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A NOTE ON FINE WUU RINGS

Peter V. Danchev

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Abstract We prove that fine WUU rings of index of nilpotence not exceeding 3 are isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. This somewhat extends a similar result due to Călugăreanu-Lam (J. Algebra Appl., 2016) which establishes that fine UU rings are always isomorphic to $\mathbb{Z}_2$.

1 Introduction and Fundamentals

Everywhere in the text of the present paper, all our rings R are assumed to be associative, containing the identity element 1, which differs from the zero element 0. Our terminology and notations are mainly in agreement with [4]. For instance, $U(R)$ denotes the group of units in R, $Nil(R)$ denotes the set of nilpotents in R and $Id(R)$ denotes the set of idempotents in R. All other key notions will be explicitly explained below.

Imitating [1], a ring R is said to be *UU* if $U(R) = 1 + Nil(R)$. On the other vein, consulting with [2], a ring R is said to be *fine* if $R \setminus \{0\} = U(R) + Nil(R)$. Interestingly, it was proved in [2] that any fine UU ring contains only two elements, that is, it is isomorphic to $\mathbb{Z}_2$.

Motivated by this statement, the aim of this short note is to generalize the assertion to a more general class of rings as follows: Mimicking [3], we recall that a ring R is called *WUU* if $U(R) = \pm 1 + Nil(R)$. Certainly, each UU ring is itself WUU, but the converse is untrue as the examples $\mathbb{Z}$ and $\mathbb{Z}_3$ show. Specifically, we shall demonstrate that the latter ring arises in our considerations of WUU fine rings. Unfortunately, at this stage, we will restrict our attention only on rings whose nilpotent elements are of exponent at most 3. Nevertheless, our proof presented below somewhat generates a strategy to attack the problem in full generality. The difficulty in proving up the general result is that WUU rings can have an arbitrary characteristic, while the corresponding one for WUU rings must be a power of the element 2.

2 Main Result and Problem

The chief result, which we can currently offer, is the following:

Theorem 2.1. *Suppose that R is a ring having index of nilpotence less than or equal to 3. Then R is a fine WUU ring if, and only if, either $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.*

Proof. "$\Rightarrow$". We claim that in R there are no non-trivial nilpotent elements which will, definitely, enable us with the pursued two isomorphisms. To that goal, we differ two basic cases:

Firstly, let $U(R) = \{\pm 1\}$. Hence, for all $r \neq 0$ it must be that $r \in 1 + Nil(R) \subseteq U(R)$ or $r \in -1 + Nil(R) \subseteq U(R)$. This automatically yields that $R = \{0, 1\} \cong \mathbb{Z}_2$ provided $2 = 0$, or that $R = \{0, 1, -1\} \cong \mathbb{Z}_3$ provided $3 = 0$, thus substantiating our claim.

Secondly, we assume that $U(R) \neq \{\pm 1\}$ and that $2 \neq 0$ as for otherwise WUU rings just coincide with UU rings and we are done utilizing the aforementioned theorem from [2]. We foremost assert that $1 \in U(R) + U(R)$. In fact, there is $1 \neq u \in U(R)$ so that $1 - u \neq 0$ and $1 - u = 1 + q_1 + q_2$ or $1 - u = -1 + q_1 + q_2$ for some $q_1, q_2 \in Nil(R)$. Therefore, $u = -(1+q_1)+(1-q_2)$ or $u = (1-q_1)+(1-q_2)$, whence either $1 = -u^{-1}(1+q_1)+u^{-1}(1-q_2) \in U(R) + U(R)$ or $1 = u^{-1}(1 - q_1) + u^{-1}(1 - q_2) \in U(R) + U(R)$ which implies the desired assertion. Further, using this presentation of the 1, one may write that $1 = 1 + q + w$ or $1 = -1 + q + w$, where $q \in Nil(R)$ and $w \in U(R)$. Seeing that the first possibility is wrong since $q = -w \in U(R) \cap Nil(R) = \emptyset$, we deduce that $2 = q + w$. So, we can write again that $2 = q - 1 + t$ and $2 = q + 1 + t$ for some $t \in Nil(R)$. The latter equality forces that $1 - q = t \in Nil(R) \cap U(R) = \emptyset$ which is manifestly wrong, so that we obtain with the other equality at hand that $3 = q + t$. We thus have that $(3 - q)^k = 0$ for some $k \in \mathbb{N}$ and, expanding by the Newton binomial formula, we derive that $3^k \in Nil(R)$ that amounts to $3 \in Nil(R)$. If now $3 \neq 0$, then writing $3 = v + h$ for some $v \in U(R)$ and $h \in Nil(R)$, we infer that $3 - v = h \in U(R) \cap Nil(R) = \emptyset$ which is false. This obvious contradiction leads to $3 = 0$.

One may also observe that $Id(R) = \{0, 1\}$, that is, in other terms R is *strongly indecomposable*. In fact, for an arbitrary $e \in Id(R)$, it follows that $1 - 2e \in U(R) = \pm 1 + Nil(R)$ giving that $1 - 2e \in 1 + Nil(R)$ or $1 - 2e \in -1 + Nil(R)$. Consequently, $e = -2e \in Nil(R)$ or $1 - e = -2(1 - e) \in Nil(R)$ and hence $e = 0$ or $e = 1$, respectively, as expected. Moreover, if $f \in R$ is such that $f^2 = -f$, then one checks that $(1 + f)^2 = 1 + f$ and thus the previous part is a guarantor that $1 + f = 0$ or $1 + f = 1$. That is why, $f = -1$ or $f = 0$, respectively.

Furthermore, for all $0 \neq r \in R$ it must be that $r = \pm 1 + a + b = \pm 1$, where $a, b \in Nil(R)$. To that purpose, given $c \in Nil(R)$ with $c^2 = 0$, we write that $c - b = \pm 1 + a$ and thus $(c - b)^3 = \pm 1$ taking into account that $3 = 0$. We shall first examine the equation $(c - b)^3 = 1$. It is equivalent to $b^2c - cbc + cb^2 + bcb = 1$. Multiplying both sides by c on the left and by b^2 on the right, we get that $(cb^2)^2 = cb^2$ which assures that cb^2 is an idempotent. However, by what we have already shown above, either $cb^2 = 0$ or $cb^2 = 1$. In the latter situation, multiplying both sides by c on the left, we deduce that $c = c^2b^2 = 0$. In the remaining situation $cb^2 = 0$, multiplying again $b^2c - cbc + cb^2 + bcb = 1$ by c on the left, we derive that $(cb)^2 = c$ and multiplying this by b on the right, it follows that $cb = 0$. But substituting $cb = 0$ in $b^2c - cbc + cb^2 + bcb = 1$ gives that $b^2c = 1$. Multiplying both sides by c on the right, we finally conclude that $c = 0$, as wanted.

As for the the equation $(c - b)^3 = -1$ we can process similarly by applying the same tricks as above to deducing that $(cb^2)^2 = -cb^2$, and hence by what is was shown above it follows that $cb^2 = 0$ or that $cb^2 = -1$. Hereafter, we may proceed by analogy.

Thus $Nil(R) = \{0\}$, indeed, proving the claim after all.

"$\Leftarrow$". It is self-evident, so we omit details. □

We close the work with the following question:

Problem 1. Does it follows that every WUU fine ring (of an arbitrary index of nilpotence) contains only two or three elements or, in other words, is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$?

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Author information

Peter V. Danchev, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, BULGARIA.
E-mail: pvdanchev@yahoo.com; danchev@math.bas.bg

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