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# ON THE GENERALIZED TOTAL GRAPH OF FIELDS AND ITS COMPLEMENT

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**Abstract** Let $R$ be a commutative ring with identity, $Z(R)$ its set of all zero-divisors, and $H$ a nonempty proper multiplicative prime subset of $R$. The generalized total graph $GT_H(R)$ of $R$ is the simple undirected graph with vertex set $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in H$. If we take $R$ as the field $F$ and $H = \{0\}$, we designate the graph as the generalized total graph of the field $F$ and denote the same as $GT(F)$. In this paper, we investigate several graph theoretical properties of the generalized total graph $GT(F)$ and its complement $\overline{GT(F)}$. In particular, we discuss about properties like Eulerian and Hamiltonian for $\overline{GT(F)}$.

## 1 Introduction

Throughout this paper $R$ denotes a commutative ring with identity, $Z(R)$ its set of zero-divisors and $Z^*(R) = Z(R) \setminus \{0\}$. Anderson and Livingston [4] introduced the *zero-divisor graph* of $R$, denoted by $\Gamma(R)$, as the simple undirected graph with vertex set $Z^*(R)$ and two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. Subsequently, Anderson and Badawi [3] introduced the concept of the *total graph* of a commutative ring. The *total graph* $T_\Gamma(R)$ of $R$ is the undirected graph with vertex set $R$ and for distinct $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. Tamizh Chelvam and Asir [6, 13, 14, 15, 16] have extensively studied about the total graph. For a complete detail about total graphs one can refer the survey [7, 12].

Recently, Anderson and Badawi [3] introduced the concept of the generalized total graph of a commutative ring $R$. A nonempty proper subset $H$ of $R$ to be a multiplicative prime subset of $R$ if the following two conditions hold: (i) $ab \in H$ for every $a \in H$ and $b \in R$; (ii) if $ab \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$. For a multiplicative prime subset $H$ of $R$, the *generalized total graph* $GT_H(R)$ of $R$ is the simple undirected graph with vertex set $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in H$. For example, every prime ideal, union of prime ideals and $H = R \setminus U(R)$ are some of the multiplicative-prime subsets of $R$. If $H = Z(R)$, then Total graph and Generalized Total graph are one and the same. The *unit graph* $G(R)$ of $R$ is the simple graph with vertex set $R$ in which two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in U(R)$. One may note that generalized total graph gives the scope to associate graph with even fields and integral domains.

Let $G = (V, E)$ be a graph. We say that $G$ is connected if there is a path between any two distinct vertices of $G$. The complement $\overline{G}$ of the graph $G$ is the simple graph with vertex set $V(G)$ and two distinct vertices $x$ and $y$ are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. For a vertex $v \in V(G)$, $deg(v)$ is the degree of $v$. For any graph $G$, $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of vertices in $G$ respectively. $K_n$ denotes the complete graph of order $n$ and $K_{m,n}$ denotes the complete bipartite graph. For basic definitions in graph theory, we refer the reader to [10]. For the terms in graph theory which are not explicitly mentioned here, one can refer [10], for the terms regarding algebra one can refer [9]. Note that if $R$ is finite, then $\overline{GT_{Z(R)}(R)}$ is the unit graph [8].

A nonempty subset $S$ of $V$ is called a *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. A subset $S$ of $V$ is called a *total dominating set* if every vertex in $V$ is adjacent to some vertex in $S$. A dominating set $S$ is called a *connected (or clique) dominating* set if the subgraph induced by $S$ is connected (or complete). A dominating set $S$ is called an *independent dominating set* if no two vertices of $S$ are adjacent. A dominating set $S$ is called a *perfect dominating set* if every vertex in $V \setminus S$ is adjacent to exactly one vertex in $S$. A dominating set $S$ is called an *efficient dominating set* if $S$ is both independent and perfect dominating set of $G$. A dominating set $S$ is called a *strong (or weak) dominating set*, if for every vertex $u \in V \setminus S$ there is a vertex $v \in S$ with $deg(v) \geq deg(u) (or\ deg(v) \leq deg(u))$ and $u$ is adjacent to $v$. A graph $G$ is called *excellent* if, for every vertex $v \in V(G)$, there is a $\gamma$-set $S$ containing $v$. The *domination number* $\gamma$ of $G$ is defined to be the minimum cardinality of a dominating set in $G$ and the corresponding dominating set is called as a $\gamma$-set of $G$. In a similar way, we define the *total domination number* $\gamma_t$, *connected domination number* $\gamma_c$, *clique domination number* $\gamma_{cl}$, *independent domination number* $\gamma_i$, *perfect domination number* $\gamma_p$, *efficient domination number* $\gamma_{eff}$, *strong domination number* $\gamma_s$ *and the weak domination number* $\gamma_w$. For all these definitions, one can refer Haynes et al., [11].

Throughout this paper $F$ denotes a finite field. In a field $F$, $\{0\}$ is the only prime ideal. When $R$ is the field $F$ and $H = \{0\}$, we designate the graph as the generalized total graph of the field $F$ and denote the same as $GT(F)$. In this paper, we investigate several graph theoretical properties of the generalized total graph $GT(F)$ and its complement $\overline{GT(F)}$. In particular, we investigate the structure of $GT(F)$ and $\overline{GT(F)}$. More specifically, we determine the domination number of $GT(F)$ and $\overline{GT(F)}$. Having determined the domination number, we characterize all gamma sets in $GT(F)$ and $\overline{GT(F)}$.

In Section 2, we study the graph theoretical properties namely clique, chromatic, independence and covering numbers of $GT(F)$, and the various domination parameters of $GT(F)$. In Section 3, we study the graph theoretical properties namely diameter, girth, radius, clique number, chromatic number, Eulerian and Hamiltonian of $\overline{GT(F)}$. In Section 4, we study about the independence and covering numbers of $\overline{GT(F)}$. In Section 5, we study about the various domination parameters of $\overline{GT(F)}$ and further obtain domatic number of $\overline{GT(F)}$.

## 2 Properties of $GT(F)$

In this section, we discuss about some special graph theoretical properties like clique, chromatic, independence, covering numbers and the various domination parameters of $GT(F)$. We make use the following Theorem, which gives the structure for the generalized total graph of a commutative ring.

**Theorem 2.1.** ([3, Theorem 2.2]) *Let $P$ be a prime ideal of a finite commutative ring $R$, and let $|P| = \lambda$ and $|R/P| = \mu$.*

(i) *If $2 \in H$, then $GT_H(R \setminus P)$ is the union of $\mu - 1$ disjoint $K_\lambda$'s;*

(ii) *If $2 \notin H$, then $GT_H(R \setminus P)$ is the union of $\frac{\mu-1}{2}$ disjoint $K_{\lambda,\lambda}$'s.*

Note that $GT(F)$ is the generalized total graph of the field $F$ with the unique multiplicative prime subset $\{0\}$. If $F$ is a field with characteristic 2, then $x + x = 0$ for every $x \in F$. When the characteristic of the field $F$ is greater than 2, for any $0 \neq x \in F$, $x \neq -x$ and $x + (-x) = 0$. In view of these, one can have the following structure for $GT(F)$.

**Lemma 2.2.** *Let $F$ be a finite field. Then*

$$GT(F) = \begin{cases} \underbrace{K_1 \cup \cdots \cup K_1}_{|F|\ copies} & \text{if } char(F) = 2; \\ K_1 \cup \underbrace{K_{1,1} \cup \cdots \cup K_{1,1}}_{\frac{|F|-1}{2}\ copies} & \text{if } char(F) > 2. \end{cases}$$

Recall that, a *clique* in a graph $G$ is a complete subgraph of $G$. The order of the largest clique in a graph $G$ is its *clique number*, which is denoted by $\omega(G)$. An assignment of colors to the vertices of a graph $G$ so that adjacent vertices are assigned different colors is called a *coloring* of $G$. The smallest number of colors in any coloring of a graph $G$ is called the *chromatic number* of $G$ and is denoted by $\chi(G)$.

The following Lemma follows from Lemma 2.2.

**Lemma 2.3.** *Let $F$ be a finite field. Then the following are true:*

(i) $\omega(GT(F)) = \begin{cases} 1 & \textit{if } char(F) = 2; \\ 2 & \textit{if } char(F) > 2. \end{cases}$

(ii) $\chi(GT(F)) = \begin{cases} 1 & \textit{if } char(F) = 2; \\ 2 & \textit{if } char(F) > 2. \end{cases}$

Note that, a set of vertices in a graph is *independent* if no two vertices in the set are adjacent. The *vertex independence number* (or the *independence number*) $\beta(G)$ of a graph $G$ is the maximum cardinality of an independent set of vertices in $G$. A *vertex cover* in a graph $G$ is a set of vertices that covers all the edges of $G$. The minimum number of vertices in a vertex cover of $G$ is the *vertex covering number* $\alpha(G)$ of $G$. The *edge independence number* $\beta_1(G)$ of a graph $G$ is the maximum cardinality of an independent set of edges. The *edge covering number* $\alpha_1(G)$ of a graph $G$ is the minimum cardinality of an edge cover of $G$. A graph $G$ is said to be well-covered if $\gamma_i(G) = \beta(G)$. In the following lemma, we obtain the vertex independence number of the generalized total graph $GT(F)$.

**Lemma 2.4.** *Let $F$ be a finite field. Then*

(i) *The vertex independence number*
$\beta(GT(F)) = \begin{cases} |F| & \textit{if } char(F) = 2; \\ \frac{|F|+1}{2} & \textit{if } char(F) > 2. \end{cases}$

(ii) *If $char(F) > 2$, then the edge independence number, $\beta_1(GT(F)) = \frac{|F|-1}{2}$.*

In the following Lemma, we obtain the vertex covering number of the generalized total graph $GT(F)$.

**Lemma 2.5.** *Let $F$ be a finite field. Then the following are true:*

(i) *The vertex covering number* $\alpha(GT(F)) = \begin{cases} 0 & \textit{if } char(F) = 2; \\ \frac{|F|-1}{2} & \textit{if } char(F) > 2. \end{cases}$

(ii) *The edge covering number, $\alpha_1(GT(F)) = 0$.*

In the following Lemma, we obtain the domination number of the generalized total graph $GT(F)$.

**Lemma 2.6.** *Let $F$ be a finite field. Then the following are true:*

(i) $\gamma(GT(F)) = \begin{cases} |F| & \textit{if } char(F) = 2; \\ \frac{|F|+1}{2} & \textit{if } char(F) > 2. \end{cases}$

(ii) *$GT(F)$ is an excellent graph;*

(iii) $\gamma_i(GT(F)) = \gamma_p(GT(F)) = \gamma_{eff}(GT(F)) = \begin{cases} |F| & \textit{if } char(F) = 2; \\ \frac{|F|+1}{2} & \textit{if } char(F) > 2. \end{cases}$

(iv) *$GT(F)$ is well-covered;*

(v) $\gamma_s(GT(F)) = \gamma_w(GT(F)) = \begin{cases} |F| & \textit{if } char(F) = 2; \\ \frac{|F|+1}{2} & \textit{if } char(F) > 2. \end{cases}$

## 3 Properties of $\overline{GT(F)}$

In this section, we discuss about some graph theoretical properties like diameter, girth, radius, Eulerian and Hamiltonian of $\overline{GT(F)}$. In view of the Lemma 2.2, we have the following structure Lemma for the complement $\overline{GT(F)}$.

**Lemma 3.1.** *Let $F$ be a finite field. Then the following are true:*

(i) *If $char(F) = 2$, then $\overline{GT(F)} = K_{|F|}$;*

(ii) *If $char(F) > 2$, then $\overline{GT(F)}$ is a connected bi-regular graph with $\Delta = |F| - 1$ and $\delta = |F| - 2$.*

In the following results, we discuss about the girth, clique and chromatic numbers of $\overline{GT(F)}$. The length of a smallest cycle in a graph is called as the *girth*. Note that if $G$ contains a cycle, then $gr(G) \leq 2\, diam(G) + 1$. Using the result on diameter, we obtain the girth of $\overline{GT(F)}$.

**Lemma 3.2.** *Let $F$ be a finite field. Then* $gr(\overline{GT(F)}) = \begin{cases} \infty & \text{if } |F| = 2, 3; \\ 3 & \text{if } |F| \geq 5. \end{cases}$

*Proof.* If $|F| = 2$, then $F \cong \mathbb{Z}_2$ and so $\overline{GT(F)} = K_2$. If $|F| = 3$, then $F \cong \mathbb{Z}_3$ and so $\overline{GT(F)} = P_3$. Therefore $gr(\overline{GT(F)}) = \infty$ for both cases. Assume that $|F| \geq 5$. Suppose $char(F) = 2$. By Lemma 3.1(i), $\overline{GT(F)} = K_{|F|}$ and so $gr(\overline{GT(F)}) = 3$. Suppose $char(F) > 2$. Consider the set $S = \{0, x, y\}$ where $x \neq y$ and $x + y \neq 0$. Clearly the subgraph induced by the set $S$ is $C_3$ and so $gr(\overline{GT(F)}) = 3$. □

In the following Lemma, we obtain the clique number of $\overline{GT(F)}$.

**Lemma 3.3.** *Let $F$ be a finite field.*
*Then* $\omega(\overline{GT(F)}) = \begin{cases} |F| & \text{if } char(F) = 2; \\ \frac{|F|+1}{2} & \text{if } char(F) > 2. \end{cases}$

*Proof.* Suppose $char(F) = 2$. Then by Lemma 3.1(i), $\overline{GT(F)} = K_{|F|}$ and so $\omega(\overline{GT(F)}) = |F|$. Suppose $char(F) > 2$. Let $S = \{0, x_1, \ldots, x_{\frac{|F|-1}{2}}\} \subset V(\overline{GT(F)})$, where no two non zero vertices are additive inverses. Then the subgraph induced by $S$ is $K_{\frac{|F|+1}{2}}$ in $\overline{GT(F)}$. Let $T \subseteq V(\overline{GT(F)})$ with $|T| > |S|$. Then there exists two distinct vertices $a, b$ in $T$ such that $a + b = 0$. Since $a, b$ are adjacent in $GT(F)$, $< T >$ is not a complete subgraph in $\overline{GT(F)}$. Therefore $\omega(\overline{GT(F)}) = \frac{|F|+1}{2}$. □

In the following Lemma, we obtain the chromatic number of $\overline{GT(F)}$.

**Lemma 3.4.** *Let $F$ be a finite field.*
*Then* $\chi(\overline{GT(F)}) = \begin{cases} |F| & \text{if } char(F) = 2; \\ \frac{|F|+1}{2} & \text{if } char(F) > 2. \end{cases}$

*Proof.* If $char(F) = 2$, then, by Lemma 3.1(i) $\overline{GT(F)} = K_{|F|}$ and so $\chi(\overline{GT(F)}) = |F|$.

Suppose $char(F) > 2$. Consider the partition $F = \{0\} \bigcup\limits_{i=1}^{\frac{|F|-1}{2}} \{x_i\} \bigcup\limits_{i=1}^{\frac{|F|-1}{2}} \{y_i\}$, where each $x_i$ is the additive inverse of $y_i$ for $1 \leq i \leq \frac{|F|-1}{2}$.

Note that $< \bigcup\limits_{i=1}^{\frac{|F|-1}{2}} \{x_i\} >=< \bigcup\limits_{i=1}^{\frac{|F|-1}{2}} \{y_i\} >= K_{\frac{|F|-1}{2}}$ and $x_i, y_i$ are not adjacent in $\overline{GT(F)}$. Assign a color to $x_i$ and $y_i$. Since 0 is adjacent to every element in $V(\overline{GT(F)}) \setminus \{0\}$, we require $\frac{|F|-1}{2} + 1$ colors for coloring the vertices of $\overline{GT(F)}$. Thus, $\chi(\overline{GT(F)}) = \frac{|F|+1}{2}$. □

Note that, a graph $G$ is said to be weakly perfect if $\chi(G) = \omega(G)$. The following Corollary follows from Lemma 3.3 and Lemma 3.4.

**Corollary 3.5.** *Let $F$ be a finite field. Then $\overline{GT(F)}$ is weakly perfect.*

In the following results, we discuss about some graph theoretical properties of $\overline{GT(F)}$ namely Eulerian and Hamiltonian. Recall that, a *circuit* in a graph $G$ is a closed trail of length 3 or more. Hence a circuit begins and ends at the same vertex but no repeat of edges. A circuit $C$ is called an *Eulerian circuit* if $C$ contains every edge of $G$. A connected graph $G$ is said to be *Eulerian* if it contains an Eulerian circuit. A characterization for a Eulerian graph is recited below.

**Corollary 3.6.** ([10, Theorem 6.1]) *A nontrivial connected graph $G$ is Eulerian if and only if every vertex of $G$ has even degree.*

Using Corollary 3.6, we obtain the following Lemma.

**Lemma 3.7.** *Let $F$ be a finite field. Then $\overline{GT(F)}$ is not Eulerian.*

*Proof.* Suppose $char(F) = 2$. By Lemma 3.1(i), $\overline{GT(F)} = K_{|F|}$. Since $F$ is a finite field and $char(F) = 2$, $|F| = 2^n$ for some $n \in \mathbb{Z}^+$. From this, $deg(v) = 2^n - 1$ is odd for every $v \in V(\overline{GT(F)})$ and so $\overline{GT(F)}$ is not Eulerian. When $char(F) > 2$, proof follows from Lemma 3.1(ii). □

The following is a known characterization for Hamiltonian graphs and the same given below for ready reference.

**Corollary 3.8.** ([10, Corollary 6.7]) *Let $G$ be a graph of order $n \geq 3$. If $deg(v) \geq \frac{n}{2}$ for each vertex of $G$, then $G$ is Hamiltonian.*

**Lemma 3.9.** *Let $F$ be a finite field and $|F| > 3$. Then $\overline{GT(F)}$ is Hamiltonian.*

*Proof.* Since $|F| > 3$, we have $|F| - 2 \geq \left\lfloor \frac{|F|}{2} \right\rfloor$.
If $char(F) = 2$, then by Lemma 3.1(i), $\delta = |F| - 1$. Now the proof follows from the Corollary 3.8.
If $char(F) > 2$, then by Lemma 3.1(ii), $\delta = |F| - 2$. Once again the proof follows from Corollary 3.8. □

The following theorems are cited to obtain the vertex covering number and edge covering number of $\overline{GT(F)}$.

**Theorem 3.10.** ([10, Theorem 8.8]) *For every graph $G$ of order $n$ containing no isolated vertices, $\alpha(G) + \beta(G) = n$.*

**Theorem 3.11.** ([10, Theorem 8.7]) *For every graph $G$ of order $n$ containing no isolated vertices, $\alpha_1(G) + \beta_1(G) = n$.*

Now let us obtain the vertex independence number $\beta$, vertex covering number $\alpha$, edge independence number $\beta_1$ and edge covering number $\alpha_1$ of $\overline{GT(F)}$.

**Lemma 3.12.** *Let $F$ be a finite field. Then* $\beta(\overline{GT(F)}) = \begin{cases} 1 & \text{if } char(F) = 2; \\ 2 & \text{if } char(F) > 2. \end{cases}$

*Proof.* Suppose $char(F) = 2$. Then $\overline{GT(F)} = K_{|F|}$. Hence $\beta(\overline{GT(F)}) = 1$.
Let $char(F) > 2$. Suppose $\beta(\overline{GT(F)}) \geq 3$. This gives that there exists a complete subgraph of order $\geq 3$ in $GT(F) = K_1 \bigcup\limits_{\frac{|F|-1}{2}} K_2$, which is a contradiction. Hence $\beta(\overline{GT(F)}) \leq 2$. For any $v \in V(\overline{GT(F)})$, $v$ and its additive inverse are only adjacent in $GT(F)$. Therefore $\beta(\overline{GT(F)}) = 2$. □

Using Theorem 3.10, we obtain the following Corollary.

**Corollary 3.13.** *Let $F$ be a finite field. Then*

$$\alpha(\overline{GT(F)}) = \begin{cases} |F| - 1 & \text{if } char(F) = 2; \\ |F| - 2 & \text{if } char(F) > 2. \end{cases}$$

**Lemma 3.14.** *Let $F$ be a finite field. Then the edge independence number* $\beta_1(\overline{GT(F)}) = \left\lfloor \frac{|F|}{2} \right\rfloor$.

*Proof.* Suppose $char(F) = 2$. By Lemma 3.1(i), $\overline{GT(F)} = K_{|F|}$ and so $\beta_1(\overline{GT(F)}) = \left\lfloor \frac{|F|}{2} \right\rfloor$. Suppose $char(F) > 2$. If $F \cong \mathbb{Z}_3$, then $\overline{GT(F)} = P_3$ and so $\beta_1(\overline{GT(F)}) = 1$. Assume that $|F| \geq 5$. List the elements of $F$ as $F = \{0, x_1, \cdots, x_{\frac{|F|-1}{2}}, y_1, \cdots, y_{\frac{|F|-1}{2}}\}$ where each $x_i$ is the additive inverse of $y_i$. Let $E = \{x_{\frac{|F|-1}{2}}\ y_1\} \cup \{x_i\ y_{i+1} : i \in \{1, 2, \ldots, \frac{|F|-3}{2}\}\}$. Then $E$ is a maximal edge independent set of order $\frac{|F|-1}{2}$ in $\overline{GT(F)}$. Therefore $\beta_1(\overline{GT(F)}) = \frac{|F|-1}{2} = \left\lfloor \frac{|F|}{2} \right\rfloor$. □

Using Theorem 3.11, we obtain the following Corollary.

**Corollary 3.15.** *Let $F$ be a finite field. Then the edge covering number* $\alpha_1(\overline{GT(F)}) = |F| - \left\lfloor \frac{|F|}{2} \right\rfloor$.

## 4 Domination Parameters of $\overline{GT(F)}$

In the following results, we discuss about various domination parameters of $\overline{GT(F)}$. More specifically, we discuss about $\gamma_t, \gamma_c, \gamma_{cl}, \gamma_p, \gamma_{eff}, \gamma_s, \gamma_w$ and independence domination number of $\overline{GT(F)}$. In the following Lemma, we obtain the domination number of $\overline{GT(F)}$.

**Lemma 4.1.** *Let $F$ be a finite field. Then $\gamma(\overline{GT(F)}) = 1$.*

*Proof.* Assume that $F$ is a finite field. By Lemma 3.1(i) and (ii), $\overline{GT(F)}$ contains a vertex of degree $|F| - 1$ and so $\gamma(\overline{GT(F)}) = 1$. □

Using Lemma 4.1, we have the following characterization of $\gamma$-sets in $\overline{GT(F)}$.

**Lemma 4.2.** *Let $F$ be a finite field.Then the following hold:*

(i) *The set $S = \{v\},\ v \in V(\overline{GT(F)})$ is a $\gamma$-set in $\overline{GT(F)}$ if and only if $char(F) = 2$.*

(ii) *The set $S = \{0\}$, is the $\gamma$-set in $\overline{GT(F)}$ if and only if $char(F) > 2$.*

Recall that when $char(F) = 2$, $\overline{GT(F)} = K_{|F|}$. Using this along with Lemma 4.2, we have the following result.

**Lemma 4.3.** *Let $F$ be a finite field. Then $\overline{GT(F)}$ is excellent if and only if $char(F) = 2$.*
*Proof.* Assume that $char(F) = 2$, $\overline{GT(F)} = K_{|F|}$ and hence it is excellent. Conversely suppose $\overline{GT(F)}$ is excellent for $char(F) > 2$. By Lemma 4.1, $\gamma(\overline{GT(F)}) = 1$. Let $v \in V(\overline{GT(F)}) \setminus \{0\}$. By Lemma 4.2(ii), there is no $\gamma$-set containing $v$ in $\overline{GT(F)}$, which is a contradiction. □

**Lemma 4.4.** *Let $F$ be a finite field. Then the following are true:*

(i) $\gamma_p(\overline{GT(F)}) = \gamma_i(\overline{GT(F)}) = 1$.

(ii) *If $char(F) = 2$, then $\gamma_s(\overline{GT(F)}) = \gamma_w(\overline{GT(F)}) = 1$;*

(iii) *If $char(F) > 2$, then $\gamma_s(\overline{GT(F)}) = 1$ and $\gamma_t(\overline{GT(F)}) = \gamma_c(\overline{GT(F)}) = \gamma_{cl}(\overline{GT(F)}) = \gamma_w(\overline{GT(F)}) = 2$.*

*Proof.* (i) is trivial.
(ii) If $char(F) = 2$, then $\overline{GT(F)} = K_{|F|}$ and so $\gamma_s(\overline{GT(F)}) = \gamma_w(\overline{GT(F)}) = 1$.
(iii) Suppose $char(F) > 2$. By Lemma 4.2(ii), $S = \{0\}$ is the $\gamma$-set in $\overline{GT(F)}$.

In $\overline{GT(F)}$, we have $deg(v) = \begin{cases} |F| - 1 & \text{if } v = 0; \\ |F| - 2 & \text{if } v \neq 0. \end{cases}$

Therefore $\gamma_s(\overline{GT(F)}) = 1$.

Consider the set $S = \{x, y\} \subset V(\overline{GT(F)}) \setminus \{0\}$ where $x + y \neq 0$. Let $z \in V(\overline{GT(F)}) \setminus S$. If $x+z = 0$, then $y, z$ are adjacent in $\overline{GT(F)}$. If $x+z \neq 0$, then $x, z$ are adjacent in $\overline{GT(F)}$. Hence $S$ is a dominating set in $\overline{GT(F)}$. Note that $x$ and $y$ are adjacent in $\overline{GT(F)}$. Hence $\gamma_t(\overline{GT(F)}) = \gamma_c(\overline{GT(F)}) = \gamma_{cl}(\overline{GT(F)}) = \gamma_w(\overline{GT(F)}) = 2$. □

**Corollary 4.5.** *Let $F$ be a finite field . Then $\gamma_{eff}(\overline{GT(F)}) = 1$.*

*Proof.* Suppose $char(F) = 2$. By Lemma 4.2(i),The set $S = \{v\}$, $v \in V(\overline{GT(F)})$ is a $\gamma$-set in $\overline{GT(F)}$. Suppose $char(F) > 2$. By Lemma 4.2(ii), $S = \{0\}$ is the $\gamma$-set in $\overline{GT(F)}$. In both cases, clearly $S$ is both independent and perfect dominating set and so $\gamma_{eff}(\overline{GT(F)}) = 1$. □

**Lemma 4.6.** *Let $F$ be a finite field.Then $\overline{GT(F)}$ is well-covered if and only if $char(F) = 2$.*

*Proof.* Proof of (i) follows from Lemma 3.12 and Lemma 4.4(i).

□

**Lemma 4.7.** *Let $F$ be a finite field. Then*

$$d(\overline{GT(F)}) = \begin{cases} F & \text{if } char(F) = 2; \\ \frac{|F|+1}{2} & \text{if } char(F) > 2. \end{cases}$$

*Proof.* If $char(F) = 2$, then $\overline{GT(F)} = K_{|F|}$ and hence $d(\overline{GT(F)}) = |F|$.

Suppose $char(F) > 2$. Consider the partition $F = \{0\} \bigcup_{i=1}^{\frac{|F|-1}{2}} \{x_i\} \bigcup_{i=1}^{\frac{|F|-1}{2}} \{y_i\}$, where each $x_i$ is the additive inverse of $y_i$ for $1 \leq i \leq \frac{|F|-1}{2}$. Let $S_i = \{x_i, y_i\} \subseteq V(\overline{GT(F)})$ for every $i \in \{1, 2, \ldots, \frac{|F|-1}{2}\}$. Clearly, each $S_i$ is a dominating set in $\overline{GT(F)}$. Hence $V(\overline{GT(F)}) = \{0\} \bigcup_{i=1}^{\frac{|F|-1}{2}} S_i$ is a maximal domatic partition of $\overline{GT(F)}$. This gives that $d(\overline{GT(F)}) = \frac{|F|+1}{2}$. □

A graph G is called *domatically full* if $d(G) = \delta(G) + 1$, which is the maximum possible order of a domatic partition of $V$.

**Lemma 4.8.** *Let $F$ be a finite field. Then the following are true:*

(i) *If $char(F) = 2$, then $\overline{GT(F)}$ is domatically full.*

(ii) *If $char(F) > 2$, then $\overline{GT(F)}$ is domatically full if and only if $|F| = 3$.*

*Proof.* (i)Proof follows from Lemma 3.1(i) and Lemma 4.7.
(ii) Proof for if part follows from Lemma 3.1(ii) and Lemma 4.7.
Conversely assume that $\overline{GT(F)}$ is domatically full. Suppose $char(F) > 2$. Then, by Lemma 3.1(ii), $\delta(\overline{GT(F)}) = |F| - 2$ and again by Lemma 4.7, $d(\overline{GT(F)}) = \frac{|F|+1}{2}$. By the assumption, $|F| - 2 + 1 = \frac{|F|+1}{2}$, which in turn implies that $|F| = 3$. □

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