf's conjecture holds if $m-\nu \le 3$. ■

The next Corollary covers the result of Sammartano for numerical semigroups with $2\nu \ge m$ ([9]) using Corollary 4.4 and Lemma 4.5.

Corollary 4.6. Let the notations be as above. If $2\nu \ge m$, then S satisfies Wilf's conjecture.

Proof. If $m-\nu > 3 = \frac{3(2)}{2}$ and $2\nu \ge m$, then by Corollary 4.4 Wilf's conjecture holds. If $m-\nu \le 3$, then, by Lemma 4.5, S satisfies Wilf's conjecture. ■

In the following Corollary, we will deduce Wilf's conjecture for numerical semigroups with $m-\nu = 4$. This will enable us later to prove the conjecture for those with $(2+\frac{1}{q})\nu \ge m$.

Corollary 4.7. Let the notations be as above. If $m-\nu = 4$, then S satisfies Wilf's conjecture.

Proof. Since Wilf's conjecture holds for $\nu \le 3$ ([4]), then we may assume that $\nu \ge 4$. Hence, $\nu \ge m-\nu$. Consequently, $2\nu \ge m$, and S satisfies Wilf's conjecture by Corollary 4.6. ■

The following technical Lemma will be used through the paper.

Lemma 4.8. Let the notations be as above. If $m-\nu \ge \frac{\alpha(\alpha-1)}{2} - 1$ for some $3 \le \alpha \le m-2$, then $w_{m-1} \ge w_1 + w_\alpha$ or $w_{m-1} \ge w_{\alpha-2} + w_{\alpha-1}$.

Proof. Suppose by the way of contradiction that $w_{m-1} < w_1 + w_\alpha$ and $w_{m-1} < w_{\alpha-2} + w_{\alpha-1}$. Let
$w \in \text{Ap}(S,m)^* \backslash \min(\text{Ap}(S,m))$, then $w \le w_{m-1}$ and $w = w_i + w_j$ for some $w_i, w_j \in \text{Ap}(S,m)^*$ (Corollary 4.2 *i*). In this case, the only possible values of w are included in $\{w_i + w_j; 1 \le i \le j \le \alpha - 1\} \setminus \{w_{\alpha-2} + w_{\alpha-1}, w_{\alpha-1} + w_{\alpha-1}\}$. Consequently, $m-\nu = |\text{Ap}(S,m)^* \backslash \min(\text{Ap}(S,m))| \le \frac{\alpha(\alpha-1)}{2} - 2$. But $\frac{\alpha(\alpha-1)}{2} - 2 < \frac{\alpha(\alpha-1)}{2} - 1$, which contradicts the hypothesis. Hence, $w_{m-1} \ge w_1 + w_\alpha$ or $w_{m-1} \ge w_{\alpha-2} + w_{\alpha-1}$. ■

In the next theorem, we will show that Wilf's conjecture holds for numerical semigroups with $m-\nu = 5$.

Theorem 4.9. Let the notations be as above. If $m-\nu = 5$, then S satisfies Wilf's conjecture.

Proof. Let $m-\nu = 5$. Since Wilf's conjecture holds for $2\nu \ge m$, then we may assume that $2\nu < m$. This implies that $\nu < \frac{m}{2} = \frac{\nu+5}{2}$ i.e. $\nu < 5$. Since the case $\nu \le 3$ is known ([4]), then we shall assume that $\nu = 4$. This also implies that $m = \nu + 5 = 9$.
Since $m-\nu = 5 = \frac{4(3)}{2} - 1$, by Lemma 4.8, it follows that $w_8 \ge w_2 + w_3$ or $w_8 \ge w_1 + w_4$.

i) If $w_8 \ge w_2 + w_3$. By taking $\alpha = 3$ in (3.2) ($m = 9$, $\nu = 4$), we get

$$\sum_{j=4}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor - \lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9) \ge (\lfloor\frac{w_8+\rho}{9}\rfloor - \lfloor\frac{w_3+\rho}{9}\rfloor)(16-9) = (\lfloor\frac{w_8+\rho}{9}\rfloor - \lfloor\frac{w_3+\rho}{9}\rfloor)(7). \tag{4.3}$$

By using (4.3), we get

$$\begin{aligned}
&\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{m}\rfloor - \lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(4j-9) + \rho \\
&= (\lfloor\frac{w_1+\rho}{9}\rfloor - \lfloor\frac{w_0+\rho}{9}\rfloor)(-5) + (\lfloor\frac{w_2+\rho}{9}\rfloor - \lfloor\frac{w_1+\rho}{9}\rfloor)(-1) + (\lfloor\frac{w_3+\rho}{9}\rfloor - \lfloor\frac{w_2+\rho}{9}\rfloor)(3) \\
&\quad + \sum_{j=4}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor - \lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9) + \rho \\
&\ge \lfloor\frac{w_1+\rho}{9}\rfloor(-4) + \lfloor\frac{w_2+\rho}{9}\rfloor(-4) + \lfloor\frac{w_3+\rho}{9}\rfloor(3) + (\lfloor\frac{w_8+\rho}{9}\rfloor - \lfloor\frac{w_3+\rho}{9}\rfloor)(7) + \rho \\
&\ge \left(\lfloor\frac{w_2+\rho}{9}\rfloor((\frac{-3}{4})4) + \lfloor\frac{w_3+\rho}{9}\rfloor((\frac{-1}{4})4)\right) + \lfloor\frac{w_2+\rho}{9}\rfloor(-4) + \lfloor\frac{w_3+\rho}{9}\rfloor(3) + (\lfloor\frac{w_8+\rho}{9}\rfloor - \lfloor\frac{w_3+\rho}{9}\rfloor)(7) \\
&\quad + \rho \\
&= \lfloor\frac{w_2+\rho}{9}\rfloor(-7) + \lfloor\frac{w_3+\rho}{9}\rfloor(2) + (\lfloor\frac{w_8+\rho}{9}\rfloor - \lfloor\frac{w_3+\rho}{9}\rfloor)(7) + \rho \\
&= \lfloor\frac{w_3+\rho}{9}\rfloor(2) + (\lfloor\frac{w_8+\rho}{9}\rfloor - \lfloor\frac{w_2+\rho}{9}\rfloor - \lfloor\frac{w_3+\rho}{9}\rfloor)(7) + \rho.
\end{aligned}$$

Hence,

$$\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(4j-9)+\rho\geq\lfloor\frac{w_3+\rho}{9}\rfloor(2)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_2+\rho}{9}\rfloor-\lfloor\frac{w_3+\rho}{9}\rfloor)(7)+\rho. \quad (4.4)$$

Since $w_8\geq w_2+w_3$, by Lemma 3.2, it follows that $\lfloor\frac{w_8+\rho}{9}\rfloor\geq\lfloor\frac{w_2+\rho}{9}\rfloor+\lfloor\frac{w_3+\rho}{9}\rfloor-1$.

- If $\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_2+\rho}{9}\rfloor-\lfloor\frac{w_3+\rho}{9}\rfloor\geq 0$, then (4.4) gives $\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor-\lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9)+\rho\geq 0.$
- If $\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_2+\rho}{9}\rfloor-\lfloor\frac{w_3+\rho}{9}\rfloor=-1$. By Lemma 3.2, we have $\rho\geq 1$. Since for $q\leq 3$ Wilf's conjecture is solved ([5], [7]), then may assume that $q\geq 4$. Since $\lfloor\frac{w_2+\rho}{9}\rfloor\leq\lfloor\frac{w_3+\rho}{9}\rfloor$ and $\lfloor\frac{w_2+\rho}{9}\rfloor+\lfloor\frac{w_3+\rho}{9}\rfloor=\lfloor\frac{w_8+\rho}{9}\rfloor+1=q+1$, in this case it follows that $\lfloor\frac{w_3+\rho}{9}\rfloor+\lfloor\frac{w_3+\rho}{9}\rfloor\geq\lfloor\frac{w_2+\rho}{9}\rfloor+\lfloor\frac{w_3+\rho}{9}\rfloor=q+1\geq 5$. Hence, $\lfloor\frac{w_3+\rho}{9}\rfloor\geq 3$. Now (4.4) gives, $\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor-\lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9)+\rho \quad\geq 3(2)-7+1 \quad\geq 0.$

Using Proposition 2.3, we get that S satisfies Wilf's conjecture in this case.

***ii*)** If $w_8\geq w_1+w_4$. We may assume that $w_8<w_2+w_3$, since otherwise we are back to case i). Hence, the possible values of $w\in\mathrm{Ap}(S,9)^*\backslash\min(\mathrm{Ap}(S,9))$ are included in $\{w_1+w_j; 1\leq j\leq 7\}\cup\{w_2+w_2\}$.

- If $\mathrm{Ap}(S,9)^*\backslash\min(\mathrm{Ap}(S,9))\subseteq\{w_1+w_j;\ 1\leq j\leq 7\}$. Then $5=m-\nu=|\mathrm{Ap}(S,9)^*\backslash\min(\mathrm{Ap}(S,9))|$. By using Corollary 4.2 (i) and (ii)), it follows that there exists at least five elements in $\mathrm{Ap}(S,9)^*$ that are not maximal (five elements from $\{w_1\ldots,w_7\}$), hence $t(S)=|\{\max(\mathrm{Ap}(S,9))\}|\leq 8-5=3=\nu-1$. Consequently, S satisfies Wilf's conjecture ([4] Proposition 2.3).
- If $w_2+w_2\in\mathrm{Ap}(S,9)^*\backslash\min(\mathrm{Ap}(S,9))$, then $w_2+w_2\in\mathrm{Ap}(S,9)$ namely $w_8\geq w_2+w_2$. By Lemma 3.2 we have $\lfloor\frac{w_8+\rho}{9}\rfloor\geq 2\lfloor\frac{w_2+\rho}{9}\rfloor-1$. In particular,

$$\lfloor\frac{w_2+\rho}{9}\rfloor\leq\frac{q+1}{2}. \quad (4.5)$$

By taking $\alpha=4$ in (3.2) ($m=9$, $\nu=4$), we get

$$\sum_{j=5}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor-\lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9)\geq(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11). \quad (4.6)$$

Now using $m=9$, $\nu=4$, (4.5) and (4.6), we get

$$\begin{aligned}
&\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor-\lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9)+\rho\\
&=(\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_0+\rho}{9}\rfloor)(-5)+(\lfloor\frac{w_2+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor)(-1)+(\lfloor\frac{w_3+\rho}{9}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor)(3)\\
&\quad+(\lfloor\frac{w_4+\rho}{9}\rfloor-\lfloor\frac{w_3+\rho}{9}\rfloor)(7)+\sum_{j=5}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor-\lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9)+\rho\\
&\geq\lfloor\frac{w_1+\rho}{9}\rfloor(-4)+\lfloor\frac{w_2+\rho}{9}\rfloor(-4)+\lfloor\frac{w_3+\rho}{9}\rfloor(-4)+\lfloor\frac{w_4+\rho}{9}\rfloor(7)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho\\
&\geq\lfloor\frac{w_1+\rho}{9}\rfloor(-4)+(\frac{q+1}{2})(-4)+\lfloor\frac{w_4+\rho}{9}\rfloor(-4)+\lfloor\frac{w_4+\rho}{9}\rfloor(7)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho\\
&=\lfloor\frac{w_1+\rho}{9}\rfloor(-4)-2(q+1)+\lfloor\frac{w_4+\rho}{9}\rfloor(3)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho\\
&=\lfloor\frac{w_1+\rho}{9}\rfloor(-4+11-11)-2(q+1)+\lfloor\frac{w_4+\rho}{9}\rfloor(3)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho\\
&=\lfloor\frac{w_1+\rho}{9}\rfloor(7)-2(q+1)+\lfloor\frac{w_4+\rho}{9}\rfloor(3)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor)(11)+\rho\\
&=(\lfloor\frac{w_1+\rho}{9}\rfloor+\lfloor\frac{w_4+\rho}{9}\rfloor)(3)+\lfloor\frac{w_1+\rho}{9}\rfloor(4)-2(q+1)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)\\
&\quad+\rho.
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(4j-9)+\rho \geq$$
$$(\lfloor\frac{w_1+\rho}{9}\rfloor+\lfloor\frac{w_4+\rho}{9}\rfloor)(3)+\lfloor\frac{w_1+\rho}{9}\rfloor(4)-2(q+1)+(\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor)(11)+\rho. \tag{4.7}$$

We have $w_8 \geq w_1+w_4$, then by Lemma 3.2 (i) $\lfloor\frac{w_8+\rho}{9}\rfloor \geq \lfloor\frac{w_1+\rho}{9}\rfloor+\lfloor\frac{w_4+\rho}{9}\rfloor-1$.

– If $\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor \geq 0$. Let $x=\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor$. Hence, $x \geq 0$ and $\lfloor\frac{w_1+\rho}{9}\rfloor+\lfloor\frac{w_4+\rho}{9}\rfloor=\lfloor\frac{w_8+\rho}{9}\rfloor-x=q-x$ (Remark 3.1 vi). Then (4.7) gives,
$$\begin{aligned}\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor-\lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9)+\rho &\geq (q-x)(3)+4-2(q+1)+11x+\rho\\ &= q+8x+2+\rho \geq 0.\end{aligned}$$

– If $\lfloor\frac{w_8+\rho}{9}\rfloor-\lfloor\frac{w_1+\rho}{9}\rfloor-\lfloor\frac{w_4+\rho}{9}\rfloor=-1$. Then $\lfloor\frac{w_1+\rho}{m}\rfloor+\lfloor\frac{w_4+\rho}{9}\rfloor=\lfloor\frac{w_8+\rho}{9}\rfloor+1=q+1$ (Remark 3.1 vi). By Lemma 3.2, we have $\lfloor\frac{w_1+\rho}{9}\rfloor \geq 2$ and $\rho \geq 1$. Since $q \geq 1$ $(S \neq \mathbb{N})$, then (4.7) gives,
$$\sum_{j=1}^{8}(\lfloor\frac{w_j+\rho}{9}\rfloor-\lfloor\frac{w_{j-1}+\rho}{9}\rfloor)(4j-9)+\rho \geq (q+1)(3)+8-2(q+1)-11+1=q-1 \geq 0.$$

By Proposition 2.3, S satisfies Wilf's conjecture in this case.

Thus, Wilf's conjecture holds if $m-\nu=5$. ■

In the next corollary, we will deduce the conjecture for $m=9$.

Corollary 4.10. If $m=9$, then S satisfies Wilf's conjecture.

Proof. By Lemma 4.5, Corollary 4.7 and Theorem 4.9, we may assume that $m-\nu>5$, hence $\nu<m-5=4$. By ([4]) S satisfies Wilf's conjecture. ■

The following Lemma will enable us later to show that Wilf's conjecture holds for numerical semigroups with $(2+\frac{1}{q})\nu \geq m$.

Lemma 4.11. Let the notations be as above. If $m-\nu=6$ and $(2+\frac{1}{q})\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. Since $m-\nu=6 \geq \frac{4(3)}{2}-1$, by Lemma 4.8, it follows that $w_{m-1} \geq w_1+w_4$ or $w_{m-1} \geq w_2+w_3$.

i) If $w_{m-1} \geq w_1+w_4$. By hypothesis $(2+\frac{1}{q})\nu \geq m$ and Theorem 4.1 Wilf's conjecture holds in this case.

ii) If $w_{m-1} \geq w_2+w_3$. We may assume that $w_{m-1}<w_1+w_4$, since otherwise we are back to case i). Hence, $\text{Ap}(S,m)^*\backslash\min(\text{Ap}(S,m))=\{w_1+w_1, w_1+w_2, w_1+w_3, w_2+w_2, w_2+w_3, w_3+w_3\}$ (as $6=m-\nu=|\text{Ap}(S,m)^*\backslash\min(\text{Ap}(S,m))|$).
By taking $\alpha=3$ in (3.2), we get
$\sum_{j=4}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m) \geq (\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu-m)$. Hence,

$$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho$$
$$=(\lfloor\frac{w_1+\rho}{m}\rfloor-\lfloor\frac{w_0+\rho}{m}\rfloor)(\nu-m)+(\lfloor\frac{w_2+\rho}{m}\rfloor-\lfloor\frac{w_1+\rho}{m}\rfloor)(2\nu-m)+(\lfloor\frac{w_3+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor)(3\nu-m)$$
$$+\sum_{j=4}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho$$
$$\geq\lfloor\frac{w_1+\rho}{m}\rfloor(-\nu)+\lfloor\frac{w_2+\rho}{m}\rfloor(-\nu)+\lfloor\frac{w_3+\rho}{m}\rfloor(3\nu-m)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu-m)+\rho$$
$$\geq(\lfloor\frac{w_2+\rho}{m}\rfloor(\frac{-\nu}{2})+\lfloor\frac{w_3+\rho}{m}\rfloor(\frac{-\nu}{2}))+\lfloor\frac{w_2+\rho}{m}\rfloor(-\nu)+\lfloor\frac{w_3+\rho}{m}\rfloor(3\nu-m)$$
$$+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu-m)+\rho$$
$$=\lfloor\frac{w_2+\rho}{m}\rfloor(\frac{-3\nu}{2})+\lfloor\frac{w_3+\rho}{m}\rfloor(\frac{5\nu}{2}-m)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu-m)+\rho$$
$$=\lfloor\frac{w_2+\rho}{m}\rfloor(\frac{-3\nu}{2}+(4\nu-m)-(4\nu-m))+\lfloor\frac{w_3+\rho}{m}\rfloor(\frac{5\nu}{2}-m)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu-m)+\rho$$
$$=\lfloor\frac{w_2+\rho}{m}\rfloor(\frac{5\nu}{2}-m)+\lfloor\frac{w_3+\rho}{m}\rfloor(\frac{5\nu}{2}-m)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor)(4\nu-m)+\rho$$
$$=\lfloor\frac{w_2+\rho}{m}\rfloor(\frac{3\nu}{2}-6)+\lfloor\frac{w_3+\rho}{m}\rfloor(\frac{3\nu}{2}-6)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor)(3\nu-6)+\rho.$$

Hence,

$$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho\geq$$
$$\lfloor\frac{w_2+\rho}{m}\rfloor(\frac{3\nu}{2}-6)+\lfloor\frac{w_3+\rho}{m}\rfloor(\frac{3\nu}{2}-6)+(\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor)(3\nu-6)+\rho. \tag{4.8}$$

We have $w_{m-1}\geq w_2+w_3$, by Lemma 3.2, it follows that $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor\geq\lfloor\frac{w_2+\rho}{m}\rfloor+\lfloor\frac{w_3+\rho}{m}\rfloor-1.$

- If $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor\geq 0$, using $\nu\geq 4$ in (4.8) ($\nu\leq 3$ is solved [4]), we get
$$\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho\geq 0.$$
- If $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor-\lfloor\frac{w_2+\rho}{m}\rfloor-\lfloor\frac{w_3+\rho}{m}\rfloor=-1$. Then, $\lfloor\frac{w_2+\rho}{m}\rfloor+\lfloor\frac{w_3+\rho}{m}\rfloor=\lfloor\frac{w_{m-1}+\rho}{m}\rfloor+1$, that is
$$\lfloor\frac{w_2+\rho}{m}\rfloor+\lfloor\frac{w_3+\rho}{m}\rfloor=q+1. \tag{4.9}$$

  We have $w_3+w_3\in \mathrm{Ap}(S,m)^*\backslash\min(\mathrm{Ap}(S,m))$ namely $w_3+w_3\in\mathrm{Ap}(S,m)$, then $w_{m-1}\geq w_3+w_3$. By Lemma 3.2, we have $\lfloor\frac{w_{m-1}+\rho}{m}\rfloor\geq 2\lfloor\frac{w_3+\rho}{m}\rfloor-1$. In particular,
$$\lfloor\frac{w_3+\rho}{m}\rfloor\leq\frac{q+1}{2}. \tag{4.10}$$

  Since Wilf's conjecture holds for $q\leq 3$ ([5], [7]), so we may assume that $q\geq 4$. Since $\lfloor\frac{w_2+\rho}{m}\rfloor\leq\lfloor\frac{w_3+\rho}{m}\rfloor$, by (4.9) and (4.10), it follows that $\lfloor\frac{w_2+\rho}{m}\rfloor=\lfloor\frac{w_3+\rho}{m}\rfloor=\frac{q+1}{2}$, in particular q is odd, so we have to assume that $q\geq 5$. Now using Now using (4.9), $q\geq 5$ and the hypothesis $(2+\frac{1}{q})\nu\geq m=\nu+6$ (in particular $-6q\geq -q\nu-\nu$) in (4.8), we get
$$\begin{aligned}\sum_{j=1}^{m-1}(\lfloor\frac{w_j+\rho}{m}\rfloor-\lfloor\frac{w_{j-1}+\rho}{m}\rfloor)(j\nu-m)+\rho &\geq (q+1)(\frac{3\nu}{2}-6)-(3\nu-6)+\rho\\ &=\nu(\frac{3q}{2}+\frac{3}{2}-3)-6q+\rho\\ &\geq\nu(\frac{3q}{2}-\frac{3}{2})-q\nu-\nu+\rho\\ &=\nu(\frac{q}{2}-\frac{5}{2})+\rho\geq 0.\end{aligned}$$
  By Proposition 2.3, S satisfies Wilf's conjecture in this case.

By the results above we get that Wilf conjecture holds for numerical semigroups satisfying $(2+\frac{1}{q})\nu\geq m$. More precisely we have the following.

Theorem 4.12. Let the notations be as above. If $(2+\frac{1}{q})\nu \geq m$, then S satisfies Wilf's conjecture.

Proof. If $m-\nu \leq 3$, then by Lemma 4.5 Wilf's conjecture holds.
If $m-\nu = 4$, then by Corollary 4.7 Wilf's conjecture holds.
If $m-\nu = 5$, then by Theorem 4.9 Wilf's conjecture holds.
If $m-\nu = 6$ and $(2+\frac{1}{q})\nu \geq m$, then by Lemma 4.11 Wilf's conjecture holds.
If $m-\nu > 6 = \frac{4(3)}{2}$ and $(2+\frac{1}{q})\nu \geq m$, then by Corollary 4.4 Wilf's conjecture holds. ■

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Received: July 12, 2017.

Accepted: December 17, 2017.