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UNIFORMLY BOUNDED COMPOSITION OPERATORS IN THE SPACES OF FUNCTIONS OF TWO VARIABLES OF BOUNDED φ-VARIATION IN THE SENSE OF RIESZ

Wadie Aziz and José A. Guerrero

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Abstract In this paper we prove that if a uniformly bounded the nonlinear composition operator maps a subset of the space of functions of bounded total φ-bidimensional variation with weight function in the sense of Riesz, into another space of that type (with the same weight function), then the generator function of the operator is an affine function in the third variable. This extends previous results (see [1, 5, 9, 10]) in the one and two-dimensional setting.

1 Introduction

Let $a = (a_1, a_2)$, $b = (b_1, b_2)$ points in $\mathbb{R}^2$ such that $a_i < b_i$, $i = 1, 2$. In the sequel, we use the symbol I_a^b to denote the basic rectangle $[a_1, b_1] \times [a_2, b_2]$, $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are real normed spaces and C is closed and convex set in X. We also denote by $X^{I_a^b}$ the algebra of all functions $f : I_a^b \longrightarrow X$, and by $\mathcal{F}$ the set of all non-decreasing continuous functions $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that

(i) $\varphi(t) = 0$ if and only if $t = 0$, and

(ii) $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

If, in addition, $\varphi \in \mathcal{F}$ is a convex map, we say that $\varphi \in \mathcal{N}$ (or that φ is an $\mathcal{N}$-function).

Given a function $h : I_a^b \times X \longrightarrow Y$, the nonlinear composition (Nemytskii or Superposition, cf. [2, 4]) operator generated by the function h,

$$H : X^{I_a^b} \longrightarrow Y^{I_a^b},$$

is defined as

$$(Hf)(t, s) := h(t, s, f(t, s)), \qquad (t, s) \in I_a^b.$$

According to a well–known result of Krasnosel'skij, H is a self–map of the set of real continuous functions into X if and only if its generator h is continuous. In this situation it is rather unexpected that there are discontinuous function $h : I_a^b \times \mathbb{R} \longrightarrow \mathbb{R}$ generating composition operators H which map the space of continuously differentiable functions $C^1(I_a^b, \mathbb{R})$ into itself (cf. [2, page 209]). Another interesting (astonishing) property of this nonlinear operator H was introduced by Matkowski in [8] which it say: if H is a Lipschitzian self–map of the Banach space $Lip(I, \mathbb{R})$, then

$$h(t, s) = A(t)s + B(t), \qquad (t \in I,\ s \in \mathbb{R}), \tag{1.1}$$

for some Lipschitz functions A and B, this mean, the generator h of H is affine in the second variable. This result has been extended to some other function Banach spaces (cf. [2]). But [4, 5] extended the results for the space of functions of bounded total φ–bidimensional variation in the sense of Riesz.

In [3] it has been demonstrated that if H maps the space $RV_{\varphi,\alpha}(I, C)$ of functions of bounded φ–variation with weight α in the sense of Riesz into the space $RV_{\psi,\alpha}(I, Y)$ and is uniformly continuous, then h, the generator function of the operator H, is affine in the second variable.

In [11] it is proved that any uniformly bounded composition operator acting between general Lipschitz function normed spaces must be of form (1.1).

For $\varphi \in \mathcal{F}$, let $(RV_\varphi(I_a^b, X), \|\cdot\|_\varphi)$ be the Banach space of all functions $f \in X^{I_a^b}$ which are of bounded total φ-bidimensional variation in the sense of Riesz, (see next section and [4]).

As usual $\mathcal{L}(X, Y)$ denotes the space of all continuous linear operators from a normed space X into a normed space Y.

The main result of this paper says that, under a weak regularity condition, the generator of every uniformly bounded-or equidistantly uniformly bounded-composition operator H maps the set of functions $f \in RV_{\varphi,\alpha}(I_a^b, X)$ such that $f(I_a^b) \subset C$ φ-bidimensional variation with weight α in the sense of Riesz into the space $RV_{\psi,\alpha}(I_a^b, Y)$ is an affine function with respect to the third variable.

2 Preliminaries

In this section we introduce useful notation and definitions and recall some results concerning the Riesz φ–bidimensional variation with weight.

Let $\xi = \{t_i\}_{i=0}^m$ and $\eta = \{s_j\}_{j=0}^n$ be partitions of two intervals $[a_1, b_1] \subset \mathbb{R}$ and $[a_2, b_2] \subset \mathbb{R}$, respectively; i.e., $m, n \in \mathbb{N}$,

$$a_1 = t_0 < t_1 < \cdots < t_m = b_1 \qquad \text{and}$$
$$a_2 = s_0 < s_1 < \cdots < s_n = b_2.$$

For $\varphi \in \mathcal{N}$ continuous strictly increasing function $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, and for each function $f \in X^{I_a^b}$, let us introduce the following notation:

$$\Delta\alpha(\ell_k) := \alpha(\ell_k) - \alpha(\ell_{k-1})$$

and

$$\begin{aligned}
\Delta_{10} f(t_i, s_j) &:= f(t_i, s_j) - f(t_{i-1}, s_j)\\
\Delta_{01} f(t_i, s_j) &:= f(t_i, s_j) - f(t_i, s_{j-1})\\
\Delta_{11} f(t_i, s_j) &:= f(t_{i-1}, s_{j-1}) - f(t_{i-1}, s_j) - f(t_i, s_{j-1}) + f(t_i, s_j).
\end{aligned}$$

Definition 2.1. ([4, 5]) Let $\varphi \in \mathcal{F}$, X be a real normed space and $f \in X^{I_a^b}$ and a continuous strictly increasing function $\alpha : I \longrightarrow \mathbb{R}$, we define:

(a) Let $x_2 \in [a_2, b_2]$ be fixed. Consider the function $f(\cdot, x_2) : [a_1, b_1] \times \{x_2\} \longrightarrow \mathbb{R}$ defined as

$$f(\cdot, x_2)(t) := f(t, x_2), \qquad t \in [a_1, b_1].$$

Then the (one-dimensional) φ-variation with weight in the sense of Riesz (see [3, 14]) of the function $f(\cdot, x_2)$, on an subinterval $[x_1, y_1] \subseteq [a_1, b_1]$, is the quantity

$$V_{\varphi,\alpha,[x_1, y_1]}(f(\cdot, x_2)) := \sup_{\Pi_1} \sum_{i=1}^m \varphi\left[\frac{|\Delta_{10} f(t_i, x_2)|}{|\Delta\alpha(t_i)|}\right] |\Delta\alpha(t_i)|,$$

where the supremum is taken over all partitions $\Pi_1 = \{t_i\}_{i=0}^m$ ($m \in \mathbb{N}$) of the interval $[x_1, y_1]$.

(b) A similar argument applies for the variation $V_{\varphi,\alpha,[x_2,y_2]}$, where $x_1 \in [a_1, b_1]$ is fixed and $[x_2, y_2]$ is a subinterval of $[a_2, b_2]$. That is, for the function $f(x_1, \cdot) : \{x_1\} \times [a_2, b_2] \longrightarrow \mathbb{R}$ the φ-variation with weight in the sense Riesz, is the quantity

$$V_{\varphi,\alpha,[x_2, y_2]}(f(x_1, \cdot)) := \sup_{\Pi_2} \sum_{j=1}^n \varphi\left[\frac{|\Delta_{01} f(x_1, s_j)|}{|\Delta\alpha(s_j)|}\right] |\Delta\alpha(s_j)|,$$

where the supremum is taken over the set of all partitions $\Pi_2 = \{s_j\}_{j=0}^n$ ($n \in \mathbb{N}$) of the interval $[x_2, y_2]$.

(c) The φ-bidimensional variation with weight in the sense of Riesz is defined by the formula

$$V_{\varphi,\alpha}(f) := \sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi\left[\frac{|\Delta_{11} f(t_i, s_j)|}{|\Delta\alpha(t_i)||\Delta\alpha(s_j)|}\right] \cdot |\Delta\alpha(t_i)||\Delta\alpha(s_j)|,$$

where the supremum is taken over the set of all partitions (Π_1, Π_2) of the rectangle $I_a^b \subset \mathbb{R}^2$.

(d) The total φ-bidimensional variation with weight α in the sense of Riesz of the function $f : I_a^b \longrightarrow \mathbb{R}$ is denoted by $TV^R_{\varphi,\alpha}(f)$ and is defined as:

$$TV^R_{\varphi,\alpha}(f) := V_{\varphi,\alpha,[a_1,b_1]}(f(\cdot, a_2)) + V_{\varphi,\alpha,[a_2,b_2]}(f(a_1, \cdot)) + V_{\varphi,\alpha}(f).$$

(e) We say that $f \in X^{I_a^b}$ has a bounded Riesz φ-variation with weight α on I_a^b, if $TV^R_{\varphi,\alpha}(f) < \infty$.

Definition 2.2. Let $\varphi \in \mathcal{F}$. We say that φ satisfies condition ∞_1 if

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

A function $\varphi \in \mathcal{F}$ is said to be in the Δ_2 class, if there exist a constant $t_0 \geq 0$ and $K > 0$ such that

$$\varphi(2t) \leq K\,\varphi(t) \quad \text{for all} \quad t > t_0.$$

Remark 2.3. It is easy to show that if $\varphi \in \mathcal{N}$ satisfies condition ∞_1, then the following equality holds:

$$\lim_{r\to 0} r\varphi^{-1}(1/r) = \lim_{\rho\to\infty} \rho/\varphi(\rho) = 0.$$

For $\varphi \in \mathcal{N} \cap \Delta_2$, we denote by $RV_{\varphi,\alpha}(I_a^b, X)$ the vector space (see [4])

$$RV_{\varphi,\alpha}(I_a^b, X) = \left\{ f \in X^{I_a^b} : \exists\, \lambda > 0,\ TV^R_{\varphi,\alpha}(\lambda f) < \infty \right\}.$$

Just as in the one dimensional situation (cf. [1]), in $RV_{\varphi,\alpha}(I_a^b, X)$ one can define the so called Luxemburg-Nakano-Orlicz seminorm [7, 12, 13, 4]

$$p_{\varphi,\alpha}(f) := \inf\left\{ \epsilon > 0 : TV^R_{\varphi,\alpha}(f/\epsilon) \leq 1 \right\},$$

and we define in $RV^R_{\varphi,\alpha}(I_a^b, X)$ the norm

$$\|f\|_{\varphi,\alpha} := |f(a)| + p_{\varphi,\alpha}(f).$$

Also, if $C \subseteq X$ we use the notation $RV_{\varphi,\alpha}(I_a^b, C)$ for the set

$$\left\{ f \in RV_{\varphi,\alpha}(I_a^b, X) :\ f(I_a^b) \subset C \right\}.$$

For $(t, s), (t', s') \in I_a^b$, let we put

$$\Omega_{t,t',s,s'} := \Big\{ |\alpha(t) - \alpha(t')|, |\alpha(s) - \alpha(s')|, |\alpha(t) - \alpha(t')||\alpha(s) - \alpha(a_2)|,$$
$$|\alpha(a_1) - \alpha(t')||\alpha(s) - \alpha(s')| \Big\}.$$

The following lemma exhibits some properties of $p_{\varphi,\alpha}$.

Lemma 2.4. *([4]) For $\varphi \in \mathcal{F}$ and $f \in RV_{\varphi,\alpha}(I_a^b; X)$, we have*

(a) *If $(t, s), (t', s') \in I_a^b$, then*

$$|f(t, s) - f(t', s')| \leq 4M\varphi^{-1}(1/m)p_{\varphi,\alpha}(f)$$

where $M := \max \Omega_{t,t',s,s'}$ *y* $m := \min \Omega_{t,t',s,s'}$.

(b) *If $p_{\varphi,\alpha}(f) > 0$, then $TV^R_{\varphi,\alpha}(f/p_{\varphi,\alpha}(f)) \leq 1$.*

(c) *If $r > 0$, then $TV^R_{\varphi,\alpha}(f/r) \leq 1$ if, and only if, $p_{\varphi,\alpha}(f) \leq r$.*

(d) *If $r > 0$ y $TV^R_{\varphi,\alpha}(f/p_{\varphi,\alpha}(f)) = 1$, then $p_{\varphi,\alpha}(f) = r$.*

Theorem 2.5. *([4]) If $\varphi \in \mathcal{F} \cap \Delta_2$, $\alpha : I \longrightarrow \mathbb{R}$ a fixed continuous strictly increasing function and X is a Banach space, then $\left(RV_{\varphi,\alpha}(I_a^b, X), \|\cdot\|_{\varphi,\alpha}\right)$ is a Banach space.*

3 Main result

In this section, we begin with an important result and get immediate a consequences as the main results of the paper, which extend the results of Matkowski and others (see [1, 9]) in the case when the Nemytskii operator is defined on the space $RV_{\varphi,\alpha}([a,b];\mathbb{R})$. The technics used for the proof are based on those of [11].

Theorem 3.1. *Assume that $I_a^b \subset \mathbb{R}^2$ is a rectangle, $\alpha : I \longrightarrow \mathbb{R}$ a fixed continuous strictly increasing function (sometimes it is called weight function) and φ, ψ are $\mathcal{N}$-functions that satisfy the ∞_1 condition, $(X, \|\cdot\|_X)$ is a real normed space, $(Y, \|\cdot\|_Y)$ is a real Banach space and C is a closed and convex set in X, and the function $h : I_a^b \times C \longrightarrow Y$ is continuous with respect to the third variable. If there exists a function $\varrho : [0,+\infty) \longrightarrow [0,+\infty)$ such that the nonlinear composition operator H of the generator by h maps the set $RV_{\varphi,\alpha}(I_a^b, C)$ into the space $RV_{\psi,\alpha}(I_a^b, Y)$ and satisfies the inequality*

$$\|H(f_1) - H(f_2)\|_{\psi,\alpha} \leq \varrho\left(\|f_1 - f_2\|_{\varphi,\alpha}\right), \qquad f_1, f_2 \in RV_{\varphi,\alpha}(I_a^b, C), \tag{3.1}$$

then there exist functions $A, B : I_a^b \longrightarrow Y$ such that

$$h(t,s,u) = A(t,s)u + B(t,s), \qquad (t,s) \in I_a^b, \quad u \in C.$$

Moreover, if $0 \in C$ and $intC \neq \emptyset$, then $A : I_a^b \longrightarrow \mathcal{L}(X,Y)$ and $B \in RV_{\psi,\alpha}(I_a^b, Y)$.

Proof. It is readily seen that for each $u \in C$, the constant function $f(t,s) := u$ belongs to $RV_{\varphi,\alpha}(I_a^b, C)$; thus, since H maps $RV_{\varphi,\alpha}(I_a^b, C)$ into $RV_{\psi,\alpha}(I_a^b, Y)$, it follows that, for each $u \in C$, the function $h_u : I_a^b \longrightarrow Y$ defined as

$$h_u(t,s) := h(t,s,u)$$

belongs to $RV_{\psi,\alpha}(I_a^b, Y)$.

From the definition of the norm $\|\cdot\|_{\psi,\alpha}$, we obtain

$$p_{\psi,\alpha}\left(H(f_1) - H(f_2)\right) \leq \|H(f_1) - H(f_2)\|_{\psi,\alpha}, \quad \text{for} \quad f_1, f_2 \in RV_{\varphi,\alpha}(I_a^b, C).$$

Hence, in view of Lemma 2.4(c) and inequality (3.1), we get that the last inequality is equivalent to

$$V_{\psi,\alpha}\left(\frac{(H(f_1) - H(f_2))}{\varrho\left(\|f_1 - f_2\|_{\varphi,\alpha}\right)}\right) \leq TV^R_{\psi,\alpha}\left(\frac{H(f_1) - H(f_2)}{\varrho\left(\|f_1 - f_2\|_{\varphi,\alpha}\right)}\right) \leq 1.$$

Now, by definition of $V_{\psi,\alpha}$ and H, it follows that for any rectangle $[t_1,t_2] \times [s_1,s_2] \subseteq I_a^b$, with $t_1 < t_2$ and $s_1 < s_2$ and for any $f_1,\ f_2 \in RV_{\varphi,\alpha}(I_a^b, C)$, we obtain:

$$\psi\left(\frac{\left|\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}(H(f_1) - H(f_2))(t_i,s_j)\right|}{\varrho\left(\|f_1 - f_2\|_{\varphi,\alpha}\right)\Delta\alpha(t_2)\cdot\Delta\alpha(s_2)}\right)\Delta\alpha(t_2)\cdot\Delta\alpha(s_2) \leq 1. \tag{3.2}$$

Let us define now, for arbitrarily fixed $\theta, \rho \in \mathbb{R}$, with $\theta < \rho$:

$$\eta_{\theta,\rho}(t) := \begin{cases} 0 & \text{for } t \leq \theta \\ \dfrac{\alpha(t) - \alpha(\theta)}{\alpha(\rho) - \alpha(\theta)} & \text{for } \theta \leq t \leq \rho \\ 1 & \text{for } t \geq \rho. \end{cases}$$

Observe that $\eta_{\theta,\rho} : \mathbb{R} \longrightarrow [0,1]$.

Next, consider two auxiliary functions: $\eta_i : [a_i, b_i] \longrightarrow [0,1]$, $i = 1,2$, defined in the following way:

$$\eta_1(t) := \begin{cases} 0 & \text{for } a_1 \leq t \leq t_1 \\ \eta_{t_1,t_2}(t) & \text{for } t_1 \leq t \leq t_2, \\ 1 & \text{for } t_2 \leq t, \end{cases}$$

$$\eta_2(s) := \begin{cases} 0 & \text{for } a_2 \le s \le s_1 \\ \eta_{s_1,s_2}(s) & \text{for } s_1 \le s \le s_2, \\ 1 & \text{for } s_2 \le s. \end{cases}$$

Finally, for arbitrary points $y_1, y_2 \in C$, $y_1 \neq y_2$, define functions $f_1,\ f_2 : I_a^b \longrightarrow C$ as follows:

$$f_j(t,s) := \frac{1}{2}\Big[\big(\eta_1(t)\cdot\eta_2(s)\big)(y_1 - y_2) + y_j + y_2\Big],\ \ (t,s) \in I_a^b,\ j = 1,2.$$

Observe, that

$$f_1(t_1,s_1) \ = \ f_1(t_1,s_2) \ = \ f_1(t_2,s_1) \ = \ \frac{y_1+y_2}{2};\ \ f_1(t_2,s_2) = y_1,$$

$$f_2(t_1,s_1) \ = \ f_2(t_1,s_2) \ = \ f_2(t_2,s_1) \ = \ y_2;\ \ f_2(t_2,s_2) = \frac{y_1+y_2}{2},$$

$$f_1(\cdot) - f_2(\cdot) = \frac{y_1-y_2}{2},\ \ \text{and consequently}\ \ \|f_1 - f_2\|_{\varphi,\alpha} = \frac{|y_1-y_2|}{2} > 0;$$

Also, by definition of H:

$$\big(H(f_1) - H(f_2)\big)(t_1,s_1) = h\Big((t_1,s_1), \frac{y_1+y_2}{2}\Big) - h\big((t_1,s_1),y_2\big) \tag{3.3}$$

$$\big(H(f_1) - H(f_2)\big)(t_1,s_2) = h\Big((t_1,s_2), \frac{y_1+y_2}{2}\Big) - h\big((t_1,s_2),y_2\big) \tag{3.4}$$

$$\big(H(f_1) - H(f_2)\big)(t_2,s_1) = h\Big((t_2,s_1), \frac{y_1+y_2}{2}\Big) - h\big((t_2,s_1),y_2\big) \tag{3.5}$$

$$\big(H(f_1) - H(f_2)\big)(t_2,s_2) = h\big((t_2,s_2),y_1\big) - h\Big((t_2,s_2), \frac{y_1+y_2}{2}\Big) \tag{3.6}$$

Now notice that, by applying the inverse function ψ^{-1} to both sides of (3.2), one gets

$$\left|\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}(H(f_1) - H(f_2))(t_i,s_j)\right|$$
$$\le \Delta\alpha(t_2)\Delta\alpha(s_2)\cdot\varrho\Big(\frac{|y_1-y_2|}{2}\Big)\psi^{-1}\Big(\frac{1}{\Delta\alpha(t_2)\Delta\alpha(s_2)}\Big). \tag{3.7}$$

Taking into account the fact that for any $u \in C$ the function $h_u \in RV_{\psi,\alpha}(I_a^b, Y)$, the identities (3.3)-(3.6), that ψ satisfies the condition ∞_1, and passing to limit in (3.7) as $\Delta\alpha(t_2)\Delta\alpha(s_2) \longrightarrow 0$, in such a way that $(t,s) \in [t_1,t_2]\times[s_1,s_2] \subseteq I_a^b$, with $t_1 < t_2$ and $s_i < s_2$, we obtain, after simplification (the first two summands cancel out each other), for all $(t,s) \in I_a^b$, $\ y_1, y_2 \in C$:

$$h\left((t,s), \frac{y_1+y_2}{2}\right) = \frac{1}{2}\left(h((t,s),y_1) + h((t,s),y_2)\right).$$

Therefore, the function $h((t,s),\cdot)$ is a solution of Jensen functional equation in C for $(t,s) \in I_a^b$. Thus, by a slight modification of a standard argument (see Kuczma [6, Th. 1,page 315]); the assumed continuity of h with respect to the third variable and for each $(t,s) \in I_a^b$ it guaranties the existence of an additive function $A : I_a^b \longrightarrow Y$ and $B : I_a^b \longrightarrow Y$ such that

$$h(\cdot,y) = A(\cdot)y + B(\cdot),\ y \in C. \tag{3.8}$$

Finally, notice that $A(t,s)(0) = 0$, for every $(t,s) \in I_a^b$. Therefore, putting $y = 0$ in (3.8), we get

$$h(t,s,0) = B(t,s), \quad (t,s) \in I_a^b,$$

which implies that $B \in RV_{\psi,\alpha}(I_a^b, Y)$. □

Remark 3.2. If the function $\gamma : [0,\infty) \longrightarrow [0,\infty)$ is right at 0 and $\gamma(0) = 0$, then the assumption of the continuity of h with respect to the third variable can be omitted, as it follows from (3.1).

Note that in the first part of the Theorem 3.1 the function $\gamma : [0,\infty) \longrightarrow [0,\infty)$ is completely arbitrary.

As an immediate corollary of the Theorem 3.1 we obtain the following

Corollary 3.3. *Let $(X, |\cdot|)$ be a real normed space, $(Y, |\cdot|)$ a real Banach space, C a convex cone in X and suppose that $\varphi,\ \psi \in \mathcal{F}$. If the composition operator H generated by the function $h : I \times C \longrightarrow Y$ maps $RV_{\varphi,\alpha}(I_a^b, C)$ into $RV_{\psi,\alpha}(I_a^b, Y)$, and there exist a function $\gamma : [0,\infty) \longrightarrow [0,\infty)$ right continuous at 0 with $\gamma(0) = 0$, such that*

$$\|H(f_1) - H(f_2)\|_\psi \leq \gamma\big(\|f_1 - f_2\|_\varphi\big), \quad f_1, f_2 \in RV_{\varphi,\alpha}(I_a^b),$$

then

$$h(t,s,x) \;=\; A(t,s)x + B(t,s), \quad (t,s) \in I_a^b, x \in C$$

for some $\ A \in \mathcal{L}(X,Y)\ $ and $\ B \in RV_{\psi,\alpha}(I_a^b, Y)$.

We now recall definitions of uniformly bounded and equidistantly uniformly bounded mapping introduced by Matkowski in [11], which plays a crucial role.

Definition 3.4 ([11]). Let $\mathbf{X}$ and $\mathbf{Y}$ be two metric (or normed) spaces. We say that a mapping $H : \mathbf{X} \longrightarrow \mathbf{Y}$ is uniformly bounded if, for any $t > 0$, there is a nonnegative real number $\varrho(t)$ such that, for any nonempty set $\mathbb{A} \subset \mathbf{X}$, we have

$$diam(\mathbb{A}) \leq t \Longrightarrow diam(H(\mathbb{A})) \leq \varrho(t).$$

Remark 3.5. Obviously, every uniformly continuous operator or Lipschitzian operator is uniformly bounded. Note that, under the assumptions of this definition, every bounded operator is uniformly bounded.

Applying the Theorem 3.1 we obtain our main results.

Theorem 3.6. *Let $I_a^b \subset \mathbb{R}^2$ be a rectangle, $\alpha :\ I \longrightarrow \mathbb{R}$ a fixed continuous strictly increasing function and φ, ψ are $\mathcal{N}$-functions that satisfy the ∞_1 condition, X is a real normed space, Y is a real Banach space and C is a closed and convex set in X. If the nonlinear composition operator H generated by the function $h : I_a^b \times C \longrightarrow Y$ maps the set $RV_{\varphi,\alpha}(I_a^b, C)$ into the Banach space $RV_{\psi,\alpha}(I_a^b, Y)$ and is uniformly bounded, then there exist functions $A(t,s) \in \mathcal{L}(I_a^b, Y)$ and $B \in RV_{\psi,\alpha}(I_a^b, Y)$ such that*

$$h((t,s),u) = A(t,s)u + B(t,s), \quad \textit{for} \quad (t,s) \in I_a^b,\ u \in C.$$

Proof. Choose any $\varepsilon \geq 0$ and arbitrary $f_1, f_2 \in RV_{\varphi,\alpha}(I_a^b, C)$ such that $\|f_1 - f_2\|_{\varphi,\alpha} = \varepsilon$. The uniform boundedness of H implies that $diam(H(\{f_1, f_2\})) \leq \varrho(\varepsilon)$, i.e.,

$$\|H(f_1) - H(f_2)\|_{\psi,\alpha} = diam(H(\{f_1, f_2\})) \leq \varrho(\|f_1 - f_2\|_{\varphi,\alpha}),$$

so it is enough to apply Theorem 3.1. □

Definition 3.7 ([11]). Let $\mathbf{X}$ and $\mathbf{Y}$ be two metric (or normed) spaces. We say that a mapping $H : \mathbf{X} \longrightarrow \mathbf{Y}$ is equidistantly uniformly bounded if, for every $t > 0$, there is a nonnegative real number $\varrho(t)$ such that, for all $z_1, z_2 \in \mathbb{A} \subset \mathbf{X}$,

$$diam\{z_1, z_2\} = t \Longrightarrow diam\Big\{H(z_1), H(z_2)\Big\} \leq \varrho(t).$$

Of course, the equidistant uniform boundedness is a weaker condition than the uniform boundedness. Similarly, by Theorem 3.1, we obtain the following result.

Theorem 3.8. *Let $I_a^b \subset \mathbb{R}^2$ be a rectangle, $\alpha :\ I \longrightarrow \mathbb{R}$ a fixed continuous strictly increasing function and φ, ψ are $\mathcal{N}$-functions that satisfy the ∞_1 condition, X is a real normed space, Y is a real Banach space and C is a closed and convex set in X. If the nonlinear composition operator H generated by the function $h : I_a^b \times C \longrightarrow Y$ maps the set $RV_{\varphi,\alpha}(I_a^b, C)$ into the Banach space $RV_{\psi,\alpha}(I_a^b, Y)$ and is equidistantly uniformly bounded, then*

$$h((t,s),u) = A(t,s)u + B(t,s), \quad \textit{for} \quad (t,s) \in I_a^b,\ u \in C.$$

for some functions $A : I_a^b \longrightarrow \mathcal{L}(I_a^b)$ and $B \in RV_{\psi,\alpha}(I_a^b, Y)$.

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Author information

Wadie Aziz and José A. Guerrero, Department of Physic and Mathematic, University of the Andes, Trujillo Estado Trujillo, Venezuela.
E-mail: `wadie@ula.ve, jaguerrero4@gmail.com`

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