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A NOTE ON LIE CENTRALIZER MAPS

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Abstract. In this sequel, some conditions are provided under which a Lie centralizer maps on a trivial extension algebra can be decomposed into the sum of a centralizer and a center valued map.Our results are examined for the triangular algebras.

1 Introduction

Let A be an unital algebra and M be an A-module. A linear mapping D from A into itself is said to be a centralizer if

$$D(ab) = D(a)b \text{ and } D(ab) = aD(b)(a, b \in A)$$

A linear mapping $L : A \to A$ is called a Lie centralizer if

$$L[a, b] = [L(a), b](a, b \in A),$$

where $[.,.]$ stands for the Lie bracket. Since L is linear, then $L[a, b] = [a, L(b)]$. If $D : A \to A$ is a centralizer and $l : A \to Z(A)$ (:=the center of A) is a linear map then $D + l$ is a Lie centralizer if and only if $l([a, b]) = 0$, for all $a, b \in A$. A problem that we are dealing with is standing those conditions on an algebra such that every Lie centralizer on it can be decomposed into the centralizer and a canter valued map.The similarity concept is studied in [2],[3],[4],[5], but Jing [1] was the first one who introduced Lie centralizer and showed that every Lie centralizer on some triangular algebras is sum a centralizer map and a center valued map. In this paper we deal with the structure of Lie centralizers on trivial extension algebra. The direct product $A \times M$ together with the pairwise addition, scalar product and the algebra product defined by

$$(a, m).(b, n) = (ab, an + mb)(a, b \in A, m, n \in M),$$

is an unital algebra which is called the trivial extension of A by M and will be denoted by $A \ltimes M$. For example,every triangular algebra is a trivial extension algebra.Indeed the triangular algebra $Tri(A, M, B)$ can be identified with the trivial extension algebra $(A \oplus B) \ltimes M$; in which M as an $(A \oplus B)$-module is equipped with the module operations

$$(a, b)m = am \text{ and } m(a, b) = mb, (a \in A, b \in B, m \in M).$$

Trivial extensions are also known as a fertile source of (counter-) examples in various situations in functional analysis. It should be remarked that in functional analysis literature this algebra was termed "module extension". If A is a Banach algebra and M is a Banach A-module then the trivial extension algebra $A \ltimes M$ equipped with the l^1-norm, is a Banach algebra with many interesting properties. Some aspects of the Banach algebra $A \ltimes M$ have been described in [6].
The main goal this paper is providing some conditions under which Lie centralizer on trivial extension algebra can be decomposed into the sum centralizer and center valued map. We are mainly dealing with those $A \ltimes M$ for which A enjoys a nontrivial idempotent p satisfying $pmq = m$,for all $m \in M$ where $q = 1 - p$. The triangular algebra $(A \oplus B) \ltimes M$ is the main example of a trivial extension algebra satisfying the previous identity.

2 Main results

The purpose of this section is whether every Lie centralizer map L on the trivial extension algebra $A \ltimes M$ into $A \ltimes M$ can be expressed as $D+l$, where D is centralizer map and $l : A \ltimes M \to Z(A \ltimes M)$ is a linear map. In the first, we describe the structures of centralizers and Lie centralizers on trivial extension algebra $A \ltimes M$.

Lemma 2.1. *Let A be a unital algebra and M be an A-module. Then every linear map $L : A \ltimes M \to A \ltimes M$ can be presented as*

$$L(a, m) = (L_A(a) + T(m), L_M(a) + S(m)), (a \in A, m \in M)$$

for some linear mappings $L_A : A \to A$,$L_M : A \to M$,$T : M \to A$ and $S : M \to M$. Moreover, L is a Lie centralizer if and only if L_A is Lie centralizer and $L_M[a, b] = [L_M(a), b]$ for all $a, b \in A$ and

(i) $[T(m), n] = 0$,$T[a, m] = 0$ *and* $T[m, a] = [T(m), a]$ *for all* $m, n \in M, a \in A$.

(ii) $S[a, m] = [L_A(a), m]$ *and* $S[m, a] = [S(m), a]$ *for all* $a \in A, m \in M$.

Furthermore, L is centralizer if and only if L_A is centralizer and $L_M(ab) = L_M(a)b = aL_M(b)$ and

(1) $mT(n) = 0 = T(m)n$ *and* $T(am) = 0 = aT(m)$ *and* $T(ma) = 0 = T(m)a$ *for all* $a \in A$,, $m, n \in M$.

(2) $S(am) = aS(m) = L_A(a)m$ *and* $S(ma) = S(m)a = mL_A(a)$ *for all* $a \in A$, $m \in M$.

Proof. Let $a, b \in A$ and $m, n \in M$. Putting $u_1 = (a, 0)$, $u_2 = (b, 0)$ and $k_1 = (0, m)$, $k_2 = (0, n)$. We have $L[u_1, u_2] = (L_A[a, b], L_M[a, b])$ and $[L(u_1)u_2] = ([L_A(a), b], [L_M(a), b])$. Also we have $L[k_1, k_2] = (0, 0)$, $[L(k_1), k_2] = (0, [T(m), n])$ and $L[u, k_1] = (T[a, m], S[a, m])$, $[L(u_1), k_1] = (0, [L_A(a), m])$ and $L[k_1, u_1] = (T[m, a], S[m, a])$, $[L(k_1), u_1] = ([T(m), a], [S(m), a])$. For the converse, let $u = (a, m)$, $v = (b, n)$ for $a, b \in A$ and $m, n \in M$. We have $L[u, v] = L([a, b], [m, b] + [a, n]) = (L_A[a, b] + T[m, b] + T[a, n], L_M[a, b] + S[m, b] + S[a, n])$ on the other hand, we have $[L(u), v] = ([L_A(a), b]+[T(m), b], L_A(a)n+T(m)n+L_M(a)b+S(m)b-bL_M(a) - bS(m) - nL_A(a) - nT(m))$, so by the hypothesis L is Lie centralizer map. With the similar method the rest of the proof is easy.

□

It can be simply verified that the center $Z(A \ltimes M)$ of $A \ltimes M$ is

$$Z(A \ltimes M) = \{(a, m); a \in Z(A), [m, b] = 0 = [n, a] forall (b, n) \in (A \ltimes M)\} = \pi_A(Z(A \ltimes M)) \times \pi_M(Z(A \ltimes M).$$

where $\pi_A : A \ltimes M \to A$ and $\pi_M : A \ltimes M \to M$ are the natural projections given by $\pi_A(a, m) = a$ and $\pi_M(a, m) = m$,respectively.

The next result is the main aim in this paper .

Theorem 2.2. *Let A be a 2-torsion free unital algebra with a nontrivial idempotent p and let M be a 2-torsion free A-module.If $pmq = m$ for all $m \in M$ then the Lie centralizer*

$$L(a, m) = (L_A(a) + T(m), L_M(a) + S(m))$$

on $A \ltimes M$ can be expressed as the sum of centralizer map and center valued map if and only if there exists a linear map $l_A : A \to Z(A)$ such that $L_A - l_A$is a centralizer on A and $[l_A(pap), m] = 0 = [l_A(qaq), m]$ for all $m \in M$, $a \in A$.

Proof. By lemma 2.1 every Lie centralizer on $A \ltimes M$ can be expressed in the form

$$L(a, m) = (L_A(a) + T(m), L_M(a) + S(m)),$$

where $L_A : A \to A$ is Lie centralizer and $L_M : A \to M$, $T : M \to A$ and $S : M \to M$ are linear map satisfying

$$L_M[a, b] = [L_M(a), b],[T(m), n] = 0, T[a, m] = 0,T[m, a] = [T(m), a],S[a, m] = [L_A(a), m] \text{ and } S[m, a] = [S(m), a]$$

for all $m, n \in M$, $a \in A$. To prove the if part,set $D(a,m) = ((L_A - l_A)(a) + T(m), L_A(a) + S(m)$ and $l(a,m) = (l_A(a), 0)$, $(a \in A, m \in M)$.Trivially $L = D + l$. That l satisfies the required properties follows from the properties of l_A.It is enough to show that $L_M(a,b) = L_M(a)b = aL_M(b)$ for all $a, b \in A$ and T, S satisfy the conditions (1), (2) of lemma 2.1. We have $T(m) = T[p,m] = 0$ for all $m \in M$. Thus,

$$T(am) = T(ma) = aT(m) = T(m)a = 0, mT(n) = 0 = T(m)n(a \in A, m \in M).$$

On the other hand, one may have $S(am) = [(L_A - l_A)(pap), m] = (L_A - l_A)(a)m$ and $S(ma) = [S(pm), qaq] = S(m)a$ and since $S[m,a] = [S(m), a]$, then we have $S(am) = aS(m)$.Also we see that $S(ma) = [pm, L_A(qaq)] = m(L_A - l_A)(a)$. Also we have $L_M(qap) = [L_M(qap, p)] = -L_M(qap)$ and since M is 2-torsion free then $L_M(qap) = 0.0 = L_M[pap, qaq] = L_M(pap)qaq$, then $L_M(pap)qaq = 0$, replace a with $pap + q$ in this relation we get

$$L_M(pap)q = 0 \tag{2.1}$$

Now set $\phi(a) = L_M(paq)$,clearly for all $a, b \in A$, one may have

$$\phi(ab) = L_M(pabq) = L_M[pa, pbq] + L_M[paq, bq] = \phi(a)b.$$

and the other hand, we have

$$\phi(ab) = L_M(pabq) = L_M[pap, bq] + L_M[paq, bq] = a\phi(b).$$

Since $0 = L_M[pap, qaq]$, then we have $papL_M(qaq) = 0$. Replace a with $qaq + p$ in this relation, we see that

$$pL_M(qaq) = 0. \tag{2.2}$$

Thus with relations (2.1), (2.2),we have $L_M(a) = \phi(a)$.

For the converse, suppose that $L = D + l$,where D is centralizer and l is a center valued linear map on $A \ltimes M$. Then there exists some suitable maps $D_A, D_M, l_A, l_M, T', S'$ such that for $a \in A, m \in M$,

$$(L_A(a) + T(m)), L_M(a) + S(m)) =$$
$$(D_A(a) + (T - T')(m), D_M(a) + (S - S)'(m)) + (l_A(a) + T'(M), l_M(a) + S'(m).$$

By lemma 2.1, $T = T'$ and since $T = 0$, therefore $T' = 0$.Since $L_A - l_A = D_A$ and l is center valued,this completes the proof. □

Now,as a consequence of Theorem 2.2, we obtain next result for the Lie centralizer maps on the triangular algebras.

Corollary 2.3. *Let that A, B be 2-torsion free unital algebras and M be a 2-torsion free (A, B)-module.Every the Lie centralizer.*

$$L((a,b), m) = (L_{A\oplus B}(a,b) + T(m), l_M(a,b) + S(m))$$

on $Tri(A, M, B)$ can be expressed as the sum of centralizer map and center valued map if and only if there exists a linear map $l_{A\oplus B} : A \oplus B \to Z(A \oplus B)$ such that $L_{A\oplus B} - l_{A\oplus B}$ is a centralizer on $A \oplus B$ and $[l_{A\oplus B}(a, 0), m] = 0 = [l_{A\oplus B}(0, b), m]$ for all $m \in M, a \in A, b \in B$.

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