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A NOTE ON INTERMEDIATE RINGS BETWEEN $D + I$ and $K[y_1]]...[y_t]]$

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Abstract Let $R = K[y_1]]...[y_t]]$ be a K-algebra (where K is a field) having krull dimension $n \geq 1$. Let I be a nonzero proper ideal of R and D be a subring of K. We characterize when $S = D + I$ is a maximal non-Jaffard subring of R.

1 Introduction

All rings considered throughout this paper are commutative integral domains with $1 \neq 0$; all subrings and ring homomorphisms are unital. Let $A \subset B$ be a ring extension. We say that A is a maximal non-Jaffard subring of B if A is not a Jaffard domain while every subring of B properly containing A is a Jaffard domain. In [4] the authors characterized when A is a maximal non-Jaffard subring of a field K. An underlying difficulty in the study of maximal non-Jaffard subrings is that the nature of the "bottom" ring A and the "top" ring B has very subtle influences on intermediate rings in between. Because of this, it appears to be too difficult to understand maximal non-Jaffard subrings in general context. Using the symbol $[t]]$ to stand for "$[t]$ or $[[t]]$" and let $R = K[y_1]]...[y_t]]$ be a K-algebra (not necessarily finitely generated over the field K) having Krull dimension $n \geq 1$. Let I be a nonzero proper ideal of R (not necessarily maximal in R) and D be a proper subring of K. The aim of this paper is to provide necessary and sufficient conditions in order that $S = D + I$ is a maximal non-Jaffard subring of R. If A is a ring, then Spec(A) (resp., Max(A)) denotes the set of prime (resp., maximal) ideals of A. As usual, A' denotes the integral closure of a domain A and qf(A) its quotient field. An *overring* of A is any A-subalgebra of qf(A), that is, any ring B such that $A \subseteq B \subseteq qf(A)$; such a ring B is termed a *proper overring* of A if $B \neq A$. For a prime ideal P of a ring A, the height of P, denoted $ht_A(P)$, is defined to be the supremum of lengths l of chains $P_0 \subset P_1 \subset ... \subset P_l = P$ in Spec(A). The supremum of such heights for $P \in$ Spec(A) is the Krull dimension of A, dim$A$. Whenever a set A is a subset of a set B and $A \neq B$ we denote this symbolically $A \subset B$. Any unexplained terminology is standard as in [7].

2 Main results

We recall that a ring A of finite (Krull) dimension n is a Jaffard ring if its valuative dimension (the limit of the sequence (dim$A[X_1, ..., X_n] - n, n \in \mathbb{N}$)) dim$_v A$, is also n. Prüfer domains and Noetherian domains are Jaffard domains. The notion of Jaffard ring is neither a local nor a residual property and thus we say that A is a locally (resp., residually) Jaffard ring if A_P (resp., A/P) is a Jaffard ring for each prime ideal P of A (cf. [1]). A domain A is said to be totally Jaffard if A/P is a locally Jaffard domain for each prime ideal P of A (cf. [6]).

In the following theorem, R is a $K-$algebra, that is, $R = K[y_1, ..., y_t]$ or $R = K[[y_1, ..., y_t]]$ having Krull dimension $n \geq 1$. Let I be a nonzero proper ideal of R and D be a proper subring of K. We determine necessary and sufficient conditions in order that $S = D + I$ is a maximal non-Jaffard subring of R.

Theorem 2.1. *The following statements are equivalent:*
(1) S is a maximal non-Jaffard subring of R.
(2) (Exactly) one of the following two conditions holds:
(a) *D is an integrally closed PVD with $dim_v(D) = dim(D) + 1$, $K = qf(D)$ and $I = (y_1 - a_1, ..., y_t - a_t)$ for some $a_1, ..., a_t \in K$.*
(b) *$I \in Max(R)$ and D is a field integrally closed in R/I and $tr.deg[K : D] = 1$.*

To prove this theorem we need the following lemmas.

Lemma 2.2. *Let $A \subset B$ be an extension of integral domains. If A is a maximal non-Jaffard subring of B, then A is integrally closed in B.*

Proof. Assume the contrary, then there exists $x \in B$ integral over A such that $x \notin A$. Since $A \subset A[x] \subseteq B$, then $A[x]$ is a Jaffard domain. As the ring extension $A \subset A[x]$ is integral, it follows from [3, Corollaire 1.6] that A is also a Jaffard domain, which is a contradiction to the assumption on A. □

Lemma 2.3. *If $S = D + I$ is a maximal non-Jaffard subring of $R = K[y_1]]...[y_t]]$, then $I \in Max(R)$.*

Proof. Assume the contrary, then there exists a maximal ideal M of R properly containing I. As $S \subset K + I \subset K + M \subseteq R$, then $K + I$ and $K + M$ are Jaffard domains. Moreover [2, Proposition 1.2] guarantees that $K + I$ and $K + M$ are catenarian and coequidimensionnal with Krull dimension n. Hence $ht_{K+I}(I) = ht_{K+M}(M) = n$. Thus, by [6, Proposition 2], $K+I$ and $K + M$ are totally Jaffard domains. Since $S \subset D + M \subseteq R$, then $D + M$ is a Jaffard domain. Thus, viewing $D + M$ as the following pullback:

$$\begin{array}{ccc} D+M & \longrightarrow & D \\ \downarrow & & \downarrow \\ K+M & \longrightarrow & K \end{array}$$

it follows from [2, Proposition 2.7] that D is a Jaffard domain and $D \subset K$ is algebraic. Now, viewing S as the following pullback:

$$\begin{array}{ccc} S & \longrightarrow & D \\ \downarrow & & \downarrow \\ K+I & \longrightarrow & K \end{array}$$

we deduce again from [2, Proposition 2.7] that S is a Jaffard domain, the desired contradiction. □

Proof of Theorem 2.1. (i)⇒(ii) Assume that S is a maximal non-Jaffard subring of R. It follows from Lemma 2.3 that $I \in Max(R)$. Hence $ht_R(I) = n = dim(R)$ and thus R is integral over $K + I$ (cf. [2, Proposition 1.7]). As in the proof of Lemma 2.3, $K + I$ is a totally Jaffard domain and $\text{ht}_{K+I}(I) = n = \text{dim}(K + I)$. Since S is not a Jaffard domain, then it follows from [2, Proposition 2.7] that either D is not a Jaffard domain or K is not algebraic over D. Thus we will discuss these two cases separately.
Case 1. D is not a Jaffard domain. In this case D is a maximal non-Jaffard subring of K. Thus it follows from [4, Theorem 1.4] that D is an integrally closed PVD with $\dim_v(D) = 1+ \dim(D)$ and $K = \text{qf}(D)$. Now, we assert that $I = (y_1 - a_1, ..., y_t - a_t)$ for some $a_1, ..., a_t \in K$. To this end, notice that as R is integral over $K + I$, then for each $1 \leq i \leq t$, there exists a monic polynomial $P_i(X) \in K[X]$ such that $P_i(y_i) \in I$. Let $P_i(y_i) = y_i^{s_i} + \alpha_1 y_i^{s_i-1} + ... + \alpha_{s_i}$ for some $\alpha_1, ..., \alpha_{s_i} \in K$. Expressing $\alpha_j = \frac{d_j}{\theta}$ (for $j = 1, ..., s_i$) for some $d_j \in D$ and $\theta \in D \setminus \{0\}$. $P_i(y_i) = y_i^{s_i} + \frac{d_1}{\theta} y_i^{s_i-1} + ... + \frac{d_{s_i}}{\theta} \in I$. So $\theta^{s_i} P_i(y_i) = \theta^{s_i} y_i^{s_i} + d_1 \theta^{s_i-1} y_i^{s_i-1} + ... + \theta^{s_i-1} d_{s_i} \in I$. This implies that θy_i is integral over S. Since S is integrally closed in R (see Lemma 2.2), then $\theta y_i \in S$. Hence $\theta y_i = \delta_i + z_i$ for some $\delta_i \in D$, $z_i \in I$. Thus $y_i - a_i \in I$ for some $a_i \in K$. Hence

$I \supseteq (y_1 - a_1, ..., y_t - a_t)$ and so $I = (y_1 - a_1, ..., y_t - a_t)$, as asserted.
Case 2. K is not algebraic over D. Let $t \in K$ be transcendental over D. It follows from case 1 that D is a Jaffard domain. Hence (D, K) is a Jaffard pair and so is the pair $(D[t], K)$. Thus, according to [3, Lemme 2.1 and Théorème 2.6], $D'[t]$ is a Prüfer domain. Therefore D' is a field and so is D. As S is integrally closed in $K + I$ (see Lemma 2.2), then it follows from [5, Lemme 2] that D is integrally closed in K. The fact that $tr.deg[K : D] = 1$ follows readily from [4, Lemma 1.2] since (D, K) is a Jaffard pair and D is a field.
(ii)⇒(i) Assume that condition (a) is satisfied. As $R = K + I$, then any ring T such that $D + I \subset T \subseteq R$ is of the form $A + I$ where A is a ring such that $D \subset A \subseteq K$. Since D is a maximal non-Jaffard subring of K, then A is a Jaffard domain; moreover K is algebraic over A. Hence T is a Jaffard domain (see [2, Théorème 3.3]). It follows that S is a maximal non-Jaffard subring of R as desired. Now assume that condition (b) is satisfied. Since $I \in Max(R)$ and $D \subset K$ is not algebraic then S is not a Jaffard domain (see [2, Théorème 3.3]). Now let T be a ring such that $S \subset T \subseteq R$. The rings T and R share the same ideal I. Hence $T := (R, I, D_1)$ where D_1 is a subring of R/I properly containing D (see [5]). As R/I is integral over K (see [2, Proposition 1.7]), then $tr.deg[R/I : D] = tr.deg[K : D] = 1$. Thus [4, Lemma 1.2] permits one to conclude that D_1 is a Jaffard domain. Moreover since D is integrally closed in R/I and $D_1 \neq D$, then necessarily R/I is algebraic over D_1. Thus it follows from [2, Théorème 3.3] that T is a Jaffard domain. The desired conclusion. □

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