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WEAKLY COHERENT PROPERTY IN AMALGAMATED ALGEBRA ALONG AN IDEAL

HAITHAM EL ALAOUI AND HAKIMA MOUANIS

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Abstract Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. In this paper, we inverstigate the weakly coherent property that the amalgamation $A \bowtie^f J$ might inherit from the ring A for some classes of ideals J and homomorphisms f. Our results generates original examples which enrich the current literature with new families of examples of non-coherent weakly coherent rings.

1 Introduction

Throughout this paper, all rings are commutative with identity element, and all modules are unitary.

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of A×B:

$$A \bowtie^f J = \{(a, f(a) + j) \ /a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [9, 10]). This constuction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [11, 12, 13] and denoted by $A \bowtie I$). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([9, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagata's idealizations(cf. [23, page 2]) and the CPI extensions (in the sense of Boisen and Sheldon [6]) are strictly related to it (see [9, Example 2.7 and Remark 2.8]).

Let R be a commutative ring. For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules:

$$F_n \longrightarrow F_{n-1} \longrightarrow \ldots F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-module are respectively, finitely generated and finitely presented R-module.

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent ring are Noetherian ring, Boolean algebras, von Neumann regular rings, and prüfer/semi-hereditary rings. For instance see [17].

In [4], Bakkari and Mahdou introduce a weakly coherent ring. A ring R is called a weakly coherent ring if any finitely generated ideal of R contained in a finitely presented proper ideal of R is itself finitely presented. If R is coherent, then R is naturally weakly coherent. For instance see [4].

Given nonnegative integers n and d, a ring R is called an (n, d)-ring if every n-presented R-module has projective dimension $\leqslant d$; and a weak (n, d)-ring if every n-presented cyclic R-module has projective dimension $\leqslant d$ (equivalently, if every $(n-1)$-presented ideal of R has projective dimension $\leqslant d-1$). See for instance [8, 18, 19, 20, 21].

In this paper, we characterize $A \bowtie^f J$ to be weakly coherent ring for some classes of ideals J and homomorphisms f. Thereby, new examples are provided which particulary, enriches the current literature with new classes of non-coherent weakly coherent rings.

2 Main result

The main result of this section (Theorem 2.5) examines the transfer of the property weakly coherent to the amalgamated algebra. Our objective is to generate new and original examples to enrich the current literature with new families of non-coherent weakly coherent rings.

Let $f : A \longrightarrow B$ be a ring homomorphism, J be an ideal of B and let n be a positive integer. Consider the function $f^n : A^n \longrightarrow B^n$ to defined by $f^n((\alpha_i)_{i=1}^{i=n}) = (f(\alpha_i))_{i=1}^{i=n}$ and $f^n(\alpha a) = f(\alpha)f^n(a)$ for all $\alpha \in A$ and $a \in A^n$. Obviously, f^n is a ring homomorphism and J^n is an ideal of B^n. This allows us to define $A^n \bowtie^{f^n} J^n$. Moreover, let $\phi : (A \bowtie^f J)^n \longrightarrow A^n \bowtie^{f^n} J^n$ defined by $\phi((a_i, f(a_i) + j_i)_{i=1}^{i=n}) = ((a_i)_{i=1}^{i=n}, f^n((a_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n})$. It is easily checked that ϕ is a ring isomorphism, so $(A \bowtie^f J)^n$ and $A^n \bowtie^{f^n} J^n$ are isomorphic as rings.

Then, before announcing some results of amalgamated algebra along an ideal. We recall by the following remark .

Remark 2.1. Let $f : A \longrightarrow B$ be a ring homomorphism and M be an B-module. Then M is a module over A, via f. Precisely, $a.m = f(a)m$ for each $a \in A$ and $m \in M$.

Proposition 2.2. *Let A be a local ring with maximal ideal M, $f : A \to B$ be a ring homomorphism and J be a ideal proper of B such that $MJ = 0$ and $J^2 = 0$. Then, $A \bowtie^f J$ is a $(2,0)$-ring provided J is a not finitely generated A-module.*

Lemma 2.3. *Let $f : A \longrightarrow B$ be a ring homomorphism, and J be an ideal proper of B. Let I and K be two ideals of A and B respectively such that $K \subseteq J$, and let U be an submodule of A^n.*

1) Assume that $IJ \subseteq K$. Then:

a) $I \bowtie^f K = \{(i, f(i) + k)/i \in I, k \in K\}$ is an ideal of $A \bowtie^f J$.

b) If U and K are finitely generated A-modules. Then $U \bowtie^{f^n} K^n$ is a finitely generated $A \bowtie^f J$-module.

c) If $U \bowtie^{f^n} K^n$ is a finitely generated $A \bowtie^f J$-module. Then U is a finitely generated A-module.

2) Assume that U is a sub-module of I^n and $IJ = 0$. If $U \bowtie^{f^n} K^n$ is a finitely generated $A \bowtie^f J$-module. Then, K is a finitely generated ideal of $f(A) + J$.

3) Under the same hypothesis of 2) and $J^2 = 0$. Then, $U \bowtie^{f^n} K^n$ is a finitely generated $A \bowtie^f J$-module if and only if U and K are finitely generated A-modules.

Proof. a) It is clear that $I \bowtie^f K$ is an ideal of $A \bowtie^f J$. Indeed :

• $(i, f(i) + k) + (i', f(i') + k') = (i + i', f(i + i') + k + k') \in I \bowtie^f K$ for all $(i, f(i) + k)$, $(i', f(i') + k') \in I \bowtie^f K$.

• $(a, f(a) + j)(i, f(i) + k) = (ai, f(ai) + jf(i) + kf(a) + kj) = (ai, f(ai) + ij + ak + kj)$ by Remark 2.1. So, $(a, f(a) + j)(i, f(i) + k) \in I \bowtie^f K$ for all $(a, f(a) + j) \in A \bowtie^f J$ and $(i, f(i) + k) \in I \bowtie^f K$, since $IJ \subseteq K$.

b) Assume that $U := \sum_{i=1}^{i=n} Au_i$ is a finitely generated A-module, where $u_i \in U$ for all $i \in \{1, \dots, n\}$ and $K^n = \sum_{i=1}^{i=m} Ae_i$, where $e_i \in K^n$ for all $i \in \{1, \dots, m\}$. Let $(x, f^n(x) + k) \in U \bowtie^{f^n} K^n$, where $x \in U$ and $k \in K^n$, so there exists $(\alpha_i)_{i=1}^{i=n} \in A^n$ and $(\beta_i)_{i=1}^{i=m} \in A^m$ such that $x = \sum_{i=1}^{i=n} \alpha_i u_i$ and $k = \sum_{i=1}^{i=m} \beta_i e_i = \sum_{i=1}^{i=m} f(\beta_i)e_i$ by Remark 2.1. So, we obtain:

$$\begin{aligned}(x, f^n(x)+k) &= (\sum_{i=1}^{i=n}\alpha_i u_i, \sum_{i=1}^{i=n} f(\alpha_i)f^n(u_i)+\sum_{i=1}^{i=m} f(\beta_i)e_i)\\ &= (\sum_{i=1}^{i=n}\alpha_i u_i, \sum_{i=1}^{i=n} f(\alpha_i)f^n(u_i))+(0,\sum_{i=1}^{i=m} f(\beta_i)e_i)\\ &= \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i))(u_i, f^n(u_i))+\sum_{i=1}^{i=m}(\beta_i, f(\beta_i))(0, e_i).\end{aligned}$$

Consequently, $(x, f^n(x)+k) \in \sum_{i=1}^{i=n} A \bowtie^f J(u_i, f^n(u_i)) + \sum_{i=1}^{i=m} A \bowtie^f J(0, e_i)$ since $(\alpha_i, f(\alpha_i)) \in A \bowtie^f J$ for all $i \in \{1, \dots, n\}$ and $(\beta_i, f(\beta_i)) \in A \bowtie^f J$ for all $i \in \{1, \dots, m\}$. Therefore, $U \bowtie^{f^n} K^n \subseteq \sum_{i=1}^{i=n} A \bowtie^f J(u_i, f^n(u_i)) + \sum_{i=1}^{i=m} A \bowtie^f J(0, e_i)$. Conversely, $\sum_{i=1}^{i=n} A \bowtie^f J(u_i, f^n(u_i)) + \sum_{i=1}^{i=m} A \bowtie^f J(0, e_i) \subseteq U \bowtie^{f^n} K^n$ since $(u_i, f^n(u_i)) \in A \bowtie^f J$ for all $i \in \{1, \dots, n\}$, $(0, e_i) \in A \bowtie^f J$ for all $i \in \{1, \dots, m\}$ and $U \bowtie^{f^n} K^n$ is a $A \bowtie^f J$-module. Hence, $U \bowtie^{f^n} K^n$= $\sum_{i=1}^{i=n} A \bowtie^f J(u_i, f^n(u_i)) + \sum_{i=1}^{i=m} A \bowtie^f J(0, e_i)$ is a finitely generated $A \bowtie^f J$-module.

c) Let $U \bowtie^{f^n} K^n$ is a finitely generated $A \bowtie^f J$-module, i.e $U \bowtie^{f^n} K^n = \sum_{i=1}^{i=r} A \bowtie^f J(u_i, f^n(u_i)+e_i)$ where $u_i \in U$ and $e_i \in K^n$ for all $i \in \{1, \dots, r\}$, let $x \in U$ and $k \in K^n$, so $(x, f^n(x)+k) \in U \bowtie^{f^n} K^n$. Then, there exists $(\alpha_i, f(\alpha_i)+j_i)_{i=1}^{i=r} \in (A \bowtie^f J)^r$ such that $(x, f^n(x)+k) = \sum_{i=1}^{i=r}(\alpha_i, f(\alpha_i)+j_i)(u_i, f^n(u_i)+e_i)$=$(\sum_{i=1}^{i=r}\alpha_i u_i, \sum_{i=1}^{i=r}(f(\alpha_i)+j_i)(f^n(u_i)+e_i)$. Therefore, $x = \sum_{i=1}^{i=r}\alpha_i u_i$, hence U is a finitely generated A-module.

2) Let $k \in K^n$. So, $(0, k) \in U \bowtie^{f^n} K^n$ i.e there exists $(\beta_i, f(\beta_i)+k_i)_{i=1}^{i=r} \in (A \bowtie^f J)^r$ such that $(0, k) = \sum_{i=1}^{i=r}(\beta_i, f(\beta_i)+k_i)(u_i, f^n(u_i)+e_i)$= $(\sum_{i=1}^{i=r}\beta_i u_i, \sum_{i=1}^{i=r} f(\beta_i)f^n(u_i) + \sum_{i=1}^{i=r} f^n(u_i)k_i+\sum_{i=1}^{i=r}(f(\beta_i)+k_i)e_i$. Then, $\sum_{i=1}^{i=r}\beta_i u_i = 0$ and $k = \sum_{i=1}^{i=r} f^n(u_i)k_i+\sum_{i=1}^{i=r}(f(\beta_i)+k_i)e_i$. Moreover, we have $u_i \in I^n$ for all $i \in \{1, \dots, r\}$, i.e $u_i = (\lambda_1, \dots, \lambda_n)$ with the $\lambda_j \in I$ for all $j \in \{1, \dots, n\}$. Then:

$$\begin{aligned}f^n(u_i)k_i &= f^n(\lambda_1, \dots, \lambda_n)k_i\\ &= (f(\lambda_1), \dots, f(\lambda_n))k_i\\ &= (f(\lambda_1)k_i, \dots, f(\lambda_n)k_i)\\ &= (\lambda_1 k_i, \dots, \lambda_n k_i).\end{aligned}$$

Hence $f^n(u_i)k_i = 0$ for all $i \in \{1, \dots, r\}$, since $(IJ = 0)$, so K^n is a finitely generated $f(A)+J$-module. Therefore, K is a finitely generated ideal of $f(A)+J$.

3) We have,$(0, k) = (\sum_{i=1}^{i=r}\beta_i u_i, \sum_{i=1}^{i=r} f(\beta_i)f^r(u_i) + \sum_{i=1}^{i=r} f^n(u_i)k_i + \sum_{i=1}^{i=r} f(\beta_i)e_i + \sum_{i=1}^{i=r} k_i e_i$ by 2). So, $k = \sum_{i=1}^{i=r} f(\beta_i)e_i$ because $e_i \in K^n$ for all $i \in \{1, \dots, r\}$ and $J^2 = 0$. Moreover, $k = \sum_{i=1}^{i=r}\beta_i e_i$ by Remark 2.1, hence K^n is a finitely generated A-module. Therefore, K is a finitely generated A-module. On the other hand U is a finitely generated A-module by (c). Conversely, let U and K are finitely generated A-modules. So, $U \bowtie^{f^n} K^n$ is a finitely generated $A \bowtie^f J$-module by (b) as desired. □

Proof of Proposition 2.2. Let E be a 2-finitely presented $A \bowtie^f J$-module and $\{e_i\}_{i=1}^{i=n}$ be a minimal generated set of E. We want to show that E is a projective $A \bowtie^f J$-module. For this, consider the exact sequence of $A \bowtie^f J$-modules :

$$0 \longrightarrow Ker(U) \longrightarrow (A \bowtie^f J)^n \xrightarrow{U} E \longrightarrow 0$$

where $U((\alpha_i, f(\alpha_i)+j_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n}(\alpha_i, f(\alpha_i)+j_i)e_i$. We prove that $Ker(U) = 0$. Otherwise, there exists $\{m_i, f^n(m_i)+k_i\}_{i=1}^{i=r}$ a minimal generated set of $Ker(U)$ where $m_i \in M^n$ and $k_i \in J^n$ for each $i \in \{1, \dots, r\}$, because $Ker(U) \subseteq (M \bowtie^f J)(A \bowtie^f J)^n$ by [24, Lemma 4.43, p.134]. On the other hand, consider the exact sequence of $A \bowtie^f J$-modules :

$$0 \longrightarrow Ker(V) \longrightarrow (A \bowtie^f J)^r \xrightarrow{V} Ker(U) \longrightarrow 0$$

where $V((\beta_i, f(\beta_i)+l_i)_{i=1}^{i=r}) = \sum_{i=1}^{i=r}(\beta_i m_i, (f(\beta_i)+l_i)(f^n(m_i)+k_i)) = (\sum_{i=1}^{i=r}\beta_i m_i, \sum_{i=1}^{i=r} f(\beta_i)(f^n(m_i)+k_i)$. But $Ker(V) \subseteq (M \bowtie^f J)^r$, hence $Ker(V) = X \bowtie^{f^r} J^r$, where $X = \{(\beta_i)_{i=1}^{i=r} \in A^r / \sum_{i=1}^{i=r}\beta_i m_i = 0\}$. On the other hand, $Ker(V)$ is a finitely generated $A \bowtie^f J$-module, because E is a 2-presented $A \bowtie^f J$-module, so J is a finitely generated A-module by Lemma 2.3 (3), (since $X \subseteq M^r$). A contradiction because J is a not finitely generated A-module, so $Ker(U) = 0$. Therefore $E \cong (A \bowtie^f J)^n$, hence E is a projective $A \bowtie^f J$-module, as desired. □

Proposition 2.4. *Let A be a local ring with maximal ideal M, $f : A \to B$ be a ring homomorphism and J an ideal proper of B. If $A \bowtie^f J$ is a $(2,0)$-ring, then there is no finitely presented proper ideal K of $A \bowtie^f J$.*

Proof. Assume that K be a finitely presented proper ideal of $A \bowtie^f J$. Then, K is projective because $A \bowtie^f J$ is a weak (2,0)-ring. So, K is free, since $A \bowtie^f J$ is a local ring by [22, Lemma 2.2]. Hence, $K = A \bowtie^f J(a, f(a)+l)$ for some regular element $(a, f(a)+l)$ of $A \bowtie^f J$. A contradiction, since $K \subseteq M \bowtie^f J$ and $(M \bowtie^f J)(0,j) = (0,0)$ for each $j \in J - \{0\}$. □

The main result of this paper is the following.

Theorem 2.5. *Let A be a local ring with maximal ideal M, $f : A \to B$ be a ring homomorphism and J be a ideal proper of B such that $MJ = 0$.*

1) If J is a finitely generated A-module and $A \bowtie^f J$ weakly coherent ring, then so is A.

2) Assume that one of the following statements holds:

a) $J^2 = 0$ and J is a not finitely generated A-module.

b) $J^2 = 0$ and J is a finitely generated A-module and A is a weakly coherent ring.

c) A is $(2,0)$-ring, M is a not finitely generated ideal of A and $J \subseteq Rad(B)$.

Then $A \bowtie^f J$ is a weakly coherent ring.

Before proving main result, we establish the following lemmas.

Lemma 2.6. *Let A be a local ring with maximal ideal M, $f : A \longrightarrow B$ be a ring homomorphism and let J be a proper ideal of B such that $J^2 = 0$ and $MJ = 0$. Let $K = \sum_{i=1}^{i=n} A \bowtie^f J(b_i, f(b_i)+k_i)$ where $k_i \in J$ for all $i \in \{1,\dots,n\}$ and $\{(b_i, f(b_i)+k_i)\}_{i=1}^{i=n}$ be a minimal generated set of K and let $I = \sum_{i=1}^{i=n} Ab_i$. Then, K is a finitely presented $A \bowtie^f J$-module if and only if I is a finitely presented ideal of A and J is a finitely generated A-module.*

Proof. We have $K = \sum_{i=1}^{i=n} A \bowtie^f J(b_i, f(b_i)+k_i)$ and $I = \sum_{i=1}^{i=n} Ab_i$. Consider the exact sequence of A-modules :

$$0 \longrightarrow Ker(U) \longrightarrow A^n \xrightarrow{U} I \longrightarrow 0 \qquad (1)$$

where U is defined by $U((\alpha_i)_{i=1}^{i=n})$=$\sum_{i=1}^{i=n}\alpha_i b_i$. On the other hand, consider the exact sequence of $A \bowtie^f J$-modules :

$$0 \longrightarrow Ker(V) \longrightarrow (A \bowtie^f J)^n \xrightarrow{V} K \longrightarrow 0 \qquad (2)$$

where V is defined by :

$$\begin{aligned} V((\beta_i, f(\beta_i)+j_i)_{i=1}^{i=n}) &= \sum_{i=1}^{i=n}(\beta_i, f(\beta_i)+j_i)(b_i, f(b_i)+k_i) \\ &= (\sum_{i=1}^{i=n}\beta_i b_i, \sum_{i=1}^{i=n}(f(\beta_i)+j_i)(f(b_i)+k_i). \end{aligned}$$

So, $Ker(V) = \{(\beta_i, f(\beta_i) + j_i)_{i=1}^{i=n} \in (A \bowtie^f J)^n / \sum_{i=1}^{i=n} \beta_i b_i = 0\}$. Indeer, $\sum_{i=1}^{i=n} j_i k_i = 0$, (since $J^2 = 0$) $\sum_{i=1}^{i=n} f(b_i) j_i = \sum_{i=1}^{i=n} b_i j_i$ by Remark 2.1, so $\sum_{i=1}^{i=n} f(b_i) j_i = 0$ because $b_i \in I \subseteq M$ for all $i \in \{i, \dots, n\}$ and $MJ = 0$, and since $A \bowtie^f J$ is a local ring with maximal $M \bowtie^f J$ by [22, Lemma 2.2]. Then, $Ker(V) \subseteq (M \bowtie^f J)(A \bowtie^f J)^n$ by [24, Lemma 4.43, p.134]. Therefore, $Ker(V) \subseteq (M \bowtie^f J)^n$ hence $\beta_i \in M$ for all $i \in \{1, \dots, n\}$. So, $\sum_{i=1}^{i=n} f(\beta_i) k_i = 0$ and since $(A \bowtie^f J)^n$ and $A^n \bowtie^{f^n} J^n$ are isomorphic as ring. Then :

$$\begin{aligned} Ker(V) &= \{((\beta_i)_{i=1}^{i=n}, f^n((\beta_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n}) \in A^n \bowtie^{f^n} J^n / (\beta_i)_{i=1}^{i=n} \in Ker(U)\} \\ &= Ker(U) \bowtie^{f^n} J^n. \end{aligned}$$

So, $Ker(V)$ is a finitely generated $A \bowtie^f J$-module if and only if $Ker(U)$ is a finitely generated A-module and J is a finitely generated A-module by Lemma 2.3 (3), since $Ker(U) \subseteq MA^n$ by [24, Lemma 4.43, p.134]. And since I is finitely presented if and only if $Ker(U)$ is finitely generated (by a sequencese (1)) and K is finitely presented if and only if $Ker(V)$ is finitely generated (by a sequencese (2)). Then, K is a finitely presented ideal of $A \bowtie^f J$ if and only if I is a finitely presented ideal of A and J is a finitely generated A-module, as desired. □

Lemma 2.7. *Let A be a local ring with maximal ideal M, $f : A \longrightarrow B$ be a ring homomorphism and let J be a proper ideal of B such that $MJ = 0$. Let I be a proper ideal of A. Then:*

1) If J is a finitely generated A-module and I is a finitely presented ideal of A. Then, $I \bowtie^f 0$ so is of $A \bowtie^f J$.

2) If $I \bowtie^f 0$ is a finitely presented ideal of $A \bowtie^f J$. Then, I so is of A.

Proof. 1) Let $I = \sum_{i=1}^{i=n} \alpha_i a_i$, where $\alpha_i \in A$ and $a_i \in I$ for each $i \in \{1, \dots, n\}$. Consider the exact sequence of A-modules :

$$0 \longrightarrow Ker(U) \longrightarrow A^n \xrightarrow{U} I \longrightarrow 0 \qquad (1)$$

where U is defined by $U((\alpha_i)_{i=1}^{i=n})$=$\sum_{i=1}^{i=n} \alpha_i a_i$. Hence, $Ker(U) = \{((\alpha_i)_{i=1}^{i=n}) \in A^n / \sum_{i=1}^{i=n} \alpha_i a_i = 0\}$ is a finitely generated A-module (since I is a finitely presented ideal of A). On the other hand, $I \bowtie^f 0 = \sum_{i=1}^{i=n} A \bowtie^f J(a_i, f(a_i))$ by Lemma 2.3 (b). Consider the exact sequence of $A \bowtie^f J$-modules :

$$0 \longrightarrow Ker(V) \longrightarrow (A \bowtie^f J)^n \xrightarrow{V} I \bowtie^f 0 \longrightarrow 0 \qquad (2)$$

where $V((\beta_i, f(\beta_i) + e_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n} (\beta_i, f(\beta_i) + e_i)(a_i, f(a_i))$. But:

$$\begin{aligned} Ker(V) &= \{(\beta_i, f(\beta_i) + e_i)_{i=1}^{i=n} \in (A \bowtie^f J)^n / \sum_{i=1}^{i=n} (\beta_i, f(\beta_i) + e_i)(a_i, f(a_i)) = 0\} \\ &= \{(\beta_i, f(\beta_i) + e_i)_{i=1}^{i=n} \in (A \bowtie^f J)^n / \sum_{i=1}^{i=n} \beta_i a_i = 0\} \\ &= Ker(U) \bowtie^{f^n} J^n. \end{aligned}$$

Since $a_i \in I \subseteq M$ wich is finitely generated $A \bowtie^f J$-module, because $Ker(U)$ and J are finitely generated A-modules. Hence, $I \bowtie^f 0$ is a finitely presented proper ideal of $A \bowtie^f J$.

2) Let $I \bowtie^f 0 = \sum_{i=1}^{i=n} A \bowtie^f J(b_i, f(b_i))$ where $b_i \in I$ for all $i \in \{1, \dots, n\}$. It is clear that $I = \sum_{i=1}^{i=n} Ab_i$, so by the same reasoning as for 1), we can show that $Ker(V) = Ker(U) \bowtie^{f^n} J^n$, where U and V as above. And since, $I \bowtie^f 0$ is a finitely presented ideal of $A \bowtie^f J$, then $Ker(V)$ is a finitely generated $A \bowtie^f J$-module. Therefore, $Ker(U)$ is a finitely generated A-module by Lemma 2.3 (c). Hence, I is a finitely presented ideal of A. □

Proof of Theorem 2.5. 1) Assume that $A \bowtie^f J$ is a weakly coherent ring and J is a finitely generated A-module. Our aim is to show that A is weakly coherent. Let $I \subseteq L$ be two proper ideals of A such that I is finitely generated and L is finitely presented. Then, $I \bowtie^f 0 \subseteq L \bowtie^f 0$ are two finitely generated proper ideals of $A \bowtie^f J$ by Lemma 2.3 (b), we claim that I is a finitely presented ideal of A. Indeed, L is a finitely presented ideal of A and J is a finitely generated A-module, so $L \bowtie^f 0$ is a finitely presented ideal of $A \bowtie^f J$ by Lemma 2.7 (1) and so $I \bowtie^f 0 \subseteq L \bowtie^f 0$ is a finitely presented ideal of $A \bowtie^f J$ since $A \bowtie^f J$ is a weakly coherent ring. Therefore I is a finitely presented ideal of A by Lemma 2.7 (2) and this shows that A is a weakly coherent ring.

2) a) Assume that J is a not finitely generated A-module and $J^2 = 0$. Then, $A \bowtie^f J$ is $(2,0)$-ring by Proposition 2.2. So, it is weakly coherent by Proposition 2.4.

b) Assume that J is an A/M-vector space with finite rank, $J^2 = 0$ and A weakly coherent. Our aim is to show that $A \bowtie^f J$ is weakly coherent. Let $I = \sum_{i=1}^{i=n} A \bowtie^f J(a_i, f(a_i) + e_i) \subseteq L = \sum_{j=1}^{i=m} A \bowtie^f J(b_j, f(b_j) + k_j)$ be two proper ideals of $A \bowtie^f J$ such that n, m are positive integers $a_i, b_j \in A$ and $e_i, k_j \in J$ for each i, j and L is finitely presented, we which to show that I is finitely presented. Two cases are then possible :

Case 1: $b_j = 0$ for all $j \in \{1, \dots, m\}$.
In this case, $a_i = 0$ for all $i \in \{1, \dots, n\}$, $I = 0 \bowtie^f E_1$ and $L = 0 \bowtie^f E_2$ for some A/M-vector subspace E_1 and E_2 of J. Assume that $\{e_i\}_{i=1}^{i=n}$ and $\{k_j\}_{j=1}^{j=m}$ are respectively basis of the (A/M)-vector subspace E_1 and E_2 of J. Consider the exact sequence of $A \bowtie^f J$-modules :

$$0 \longrightarrow Ker(U) \longrightarrow (A \bowtie^f J)^m \xrightarrow{U} L := 0 \bowtie^f E_2 \longrightarrow 0$$

Where

$$\begin{aligned} U((c_j, f(c_j) + g_j)_{j=1}^{j=m}) &= \sum_{j=1}^{j=m} (c_j, f(c_j) + g_j)(0, k_j) \\ &= \sum_{j=1}^{j=m} f(c_j)k_j = \sum_{j=1}^{j=m} c_j k_j \\ &= \sum_{j=1}^{j=m} \bar{c}_j k_j \end{aligned}$$

since E_2 is a A/M-vector space. Hence, $Ker(U) = \{(c_j, f(c_j) + g_j)_{j=1}^{j=m} \in (A \bowtie^f J)^m / \sum_{j=1}^{j=m} \bar{c}_j k_j = 0\} = \{((c_j)_{j=1}^{j=m}, f^m((c_j)_{j=1}^{j=m}) + (g_j)_{j=1}^{j=m}) \in A^m \bowtie^{f^m} J^m / c_j \in M\}$, since $\{k_j\}_{j=1}^{j=m}$ is a basis of the A/M-vector space E_2. Then, $Ker(U) = M^m \bowtie^{f^m} J^m$, so M is a finitely generated ideal of A (since L is a finitely presented ideal of $A \bowtie^f J$). Therefore, the exact sequence of $A \bowtie^f J$-modules :

$$0 \longrightarrow Ker(V) \longrightarrow (A \bowtie^f J)^n \xrightarrow{V} I := 0 \bowtie^f E_1 \longrightarrow 0 \ .$$

Where, $V((c_i, f(c_i) + g_i)_{i=1}^{i=n}) = \sum_{i=1}^{n} (c_i, f(c_i) + g_i)(0, e_i) = \sum_{i=1}^{n} \bar{c}_i e_i$ shows that I is a finitely presented ideal of $A \bowtie^f J$ (since $M^n \bowtie^{f^n} J^n$ is a finitely generated $A \bowtie^f J$-module) by lemma 2.3 (b), as desired.

Case 2: $b_j \neq 0$ for some $j = 1, \dots, m$.
We may assume that $\{a_i, f(a_i) + e_i\}_{i=1}^{i=n}$ and $\{b_j, f(b_j) + k_j\}_{j=1}^{j=m}$ are minimal generating sets respectively of I and L. Let $I_0 = \sum_{i=1}^{i=n} Aa_i$ and $L_0 = \sum_{j=1}^{j=m} Ab_j$, we have L is a finitely presented ideal of $A \bowtie^f J$, so L_0 is a finitely presented ideal of A by Lemma 2.6. Hence $I_0 \subseteq L_0$ is a finitely presented ideal of A since A is a weakly coherent ring. Therefore, I is a finitely presented ideal of $A \bowtie^f J$ by Lemma 2.6.

c) Assume that A is $(2,0)$-ring, M is a not finitely generated ideal of A and $J \subseteq Rad(B)$. Then, $A \bowtie^f J$ is a $(2,0)$-ring by [1, Theorem 2.2 (2)(a)] . So, $A \bowtie^f J$ is weakly coherent by Proposition 2.4. This completes the proof of main result. □

The following Corollaries are an immediate consequence of Theorem 2.1.

Corollary 2.8. *Let A be a local ring with maximal ideal M and I and J be two proper ideals of A and let $B = A/I$ and $f : A \longrightarrow B$ be the canonical homomorphism $(f(x) = \overline{x})$. Assume that $M\overline{J} = 0$. Then, $A \bowtie^f \overline{J}$ is a weakly coherent ring if and only if one of the following two properties holds:*
1) $\overline{J}$ is a not finitely generated A-module.
2) $\overline{J}$ is a finitely generated A-module and A is a weakly coherent ring.

The next Corollary examines the case of the amalgamated duplication.

Corollary 2.9. *Let A be a local ring with maximal ideal M and let I be a ideal of A, such that $MI = 0$.*
1) If I is a finitely generated ideal of A and $A \bowtie I$ weakly coherent ring, then so is A.
2) Assume that one of the following statements holds:
a) I is a not finitely generated ideal of A.
b) I is a finitely generated ideal of A and A is a weakly coherent ring.
c) A is $(2,0)$-ring, M is a not finitely generated ideal of A.

Then $A \bowtie I$ is a weakly coherent ring.

Now, we give examples of non-coherent weakly coherent ring.

Example 2.10. Let A be a local ring with maximal ideal M and E be a A/M-vector space with infinite rank. Let $B = A \propto E$ and $J = 0 \propto E$. Consider the ring homomorphism $f : A \longrightarrow B$ $(f(a) = (a, 0))$. Then:
1) $A \bowtie^f J$ is a weakly coherent ring.
2) $A \bowtie^f J$ is not a coherent ring.

Proof. 1) By Theorem 2.5 (2)(a).
2) Assume that $A \bowtie^f J$ is coherent. But, $A \bowtie^f J$ is a $(2,0)$-ring by Proposition 2.2. Hence, $A \bowtie^f J$ is a $(1,0)$-ring by [8, Theorem 2.4]. So, it is $(1,2)$-ring. A contradiction by [1, Theorem 2.2 (1)]. Hence, $A \bowtie^f J$ is not coherent. □

Example 2.11. Let $R = \mathbb{Z}/4\mathbb{Z}$, $m = 2\mathbb{Z}/4\mathbb{Z}$ and E be an R/m-vector space with infinite rank. Let $A = R \propto E$, $M = m \propto E$, $B = A/M^2$ and $J = M/M^2$. We consider the canonical ring homomorphism $f : A \longrightarrow B$ $(f(x) = \overline{x})$. Then:
1) $A \bowtie^f J$ is a weakly coherent ring .
2) $A \bowtie^f J$ is not a coherent ring.

Proof. 1) By Theorem 2.5 (2)(a).
2) If $A \bowtie^f J$ is a coherent ring, then so is A, because A a retract of $A \bowtie^f J$, by [17, Theorem 4.1.5]. A contradiction by [19, Theorem 2.6 (2)] since E is not finite rank . □

Example 2.12. Let $R = \mathbb{Z}/8\mathbb{Z}$, $m = 2\mathbb{Z}/8\mathbb{Z}$ and E be an R/m-vector space with infinite rank and let $A = R \propto E$, $M = m \propto E$, $I = 0 \propto E$ and $J = 4\mathbb{Z}/8\mathbb{Z} \propto E$. Let $B = A/I$ and $f : A \longrightarrow B$ be the canonical homomorphism $(f(x) = \overline{x})$. Then :
1) $A \bowtie^f \overline{J}$ is a weakly coherent ring .
2) $A \bowtie^f \overline{J}$ is not a coherent ring.

Proof. 1) By Corollary 2.8 (1).
2) If $A \bowtie^f \overline{J}$ is a coherent ring, then so is A, because A a retract of $A \bowtie^f \overline{J}$, by [17, Theorem 4.1.5]. A contradiction by [19, Theorem 2.6 (2)] since E is not finite rank. □

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Author information

HAITHAM EL ALAOUI AND HAKIMA MOUANIS, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar Al Mahraz, Laboratory of Geometric and Arithmetic Algebra, Fez, Morocco..
E-mail: hmouanis@yahoo.fr

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