Palestine Journal of Mathematics

Vol. 7(2)(2018) , 425–429

A GENERALIZATION OF FINE RINGS

Peter V. Danchev

Communicated by Mohamed Yousif

MSC 2010 Classifications: Primary 16D 60; Secondary 16S 34; 16U 60.

Keywords and phrases: Nilpotents, Involutions, Fields.

The author is grateful to the expert referee for his/her professional suggestions made. The author is also very obliged to the handling editor, Professor Mohamed Yousif, for his professional management of the article.

Abstract We define the classes of invo-nil and weakly invo-nil unital rings and explore their algebraic structure. Our achieved results somewhat supply and continue the study of fine rings, introduced by Călugăreanu-Lam (J. Algebra Appl., 2016).

1 Introduction and Background

Everywhere in the text of the present article, all our rings R are assumed to be associative, containing the identity element 1, which differs from the zero element 0. Our terminology and notations are mainly in agreement with [12]. For instance, $Nil(R)$ denotes the set of all nilpotent elements in R, and $Nil_2(R)$ is its subset consisting of all nilpotents of order ≤ 2 – thus $0 \in Nil_2(R)$ (cf. [5] and [6]). Likewise, $U(R)$ stands for the unit group of R with a subset $Inv(R)$ consisting of all invertible elements of order ≤ 2 – thus $1 \in Inv(R)$. As usual, $J(R)$ designates the Jacobson radical of R and $Id(R)$ designates the set of all idempotents in R. We shall say that a ring R is *strongly indecomposable*, provided $Id(R) = \{0, 1\}$.

On the one side, as defined in [13] and [1], respectively, a ring R is said to be *clean* if $R = U(R) + Id(R)$ and *weakly clean* if $R = U(R) \pm Id(R)$. In this way, as stated in [7] and [8], respectively, a ring R is said to be *invo-clean* if $R = Inv(R) + Id(R)$ and *weakly invo-clean* if $R = Inv(R) \pm Id(R)$.

Similarly, a ring R is said to be *nil-clean* in [11] if $R = Nil(R) + Id(R)$ and *weakly nil-clean* in [2] if $R = Nil(R) \pm Id(R)$.

The next relationships hold:

$$\{\text{nil-clean}\} \Rightarrow \{\text{weakly nil-clean}\} \Rightarrow \{\text{clean}\} \Rightarrow \{\text{weakly clean}\}$$

On the other side, imitating [3], a ring R is called *fine* if the equality $R \setminus \{0\} = U(R) + Nil(R)$ holds (a significant generalization was done in [4]). However, this class of rings is quite large and so difficult for a structural characterization. That is why, mimicking [5] or [6], a ring R is called *invo-fine* if the equality $R \setminus \{0\} = Inv(R) + Nil(R)$ holds. Contrasting, this class of rings is rather small because they are isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Nevertheless, this result allows us to consider the ring $\mathbb{Z}_4$ which is then **not** invo-fine and hence **not** fine since $U(\mathbb{Z}_4) = Inv(\mathbb{Z}_4)$.

In order to include this ring somewhere, of some interest is to consider those rings whose elements are expressed as special sums of units and nilpotents. So, we are now in a position to state the following new notion in the manner that the next ring class properly encompasses the class of fine rings introduced in [3] (for an other significant generalization of fine rings the interested reader can see also [4]).

Definition 1.1. We call a ring R *unit-nil* if, for each $r \in R$, there exist $u \in U(R)$ and $q \in Nil(R)$ such that $r = u + q$ or $r = u + q + 1$.

Definition 1.2. We call a ring R *weakly unit-nil* if, for each $r \in R$, there exist $u \in U(R)$ and $q \in Nil(R)$ such that $r = u + q$ or $r = u + q + 1$ or $r = u + q - 1$.

Expectingly, these two classes of rings remain extremely big, so that we will restrict our attention to the examination of the following two other expansions of invo-fine rings.

Definition 1.3. We call a ring R *invo-nil* if, for each $r \in R$, there exist $v \in Inv(R)$ and $q \in Nil(R)$ such that $r = v + q$ or $r = v + q + 1$.

Definition 1.4. We call a ring R *weakly invo-nil* if, for each $r \in R$, there exist $v \in Inv(R)$ and $q \in Nil(R)$ such that $r = v + q$ or $r = v + q + 1$ or $r = v + q - 1$.

Certainly, if 2 is a nilpotent, then these two concepts do coincide.

It is also readily seen that $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$ are invo-nil rings, while $\mathbb{Z}_5$, $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are weakly invo-nil but *not* invo-nil. Besides, it is not too hard to check that $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ are *not* weakly invo-nil. In this direction, the element $(1, 2)$ manifestly shows that the direct products $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ need not be weakly invo-nil, too. So, the question of finding a suitable criterion when the direct product of two weakly invo-nil rings is again a weakly invo-nil ring is worthwhile. These examples will be substantiated in the sequel by the utilization of some statement (see, e.g., Lemma 2.1).

Our motivation for writing up the current paper is to try to describe the isomorphic structure of these two ring types, stated in Definitions 1.3 and 1.4. However, we have not done this completely, because of the very intricate situation when the element 2 is nilpotent. Nevertheless some basic affirmations are proved, which allow us to obtain a more close look into the substantial properties of these newly defined rings.

2 Main Results

The next technicality is useful as a starting point of view.

Lemma 2.1. *The following three items are valid:*
(i) Invo-nil rings are strongly indecomposable, whereas weakly invo-nil rings may be not.
(ii) In invo-nil rings either 2 *or* 3 *are nilpotent elements.*
(iii) In weakly invo-nil rings either 2 *or* 3 *or* 15 *are nilpotent elements.*

Proof. Point (i) follows thus: For an arbitrary $e \in Id(R)$ we write $e = v + q$ or $e = v + q + 1$ for some involution v and nilpotent q. Since $v = (-q) + e$ or $-v = q + (1 - e)$, the *lemma on involutions* from ([4], [5], [6], [7]) applies to get that $e = 1$ or $e = 0$, as required. In this direction, plain calculations show that the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3$ is weakly invo-nil.

Point (ii) has an analogous treatment to that of point (iii) below.

Point (iii) follows thus: Writing $3 = v+q$ or $3 = v+q+1$ or $3 = v+q-1$, we subsequently deduce that $3-q = v$ or $2-q = v$ or $4-q = v$. By squaring both sides, we have that $8 \in Nil(R)$ or $3 \in Nil(R)$ or $15 \in Nil(R)$, i.e., $2 \in Nil(R)$ or $3 \in Nil(R)$ or $15 \in Nil(R)$, as stated. ◻

This lemma exhibits some new non-commutative examples and relations as follows: Each strongly indecomposable nil-clean ring is obviously invo-nil – in fact, $R/J(R) \cong \mathbb{Z}_2$ and $J(R)$ is nil. Reversibly, the matrix ring $\mathbb{M}_2(\mathbb{Z}_2)$ and its triangular subring $\mathbb{T}_2(\mathbb{Z}_2)$ are both nil-clean but not invo-nil since they contain non trivial, non central idempotents. Moreover, as already noticed above, $\mathbb{Z}_3 \times \mathbb{Z}_3$ is a weakly invo-nil ring which is not weakly nil-clean (see [10]). However, easy computations demonstrate that the ring $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is not weakly invo-nil. In fact, in this ring the only nilpotent is the zero one. Considering the non-involution element $(-1, 1, 0)$, it follows that $(-1, 1, 0) - (1, 1, 1) = (1, 0, -1)$ and that $(-1, 1, 0) + (1, 1, 1) = (0, -1, 1)$ are not both involutions, so that the direct product in question need not be weakly invo-nil, as asserted.

The following comments could be helpful.

Remark 2.2. In weakly invo-nil rings there exist non-trivial idempotents. In fact, if $3 = 0$, then one checks that $-w - 1$ is always a non-trivial idempotent whenever $w^2 = 1$. Thus the nonstandard presentation $e = -w + 0 - 1$ is fulfilled when $e \in Id(R) \setminus \{0, 1\}$ and $w \in Inv(R) \setminus \{-1, 1\}$.

We have now accumulated all the ingredients necessary to proceed by proving the following central statement.

Theorem 2.3. *Suppose that R is a ring. Then R is invo-nil with* $3 \in Nil(R)$ *if, and only if, $J(R)$ is nil and $R/J(R) \cong \mathbb{Z}_3$. In particular, invo-nil rings having* 3 *as nilpotent are weakly nil-clean.*

Proof. **"Necessity"**. We will show that each element in R is either a nilpotent or a unit. First, we claim that all involutions are the trivial ones -1 and 1. In fact, since $2 \in U(R)$, it follows by a direct check that $\frac{v+1}{2}$ is always an idempotent whenever $v^2 = 1$. Employing now Lemma 2.1 (i), it must be that either $\frac{v+1}{2} = 0$ or $\frac{v+1}{2} = 1$. Therefore, either $v = -1$ or $v = 1$. So, for all $r \in R$, we have four possibilities $r = q \in Nil(R)$, $r = -1 + q \in U(R)$, $r = 1 + q \in U(R)$ and $r = 2 + q \in U(R)$, as promised. Notice that $2 + q \in -1 + Nil(R)$. It is now clear that $J(R)$ has to be nil. Also, it is obvious that the ring R is local, that is, $R/J(R)$ is a division ring. Since this quotient does not have non-trivial nilpotents, it follows at once that it must contain only three elements, as pursued.

"Sufficiency". It follows immediately that $3 \in J(R) \subseteq Nil(R)$ and thus $3 \in Nil(R)$. Letting $r \in R$, we can write $r+J(R) = J(R)$ or $r+J(R) = -1+J(R)$ or $r+J(R) = 1+J(R)$. Hence $r \in J(R)$ or $r \in -1 + J(R)$ or $r \in 1 + J(R)$, as wanted.

By what we have proved so far, the second part is now immediate. □

The nil property of the Jacobson radical can be strengthened by the following observation.

Proposition 2.4. *Let R be a weakly invo-nil ring such that $2 \in U(R)$. Then $J(R)$ is nil.*

Proof. For an arbitrary element $z \in J(R)$, we write $2z = q+v+1$ or $2z = q+v-1$ for some $q \in Nil(R)$ and $v \in Inv(R)$ since one sees that the representation $2z = q + v$ is impossible because $2z - v = q \in Nil(R) \cap U(R) = \emptyset$. Therefore, one writes that $z = \frac{q}{2} + \frac{v+1}{2} \in Nil(R) + Id(R)$ or $z = \frac{q}{2} - \frac{1-v}{2} \in Nil(R) - Id(R)$. Thus writing $z = t + e$ or $z = t - f$ for some $t \in Nil(R)$ and $e, f \in Id(R)$, we deduce that $(z - e)^n = 0$ or $(z + f)^n = 0$ for some $n \in \mathbb{N}$. Consequently, by expanding these two binomials, we get that $e \in J(R)$ or $f \in J(R)$. So, in both cases, $e = 0$ or $f = 0$ giving us that $z = t$, as required. □

In closing our investigation whether or not $J(R)$ is nil in general, one may state a few more comments concerning that theme for invo-nil rings in the case when eventually $2 \in Nil(R)$.

Remark 2.5. We shall now discuss invo-nil rings and will illustrate that even in the case when the index of nilpotence is at most 2 the situation is rather complicated. In fact, for such a ring R, we will prove now that $J(R)$ is nil if, and only if, for each its element z the record $z = v + q + 1$ for some $v \in Inv(R)$ and $q \in Nil(R)$ with $q^2 = 0$ implies that vq is a nilpotent (note that the presentation $z = v + q$ is impossible, because $z - v = q \in U(R) \cap Nil(R) = \emptyset$). To that goal, suppose first that $z \in J(R) \subseteq Nil(R)$ with $z = v + q + 1$. Squaring $z - 1 = v + q$, we get that $vq + qv \in Nil(R)$. Since vq and qv are orthogonal elements, there exists $i \in \mathbb{N}$ such that $(vq)^i + (qv)^i = 0$, i.e., $(vq)^i = -(qv)^i$. The multiplication of both sides by vq on the left allows us to establish that $(vq)^{i+1} = 0$, which means that $vq \in Nil(R)$, thus substantiating our claim. Next, to treat the reverse, assume that for each element $z = v + q + 1$ in $J(R)$ the product vq is a nilpotent. Hence it is straightforward that qv is a nilpotent as well and, consequently, by squaring $z - 1 = v + q$ we deduce that $z^2 - 2z = vq + qv \in Nil(R)$ for all $z \in J(R)$, since $(vq).(qv) = (qv).(vq) = 0$. Replacing subsequently z by $2z$ and by z^2 in the above containment, we derive that $4z(z - 1) \in Nil(R)$, whence $4z \in Nil(R)$ because $z - 1 \in U(R)$ and so $2z^2 - 4z \in Nil(R)$ assures that $2z^2 \in Nil(R)$, as well as that $z^4 - 2z^3 \in Nil(R)$, whence $z^4 \in Nil(R)$ because $2z^3 \in Nil(R)$. We finally conclude that $z \in Nil(R)$, as desired.

Finally, one may observe that the record $z = v+q+1$ forces that z is a sum of two nilpotents whenever 2 is a nilpotent, because $(v + 1)^2 = 2(v + 1) \in Nil(R)$.

As noted above, because of the existence of non-trivial idempotents, the situation when 3 is nilpotent in weakly invo-nil rings seems to be more complicated. Nevertheless the following is true:

Proposition 2.6. *If R is a weakly invo-nil ring with $3 \in Nil(R)$, then $J(R)$ is nil and each element in R is either clean or nil-clean. In particular, R is weakly clean.*

Proof. Since 3 is a nilpotent, it follows that 2 is an invertible, so that $J(R)$ is nil follows at once owing to Proposition 2.4. For any $r \in R$ writing $2r - 3 = v + q$ or $2r - 3 = v + q - 1$ or $2r - 3 = v + q + 1$, where $q \in Nil(R)$, we obtain respectively that

- $r = \frac{v+1}{2} + \frac{q+2}{2} \in Id(R) + U(R)$;

• $r = \frac{v+1}{2} + \frac{q+1}{2} \in Id(R) + U(R)$;

• $r = \frac{v+1}{2} + \frac{q+3}{2} \in Id(R) + Nil(R)$, which can also be interpreted as $r \in -(1 - Id(R)) + (1 + Nil(R)) \subseteq -Id(R) + U(R)$.

The second part-half is now immediate. □

A little more additional information is provided in the next commentary.

Remark 2.7. Similarly to Theorem 2.3 and bearing in mind Remark 2.2, what can be happen is that if R is a weakly invo-nil ring with $3 \in Nil(R)$, then $J(R)$ is nil and $R/J(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. In particular, is is well known that then $R/J(R)$ has to be a clean ring which will enable us by [13] that R is a clean ring, too.

What we can offer now for weakly invo-nil rings in the case when 5 is a nilpotent is the following one:

Proposition 2.8. *Let R be a weakly invo-nil ring in which $5 \in Nil(R)$. Then $J(R)$ is nil and R is clean. In addition, if R is with index of nilpotence ≤ 2, then $R/J(R) \cong \mathbb{Z}_5$.*

Proof. First of all, that $J(R)$ is nil follows from Proposition 2.4.

For any $r \in R$ one writes that $2r - 3 = v + q$ or $2r - 3 = v + q + 1$ or $2r - 3 = v + q - 1$ for some involution v and nilpotent q. Thus $2r = (v + 1) + (q + 2)$ or $2r = (v + 1) + (q + 3)$ or $2r = (v+1)+(q+1)$. Since $1+Nil(R) \subseteq U(R)$, one infers that $6 \in U(R)$, i.e., both $2 \in U(R)$ and $3 \in U(R)$. Finally, one has that $r = \frac{v+1}{2} + \frac{q+2}{2}$ or $r = \frac{v+1}{2} + \frac{q+3}{2}$ or $r = \frac{v+1}{2} + \frac{q+1}{2}$. But one follows that $\frac{v+1}{2} \in Id(R)$ and $\frac{q+2}{2}, \frac{q+3}{2}, \frac{q+1}{2} \in U(R)$, which substantiates our claim.

Assume now that all nilpotents in R are of order at most 2. Since $5 \in J(R)$ and $R/J(R)$ is obviously weakly invo-nil of characteristic 5, we may without loss of generality assume that $5 = 0$ in R. We will show that R is strongly indecomposable. Given $e \in Id(R)$, we write that $e = v + q$ or $e = v + q + 1$ or $e = v + q - 1$ for some $v \in Inv(R)$ and $q \in Nil_2(R)$. In the first two cases, as observed above, the *involution lemma* from ([4], [5], [6], [7]) can be applied to get that either $e = 0$ or $e = 1$. As for the third case, writing $e = v+q-1$, we have $1+e = v+q$ and by squaring both cases we obtain that $3e = vq+qv$. Therefore, $3eq = qvq = 3qe$ and so $eq = qe$ because $3 \in U(R)$ with the inverse 2. That is why, $ev = ve$ and $qv = vq$. But one verifies that $1 + e$ is a unit with the inverse $1 + 2e$, say $(1 + e)(1 + 2e) = 1 = (1 + 2e)(1 + e)$, and so by substituting we deduce that $(v + q)(2v + 2q - 1) = 1$ which is equivalent to $-vq - v - q = -1$, i.e., $-vq = e \in Nil(R) \cap Id(R) = \{0\}$. Finally, $Id(R) = \{0, 1\}$, as asserted.

Since $2 \in U(R)$, as in the proof of Theorem 2.3, one derives that $v = \{-1, 1\}$ and hence all elements of R are of the kind $1 + q$, $-1 + q$, $2 + q$, q, $-2 + q$, so that they are either units or nilpotents. But thereby $R/J(R)$ must be a local rings, that is, $R/J(R)$ has to be a division ring. That is why in $R/J(R)$ there are no non-zero nilpotent elements, whence all elements in this factor-ring are the fifth different elements $\{-2, -1, 0, 1, 2\}$. This observation leads us to $R/J(R)$ is necessarily isomorphic to the five element field, as stated. □

As an immediate consequence, combining Propositions 2.6 and 2.8, we can deduce:

Corollary 2.9. *If R is a weakly invo-nil ring with $15 \in Nil(R)$, then $J(R)$ is nil and R is weakly clean.*

Summarizing all of the above structural statements, one can state the following unifiable result.

Theorem 2.10. *A ring R is weakly invo-nil $\iff$ $R \cong R_1 \times R_2$, where either $R_1 = \{0\}$ or R_1 is an invo-nil ring in which 2 is a nilpotent and either $R_2 = \{0\}$ or R_2 is a weakly invo-nil ring that is a weakly clean ring in which $J(R)$ is nil and 15 is a nilpotent.*

Proof. It follows at once in virtue of a subsequent application of Lemma 2.1 (iii), the Chinese Reminder Theorem and the previous corollary. □

3 Left-Open Problems

We finish off our work with the next two questions of some importance, which treat a common generalization of the rings from Definitions 1.3 and 1.4.

Problem 3.1. Classify *invo-nil-clean* rings, that are rings R for which each element $r \in R$ satisfies the equality $r = v + q + e$ for some $v \in Inv(R)$, $q \in Nil(R)$ and $e \in Id(R)$.

Apparently, invo-clean rings defined as in [7] are invo-nil-clean when $q = 0$. Moreover, invo-fine rings from ([5], [6]) are invo-nil-clean by taking $e = 0$ for $r \neq 0$, and $v = -1$, $q = 0$, $e = 1$ for $r = 0$. Besides, nil-clean rings as stated in [11] are invo-nil-clean; in fact, $r = q - e$ could be written like this $r = (-1) + q + (1 - e)$, as needed.

In a way of similarity, we may state:

Problem 3.2. Classify *weakly invo-nil-clean* rings, that are rings R for which each element $r \in R$ satisfies the equalities $r = v + q + e$ or $r = v + q - e$ for some $v \in Inv(R)$, $q \in Nil(R)$ and $e \in Id(R)$.

Evidently, weakly invo-clean rings defined as in [8] are weakly invo-nil-clean when $q = 0$. Likewise, weakly nil-clean rings as posed in [2] are weakly invo-nil-clean; indeed, $r = q + e$ could be written like this $r = 1 + q - (1 - e)$, as required.

References

[1] M. S. Ahn and D. D. Anderson, *Weakly clean rings and almost clean rings*, Rocky Mount. J. Math. **36** (2006), 783–798.

[2] S. S. Breaz, P. V. Danchev and Y. Zhou, *Rings in which every element is either a sum or a difference of a nilpotent and an idempotent*, J. Algebra Appl. (8) **15** (2016).

[3] G. Călugăreanu and T. Y. Lam, *Fine rings: A new class of simple rings*, J. Algebra Appl. (9) **15** (2016).

[4] P. V. Danchev, *Nil-good unital rings*, Internat. J. Algebra **10** (2016), 239–252.

[5] P. V. Danchev, *Invo-fine unital rings*, Internat. Math. Forum **11** (2016), 841–843.

[6] P. V. Danchev, *The complete characterization of invo-fine rings*, Internat. Math. Forum **12** (2017), 481–484.

[7] P. V. Danchev, *Invo-clean unital rings*, Commun. Korean Math Soc. **32** (2017), 19–27.

[8] P. V. Danchev, *Weakly invo-clean unital rings*, Afrika Mat. (7-8) **28** (2017), 1285–1295.

[9] P. V. Danchev and T. Y. Lam, *Rings with unipotent units*, Publ. Math. (Debrecen) **88** (2016), 449–466.

[10] P. V. Danchev and W. Wm. McGovern, *Commutative weakly nil clean unital rings*, J. Algebra (5) **425** (2015), 410–422.

[11] A. J. Diesl, *Nil clean rings*, J. Algebra **383** (2013), 197–211.

[12] T. Y. Lam, A First Course in Noncommutative Rings, Second Edition, Graduate Texts in Math., Vol. **131**, Springer-Verlag, Berlin-Heidelberg-New York, 2001.

[13] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278.

Author information

Peter V. Danchev, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, BULGARIA.
E-mail: `pvdanchev@yahoo.com; danchev@math.bas.bg`

Received: September 3, 2017.

Accepted: October 17, 2017.