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ESTIMATES ON INITIAL COEFFICIENTS OF CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE HOHLOV OPERATOR

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Abstract. In the present investigation, we introduce certain subclasses of the function class Σ of bi-univalent functions defined in the open unit disk $\mathbb{U}$, which are associated with the Hohlov operator. Also we obtain estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions in these subclasses and pointed out several consequences of these results.

1 Introduction

Let $\mathcal{A}$ denote the class of all normalized analytic functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

defined in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\}$, where $\mathbb{C}$ being the set of complex numbers. Further, the subclass of $\mathcal{A}$ consisting of all functions which are also univalent in $\mathbb{U}$ is denoted by $\mathcal{S}$ (for details, see [4]).

Due to the well known Koebe one quarter theorem (see [4]) it is clear that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by:

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, some computations using (1.1) gives:

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let the class of all bi-univalent functions f in $\mathbb{U}$ given by (1.1) is denoted by Σ.

If the functions ϕ and ψ are analytic in $\mathbb{U}$, then ϕ is said to be subordinate to ψ, written as $\phi(z) \prec \psi(z),\ z \in \mathbb{U}$ if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, such that $\phi(z) = \psi(w(z)),\ z \in \mathbb{U}$.

For the functions $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product or convolution is denoted by $f * g$ and is defined by:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \tag{1.3}$$

and the Gaussian hypergeometric function ${}_2F_1(a,b,c;z)$ for the complex parameters a, b and c with $c \neq 0, -1, -2, -3, \cdots$, is defined by:

$$\begin{aligned} {}_2F_1(a,b,c;z) &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \\ &= 1 + \sum_{k=2}^{\infty} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}} \frac{z^{k-1}}{(k-1)!} \quad (z \in \mathbb{U}), \end{aligned} \tag{1.4}$$

where $(l)_k$ denotes the Pochhammer symbol (the shifted factorial) defined by:

$$(l)_k = \frac{\Gamma(l+k)}{\Gamma(l)} = \begin{cases} 1, & if\ k = 0, l \in \mathbb{C} \setminus \{0\} \\ l(l+1)(l+2)\cdots(l+k-1), & if\ k = 1,2,3,\cdots. \end{cases} \tag{1.5}$$

Hohlov [8, 9] introduced a convolution operator $\mathcal{I}_{a,b;c}$ by using the Gaussian hypergeometric function ${}_2F_1(a,b,c;z)$ given by (1.4) as follows:

$$\begin{aligned} \mathcal{I}_{a,b;c} f(z) &= z\, {}_2F_1(a,b,c;z) * f(z) \\ &= z + \sum_{k=2}^{\infty} y_k a_k z^k, \quad (z \in \mathbb{U}), \end{aligned} \tag{1.6}$$

where

$$y_k = \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}. \tag{1.7}$$

Observe that, if $b = 1$ in (1.6), then the Hohlov operator $\mathcal{I}_{a,b;c}$ reduces to the Carlson-Shaffer operator. Also it can be easily seen that the Hohlov operator is a generalization of the Ruscheweyh derivative operator and the Bernardi-Libera-Livingston operator.

For functions in the class Σ, Lewin [10] proved that $|a_2| < 1.51$, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$ and Netanyahu [12] proved that $max_{f\in\Sigma}|a_2| = 4/3$. However the coefficient estimate problem for each $|a_n|$, $(n = 3, 4, \cdots)$ is still an open problem. Brannan and Taha [3] (see also [22]) introduced certain subclasses of the bi-univalent function class Σ such as $\mathcal{S}^*_\Sigma[\alpha]$ where $0 < \alpha \leq 1$, the class of strongly bi-starlike functions of order α and $\mathcal{S}^*_\Sigma(\beta)$ where $0 \leq \beta < 1$, the class of bi-starlike functions of order β.

Following Brannan and Taha [3], Srivastava et al. [20] and many other researchers (viz. [5, 7, 10, 11, 13, 14, 16, 17, 18, 19, 20, 21, 23, 24, 25] etc.) have investigated several subclasses of the bi-univalent function class Σ and found the estimate on the initial coefficients $|a_2|$ and $|a_3|$. The purpose of the present investigation is to introduce certain subclasses of the function class Σ, which are associated with the Hohlov operator and to find the estimate on the initial coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses.

Let ϕ be an analytic function with positive real part in $\mathbb{U}$ such that $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Hence we have,

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{1.8}$$

In order to prove our main results, we shall need the following Lemma .

Lemma 1.1. *(see [4], [6], [15]) If $h(z) \in \mathcal{P}$, the class of functions analytic in $\mathbb{U}$ with*

$$\Re(h(z)) > 0,$$

then $|c_n| \leq 2$ for each $n \in \mathbb{N}$, where

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad (z \in \mathbb{U}). \tag{1.9}$$

2 Coefficient Estimates for the Function Class $\mathcal{J}_{\Sigma}^{a,b;c}(\alpha,\phi)$

Definition 2.1. A function $f(z)\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{J}_{\Sigma}^{a,b;c}(\alpha,\phi)$ if the following conditions are satisfied:

$$\left[\frac{z\left(\mathcal{I}_{a,b;c}f(z)\right)'}{\mathcal{I}_{a,b;c}f(z)}\right]\left[\frac{\mathcal{I}_{a,b;c}f(z)}{z}\right]^{\alpha}\prec\phi(z)$$

and

$$\left[\frac{w\left(\mathcal{I}_{a,b;c}g(w)\right)'}{\mathcal{I}_{a,b;c}g(w)}\right]\left[\frac{\mathcal{I}_{a,b;c}g(w)}{w}\right]^{\alpha}\prec\phi(w)$$

where $z,w\in\mathbb{U}$, $\alpha\geq 0$ and the functions $g\equiv f^{-1}$ and ϕ are given by (1.2) and (1.8) respectively.

Theorem 2.2. *Let $f(z)\in\Sigma$ given by (1.1) be in the class $\mathcal{J}_{\Sigma}^{a,b;c}(\alpha,\phi)$. Then,*

$$|a_2|\leq min\left\{\frac{B_1}{(\alpha+1)y_2},\sqrt{\frac{2(B_1+|B_2-B_1|)}{(\alpha+2)\left|2y_3+(\alpha-1)y_2^2\right|}},\right.$$
$$\left.\frac{B_1\sqrt{2B_1}}{\sqrt{\left|(\alpha+2)[2y_3+(\alpha-1)y_2^2]B_1^2+2(\alpha+1)^2y_2^2(B_1-B_2)\right|}}\right\} \tag{2.1}$$

and

$$|a_3|\leq min\left\{\frac{B_1}{(\alpha+2)y_3}+\frac{B_1^2}{(\alpha+1)^2y_2^2},\frac{2(B_1+|B_2-B_1|)}{(\alpha+2)\left|2y_3+(\alpha-1)y_2^2\right|}\right\}. \tag{2.2}$$

Proof. Since $f\in\mathcal{J}_{\Sigma}^{a,b;c}(\alpha,\phi)$, there exist two analytic functions $u,v:\mathbb{U}\to\mathbb{U}$, with $u(0)=v(0)=0$, such that:

$$\left[\frac{z\left(\mathcal{I}_{a,b;c}f(z)\right)'}{\mathcal{I}_{a,b;c}f(z)}\right]\left[\frac{\mathcal{I}_{a,b;c}f(z)}{z}\right]^{\alpha}=\phi(u(z)) \tag{2.3}$$

and

$$\left[\frac{w\left(\mathcal{I}_{a,b;c}g(w)\right)'}{\mathcal{I}_{a,b;c}g(w)}\right]\left[\frac{\mathcal{I}_{a,b;c}g(w)}{w}\right]^{\alpha}=\phi(v(w)) \tag{2.4}$$

where $z,w\in\mathbb{U}$. Define the functions s and t as:

$$s(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+c_3z^3+\cdots$$

and

$$t(w)=\frac{1+v(w)}{1-v(w)}=1+d_1w+d_2w^2+d_3w^3+\cdots.$$

Clearly s and t are analytic in $\mathbb{U}$ and $s(0)=t(0)=1$. Since $u,v:\mathbb{U}\to\mathbb{U}$, the functions s and t have positive real part in $\mathbb{U}$. Hence by Lemma 1.1,

$$|c_n|\leq 2,\quad |d_n|\leq 2,\quad (n\in\mathbb{N}). \tag{2.5}$$

Solving for $u(z)$ and $v(w)$, we get:

$$u(z)=\frac{1}{2}\left[c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right],\quad (z\in\mathbb{U})$$

and

$$v(w)=\frac{1}{2}\left[d_1w+\left(d_2-\frac{d_1^2}{2}\right)w^2+\cdots\right],\quad (w\in\mathbb{U}).$$

Using these expansions in (1.8), we obtain:

$$\phi(u(z)) = 1 + \frac{1}{2}B_1c_1z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right]z^2 + \cdots \tag{2.6}$$

and

$$\phi(v(w)) = 1 + \frac{1}{2}B_1d_1w + \left[\frac{1}{2}B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{1}{4}B_2d_1^2\right]w^2 + \cdots. \tag{2.7}$$

Expanding the LHS of (2.3) and (2.4) and then equating the coefficients of z, z^2, w, w^2; we get:

$$(\alpha+1)y_2a_2 = \frac{B_1c_1}{2}, \tag{2.8}$$

$$(\alpha+2)y_3a_3 + \frac{1}{2}(\alpha-1)(\alpha+2)y_2^2a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.9}$$

$$-(\alpha+1)y_2a_2 = \frac{B_1d_1}{2}, \tag{2.10}$$

$$(\alpha+2)y_3(2a_2^2 - a_3) + \frac{1}{2}(\alpha-1)(\alpha+2)y_2^2a_2^2 = \frac{1}{2}B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{1}{4}B_2d_1^2. \tag{2.11}$$

From (2.8) and (2.10), we get:

$$c_1 = -d_1 \tag{2.12}$$

and

$$8(\alpha+1)^2y_2^2a_2^2 = B_1^2(c_1^2 + d_1^2). \tag{2.13}$$

Adding (2.9) and (2.11), we obtain:

$$4(\alpha+2)\left[2y_3 + (\alpha-1)y_2^2\right]a_2^2 = 2B_1(c_2+d_2) + (B_2 - B_1)(c_1^2 + d_1^2). \tag{2.14}$$

This on using (2.13) gives:

$$a_2^2 = \frac{B_1^3(c_2+d_2)}{2(\alpha+2)[2y_3 + (\alpha-1)y_2^2]B_1^2 + 4(\alpha+1)^2y_2^2(B_1 - B_2)}. \tag{2.15}$$

Clearly (2.13), (2.14) and (2.15) in light of (2.5) gives us the desired estimate on $|a_2|$ as asserted in (2.1).

Next, to find the estimate on $|a_3|$, subtracting (2.11) from (2.9), we get:

$$2\,(\alpha+2)\,y_3\left[a_3 - a_2^2\right] = \frac{2B_1\,(c_2 - d_2) + (B_2 - B_1)\left(c_1^2 - d_1^2\right)}{4},$$

which on using (2.12), gives:

$$a_3 = a_2^2 + \frac{B_1(c_2 - d_2)}{4(\alpha+2)y_3}. \tag{2.16}$$

Using (2.13) in (2.16), we get:

$$a_3 = \frac{B_1^2(c_1^2 + d_1^2)}{8(\alpha+1)^2y_2^2} + \frac{B_1(c_2 - d_2)}{4(\alpha+2)y_3}. \tag{2.17}$$

Similarly, using (2.14) in (2.16), we get:

$$a_3 = \frac{2B_1(c_2+d_2) + (B_2 - B_1)(c_1^2 + d_1^2)}{4(\alpha+2)\left[2y_3 + (\alpha-1)y_2^2\right]} + \frac{B_1(c_2 - d_2)}{4(\alpha+2)y_3}$$

or

$$a_3 = \frac{\left[2B_1(c_2+d_2) + (B_2 - B_1)(c_1^2 + d_1^2)\right]y_3 + B_1(c_2 - d_2)\left[2y_3 + (\alpha-1)y_2^2\right]}{4(\alpha+2)y_3\left[2y_3 + (\alpha-1)y_2^2\right]}.$$

Which, on separating the coefficients of c_2 and d_2, gives:

$$a_3 = \frac{\left[(4y_3 + (\alpha-1)y_2^2)c_2 - (\alpha-1)y_2^2 d_2\right] B_1 + y_3(c_1^2 + d_1^2)(B_2 - B_1)}{4(\alpha+2)y_3\left[2y_3 + (\alpha-1)y_2^2\right]}. \tag{2.18}$$

Clearly (2.17) and (2.18) in light of (2.5) gives us the desired estimate on a_3 as asserted in (2.2). This completes the proof of Theorem 2.2. □

Taking $a = c$ and $b = 1$ in Theorem 2.2, we get the class $\mathcal{J}_\alpha(\phi)$, $(\alpha \geq 0)$ (generalized class is $\mathcal{J}_\alpha^q(\phi)$ which is associated with quasi-subordination, defined and studied by Goyal et al. [7]). Hence we have the following Corollary.

Corollary 2.3. *Let $f(z) \in \Sigma$ given by (1.1) be in the class $\mathcal{J}_\alpha(\phi)$. Then,*

$$|a_2| \leq min\left\{\frac{B_1}{(\alpha+1)}, \sqrt{\frac{2(B_1 + |B_2 - B_1|)}{(\alpha+1)(\alpha+2)}}, \frac{B_1\sqrt{2B_1}}{\sqrt{(\alpha+1)\left|(\alpha+2)B_1^2 + 2(\alpha+1)(B_1 - B_2)\right|}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{B_1}{(\alpha+2)} + \frac{B_1^2}{(\alpha+1)^2}, \frac{2(B_1 + |B_2 - B_1|)}{(\alpha+1)(\alpha+2)}\right\}.$$

Putting $\alpha = 0$ in Corollary 2.3, we get the class $\mathcal{S}_\Sigma^*(1;\phi)$ (a branch of the class $\mathcal{S}_\Sigma^*(\gamma;\phi)$ whose generalization is the class $\mathcal{S}_\Sigma(\lambda,\gamma;\phi)$ defined and studied by Erhan Deniz [5]) or the class $\mathcal{S}_\Sigma^*(\phi)$ defined and studied by Brannan and Taha [3]. Also, see Corollary 2.4 given by Tang et al. [23]. Hence we have the following Corollary.

Corollary 2.4. *Let $f(z) \in \Sigma$ given by (1.1) be in the class $\mathcal{S}_\Sigma^*(\phi)$. Then,*

$$|a_2| \leq min\left\{B_1, \sqrt{B_1 + |B_2 - B_1|}, \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2 + B_1 - B_2\right|}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{B_1}{2} + B_1^2, B_1 + |B_2 - B_1|\right\}.$$

Putting $\alpha = 1$ in Corollary 2.3, we get the class $\mathcal{H}_\sigma(\phi)$ defined and studied by Ali et al [1]. Similarly, we get the class $\Sigma(1,0,\phi)$ (whose generalization is the class $\Sigma(\tau,\gamma,\phi)$, defined and studied by Srivastava and Bansal [16]). Also, see Corollary 2.2 given by Tang et al. [23]. Hence we have the following Corollary as an improvement in Theorem 2.1 given by Ali et al. [1].

Corollary 2.5. *Let $f(z) \in \Sigma$ given by (1.1) be in the class $\mathcal{H}_\sigma(\phi)$. Then,*

$$|a_2| \leq min\left\{\frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2 - B_1|}{3}}, \frac{B_1\sqrt{B_1}}{\sqrt{\left|3B_1^2 + 4(B_1 - B_2)\right|}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{B_1}{3} + \frac{B_1^2}{4}, \frac{B_1 + |B_2 - B_1|}{3}\right\}.$$

3 Coefficient Estimates for the Function Class $\mathcal{K}_\Sigma^{a,b;c}(\beta,\gamma,\phi)$

Definition 3.1. A function $f(z)\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{K}_\Sigma^{a,b;c}(\beta,\gamma,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\left(\mathcal{I}_{a,b;c}f(z)\right)^{'}+\beta z\left(\mathcal{I}_{a,b;c}f(z)\right)^{''}-1\right]\prec\phi(z)$$

and

$$1+\frac{1}{\gamma}\left[\left(\mathcal{I}_{a,b;c}g(w)\right)^{'}+\beta w\left(\mathcal{I}_{a,b;c}g(w)\right)^{''}-1\right]\prec\phi(w)$$

where $z,w\in\mathbb{U}$, $0\le\beta<1$, $\gamma\in\mathbb{C}\setminus\{0\}$ and the functions $g\equiv f^{-1}$ and ϕ are given by (1.2) and (1.8) respectively.

Theorem 3.2. *Let $f(z)\in\Sigma$ given by (1.1) be in the class $\mathcal{K}_\Sigma^{a,b;c}(\beta,\gamma,\phi)$. Then,*

$$|a_2|\le min\left\{\frac{|\gamma|B_1}{2(1+\beta)y_2},\sqrt{\frac{|\gamma|(B_1+|B_2-B_1|)}{3(1+2\beta)y_3}},\right.\left.\frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|3\gamma(1+2\beta)y_3B_1^2+4(1+\beta)^2y_2^2(B_1-B_2)|}}\right\} \tag{3.1}$$

and

$$|a_3|\le min\left\{\frac{|\gamma|B_1}{3(1+2\beta)y_3}+\frac{\gamma^2B_1^2}{4(1+\beta)^2y_2^2},\frac{|\gamma|(B_1+|B_2-B_1|)}{3(1+2\beta)y_3}\right\}. \tag{3.2}$$

Proof. Since $f\in\mathcal{K}_\Sigma^{a,b;c}(\beta,\gamma,\phi)$, there exist two analytic functions $u,v:\mathbb{U}\to\mathbb{U}$, with $u(0)=v(0)=0$, such that:

$$1+\frac{1}{\gamma}\left[\left(\mathcal{I}_{a,b;c}f(z)\right)^{'}+\beta z\left(\mathcal{I}_{a,b;c}f(z)\right)^{''}-1\right]=\phi(u(z)) \tag{3.3}$$

and

$$1+\frac{1}{\gamma}\left[\left(\mathcal{I}_{a,b;c}g(w)\right)^{'}+\beta w\left(\mathcal{I}_{a,b;c}g(w)\right)^{''}-1\right]=\phi(v(w)), \tag{3.4}$$

where $z,w\in\mathbb{U}$. Define the functions s and t as in Theorem 2.2 and then proceed similarly up to (2.7).
Expanding the LHS of (3.3) and (3.4), we obtain:

$$\begin{aligned}&1+\frac{1}{\gamma}\left[\left(\mathcal{I}_{a,b;c}f(z)\right)^{'}+\beta z\left(\mathcal{I}_{a,b;c}f(z)\right)^{''}-1\right]\\&=1+\frac{1}{\gamma}\left[2(1+\beta)y_2a_2z+3(1+2\beta)y_3a_3z^2+\cdots\right]\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left[\left(\mathcal{I}_{a,b;c}g(w)\right)^{'}+\beta w\left(\mathcal{I}_{a,b;c}g(w)\right)^{''}-1\right]\\&=1+\frac{1}{\gamma}\left[-2(1+\beta)y_2a_2w+3(1+2\beta)y_3(2a_2^2-a_3)z^2+\cdots\right].\end{aligned} \tag{3.6}$$

Now, using (2.6), (2.7), (3.5), (3.6) in (3.3) and (3.4) and then equating the coefficients of z, z^2, w, w^2; we get:

$$2(1+\beta)y_2a_2=\frac{\gamma B_1c_1}{2}, \tag{3.7}$$

$$3(1+2\beta)y_3a_3=\gamma\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right], \tag{3.8}$$

$$-2(1+\beta)y_2 a_2 = \frac{\gamma B_1 d_1}{2}, \tag{3.9}$$

$$3(1+2\beta)y_3(2a_2^2 - a_3) = \gamma\left[\frac{1}{2}B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{1}{4}B_2 d_1^2\right]. \tag{3.10}$$

From (3.7) and (3.9), we get:

$$c_1 = -d_1 \tag{3.11}$$

and

$$32(1+\beta)^2 y_2^2 a_2^2 = \gamma^2 B_1^2 (c_1^2 + d_1^2). \tag{3.12}$$

Adding (3.8) and (3.10), we obtain:

$$24(1+2\beta)y_3 a_2^2 = \gamma\left[2B_1(c_2+d_2) + (B_2 - B_1)(c_1^2 + d_1^2)\right]. \tag{3.13}$$

Also, using (3.12) in (3.13), we get:

$$a_2^2 = \frac{\gamma^2 B_1^3 (c_2 + d_2)}{[12\gamma(1+2\beta)y_3 B_1^2 + 16(1+\beta)^2 y_2^2 (B_1 - B_2)]}. \tag{3.14}$$

Clearly (3.12), (3.13) and (3.14) in light of (2.5) gives us the desired estimate on $|a_2|$ as asserted in (3.1).

Next, to find the estimate on $|a_3|$, subtracting (3.10) from (3.8) and then using (3.11), we get:

$$a_3 = a_2^2 + \frac{\gamma B_1 (c_2 - d_2)}{12(1+2\beta)y_3}. \tag{3.15}$$

Using (3.12) in (3.15), we get:

$$a_3 = \frac{\gamma^2 B_1^2 (c_1^2 + d_1^2)}{32(1+\beta)^2 y_2^2} + \frac{\gamma B_1 (c_2 - d_2)}{12(1+2\beta)y_3}. \tag{3.16}$$

Similarly, using (3.13) in (3.15), we get:

$$a_3 = \frac{\gamma\left[2B_1(c_2+d_2) + (B_2 - B_1)(c_1^2 + d_1^2)\right]}{24(1+2\beta)y_3} + \frac{\gamma B_1 (c_2 - d_2)}{12(1+2\beta)y_3}.$$

Which, on simplification, yields:

$$a_3 = \frac{\gamma\left[4c_2 B_1 + (B_2 - B_1)(c_1^2 + d_1^2)\right]}{24(1+2\beta)y_3}. \tag{3.17}$$

Clearly (3.16) and (3.17) in light of (2.5) gives us the desired estimate on $|a_3|$ as asserted in (3.2). This completes the proof of Theorem 3.2. □

Taking $a = c$ and $b = 1$ in Theorem 3.2, we get the class $\Sigma(\gamma, \beta, \phi)$, $0 \le \beta < 1$, $\gamma \in \mathbb{C} \setminus \{0\}$ defined and studied by Srivastava and Bansal [16]. Hence we get the following Corollary as an improvement in Theorem 1 given by Srivastava and Bansal [16].

Corollary 3.3. *Let $f(z) \in \Sigma$ given by (1.1) be in the class $\Sigma(\gamma, \beta, \phi)$. Then,*

$$|a_2| \le min\left\{\frac{|\gamma| B_1}{2(1+\beta)}, \sqrt{\frac{|\gamma|(B_1 + |B_2 - B_1|)}{3(1+2\beta)}}, \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|3\gamma(1+2\beta)B_1^2 + 4(1+\beta)^2(B_1 - B_2)|}}\right\}$$

and

$$|a_3| \le min\left\{\frac{|\gamma| B_1}{3(1+2\beta)} + \frac{\gamma^2 B_1^2}{4(1+\beta)^2}, \frac{|\gamma|(B_1 + |B_2 - B_1|)}{3(1+2\beta)}\right\}.$$

Putting $\gamma = 1$ and $\beta = 0$ in Corollary 3.3, we get the class $\Sigma(1, 0, \phi) \equiv \mathcal{H}_\sigma(\phi)$ defined and studied by Ali et al [1]. Also, see Corollary 2.2 given by Tang et al. [23]. Hence we have Corollary 2.5 as an improvement in the Theorem 2.1 given by Ali et al. [1].

4 Coefficient Estimates for the Function Class $\mathcal{S}_{\Sigma}^{a,b;c}(\lambda,\gamma,\phi)$

Definition 4.1. A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{a,b;c}(\lambda,\gamma,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{I}_{a,b;c}f(z)\right)^{'}+\lambda z^{2}\left(\mathcal{I}_{a,b;c}f(z)\right)^{''}}{\lambda z\left(\mathcal{I}_{a,b;c}f(z)\right)^{'}+(1-\lambda)\left(\mathcal{I}_{a,b;c}f(z)\right)}-1\right]\prec\phi(z)$$

and

$$1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{I}_{a,b;c}g(w)\right)^{'}+\lambda w^{2}\left(\mathcal{I}_{a,b;c}g(w)\right)^{''}}{\lambda w\left(\mathcal{I}_{a,b;c}g(w)\right)^{'}+(1-\lambda)\left(\mathcal{I}_{a,b;c}g(w)\right)}-1\right]\prec\phi(w)$$

where $z,w\in\mathbb{U}$, $0\le\lambda\le1$, $\gamma\in\mathbb{C}\setminus\{0\}$ and the functions $g\equiv f^{-1}$ and ϕ are given by (1.2) and (1.8) respectively.

Theorem 4.2. *Let $f(z)\in\Sigma$ given by (1.1) be in the class $\mathcal{S}_{\Sigma}^{a,b;c}(\lambda,\gamma,\phi)$. Then,*

$$|a_2|\le min\left\{\frac{|\gamma|B_1}{(1+\lambda)y_2},\sqrt{\frac{|\gamma|(B_1+|B_2-B_1|)}{|2(1+2\lambda)y_3-(1+\lambda)^2y_2^2|}},\frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{\left|\left[2(1+2\lambda)y_3-(1+\lambda)^2y_2^2\right]\gamma B_1^2+(1+\lambda)^2y_2^2(B_1-B_2)\right|}}\right\} \tag{4.1}$$

and

$$|a_3|\le min\left\{\frac{|\gamma|B_1}{2(1+2\lambda)y_3}+\frac{\gamma^2B_1^2}{(1+\lambda)^2y_2^2},\frac{|\gamma|(B_1+|B_2-B_1|)}{|2(1+2\lambda)y_3-(1+\lambda)^2y_2^2|}\right\}. \tag{4.2}$$

Proof. Since $\mathcal{S}_{\Sigma}^{a,b;c}(\lambda,\gamma,\phi)$, there exist two analytic functions $u,v:\mathbb{U}\to\mathbb{U}$, with $u(0)=v(0)=0$, such that:

$$1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{I}_{a,b;c}f(z)\right)^{'}+\lambda z^{2}\left(\mathcal{I}_{a,b;c}f(z)\right)^{''}}{\lambda z\left(\mathcal{I}_{a,b;c}f(z)\right)^{'}+(1-\lambda)\left(\mathcal{I}_{a,b;c}f(z)\right)}-1\right]=\phi(u(z)) \tag{4.3}$$

and

$$1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{I}_{a,b;c}g(w)\right)^{'}+\lambda w^{2}\left(\mathcal{I}_{a,b;c}g(w)\right)^{''}}{\lambda w\left(\mathcal{I}_{a,b;c}g(w)\right)^{'}+(1-\lambda)\left(\mathcal{I}_{a,b;c}g(w)\right)}-1\right]=\phi(v(w)), \tag{4.4}$$

where $z,w\in\mathbb{U}$. Define the functions s and t as in Theorem 2.2 and then proceed similarly up to (2.7).
Expanding the LHS of (4.3) and (4.4), we obtain:

$$\begin{aligned}&1+\frac{1}{\gamma}\left[\frac{z\left(\mathcal{I}_{a,b;c}f(z)\right)^{'}+\lambda z^{2}\left(\mathcal{I}_{a,b;c}f(z)\right)^{''}}{\lambda z\left(\mathcal{I}_{a,b;c}f(z)\right)^{'}+(1-\lambda)\left(\mathcal{I}_{a,b;c}f(z)\right)}-1\right]\\&=1+\frac{1}{\gamma}\left[(1+\lambda)y_2a_2z+[2(1+2\lambda)y_3a_3-(1+\lambda)^2y_2^2a_2^2]z^2+\cdots\right]\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left[\frac{w\left(\mathcal{I}_{a,b;c}g(w)\right)^{'}+\lambda w^{2}\left(\mathcal{I}_{a,b;c}g(w)\right)^{''}}{\lambda w\left(\mathcal{I}_{a,b;c}g(w)\right)^{'}+(1-\lambda)\left(\mathcal{I}_{a,b;c}g(w)\right)}-1\right]\\&=1+\frac{1}{\gamma}\left[-(1+\lambda)y_2a_2w+[2(1+2\lambda)y_3(2a_2^2-a_3)-(1+\lambda)^2y_2^2a_2^2]w^2+\cdots\right].\end{aligned} \tag{4.6}$$

Now, using (2.6), (2.7), (4.5), (4.6) in (4.3) and (4.4) and then equating the coefficients of z, z^2, w, w^2; we get:

$$(1+\lambda)y_2a_2=\frac{\gamma B_1c_1}{2}, \tag{4.7}$$

$$[2(1+2\lambda)y_3a_3-(1+\lambda)^2y_2^2a_2^2]=\gamma\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right], \tag{4.8}$$

$$-(1+\lambda)y_2a_2=\frac{\gamma B_1d_1}{2}, \tag{4.9}$$

$$[2(1+2\lambda)y_3(2a_2^2-a_3)-(1+\lambda)^2y_2^2a_2^2]=\gamma\left[\frac{1}{2}B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}B_2d_1^2\right]. \tag{4.10}$$

From (4.7) and (4.9), we get:

$$c_1=-d_1 \tag{4.11}$$

and

$$8(1+\lambda)^2y_2^2a_2^2=\gamma^2B_1^2(c_1^2+d_1^2)=2\gamma^2c_1^2B_1^2. \tag{4.12}$$

Adding (4.8) and (4.10), we obtain:

$$\left[2(1+2\lambda)y_3-(1+\lambda)^2y_2^2\right]a_2^2=\frac{1}{4}\gamma\left[B_1(c_2+d_2)+(B_2-B_1)c_1^2\right]. \tag{4.13}$$

Which, on using (4.12), yields:

$$a_2^2=\frac{\gamma^2B_1^3(c_2+d_2)}{4\left[2(1+2\lambda)y_3-(1+\lambda)^2y_2^2\right]\gamma B_1^2+4(1+\lambda)^2y_2^2(B_1-B_2)}. \tag{4.14}$$

Clearly (4.12), (4.13) and (4.14) in light of (2.5) gives us the desired estimate on $|a_2|$ as asserted in (4.1).

Next, to find the estimate on $|a_3|$, subtracting (4.10) from (4.8) and then using (4.11), we get:

$$a_3=a_2^2+\frac{\gamma B_1(c_2-d_2)}{8(1+2\lambda)y_3}. \tag{4.15}$$

Using (4.12) in (4.15), we get:

$$a_3=\frac{\gamma^2c_1^2B_1^2}{4(1+\lambda)^2y_2^2}+\frac{\gamma B_1(c_2-d_2)}{8(1+2\lambda)y_3}. \tag{4.16}$$

Similarly, using (4.13) in (4.15), we get:

$$a_3=\frac{\gamma\left[B_1(c_2+d_2)+(B_2-B_1)c_1^2\right]}{4\left[2(1+2\lambda)y_3-(1+\lambda)^2y_2^2\right]}+\frac{\gamma B_1(c_2-d_2)}{8(1+2\lambda)y_3}.$$

Which, on simplification, yields:

$$a_3=\frac{\gamma B_1\left[c_2\left(4(1+2\lambda)y_3-(1+\lambda)^2y_2^2\right)+d_2\left((1+\lambda)^2y_2^2\right)\right]}{8(1+2\lambda)y_3\left[2(1+2\lambda)y_3-(1+\lambda)^2y_2^2\right]}+\frac{\gamma c_1^2(B_2-B_1)}{4\left[2(1+2\lambda)y_3-(1+\lambda)^2y_2^2\right]}. \tag{4.17}$$

Clearly (4.16) and (4.17) in light of (2.5) gives us the desired estimate on $|a_3|$ as asserted in (4.2). This completes the proof of Theorem 4.2. □

Taking $a=c$ and $b=1$ in Theorem 4.2, we get the class $\mathcal{S}_\Sigma(\lambda,\gamma;\phi)$, $0\le\lambda\le1$, $\gamma\in\mathbb{C}\setminus\{0\}$ defined and studied by Erhan Deniz [5]. Hence we get the following Corollary as an improvement in Theorem 2.1 given by Erhan Deniz [5].

Corollary 4.3. *Let $f(z) \in \Sigma$ given by (1.1) be in the class $\mathcal{S}_\Sigma(\lambda, \gamma; \phi)$. Then,*

$$|a_2| \leq min\left\{ \frac{|\gamma|B_1}{(1+\lambda)}, \sqrt{\frac{|\gamma|(B_1 + |B_2 - B_1|)}{1+2\lambda-\lambda^2}}, \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|\gamma(1+2\lambda-\lambda^2)B_1^2 + (1+\lambda)^2(B_1 - B_2)|}} \right\}$$

and

$$|a_3| \leq min\left\{ \frac{|\gamma|B_1}{2(1+2\lambda)} + \frac{\gamma^2 B_1^2}{(1+\lambda)^2}, \frac{|\gamma|(B_1 + |B_2 - B_1|)}{1+2\lambda-\lambda^2} \right\}.$$

Observe that for $\lambda = 0$ and $\gamma = 1$, we have the class $\mathcal{S}_\Sigma(0, 1; \phi) \equiv \mathcal{S}^*_\Sigma(1; \phi)$ and the Corollary 4.3 reduces to the Corollary 2.4. Also for $\lambda = 1$ and $\gamma = 1$ we have the class $\mathcal{S}_\Sigma(1, 1; \phi) \equiv \mathcal{C}_\Sigma(1; \phi)$ and the Corollary 4.3 reduces to the following Corollary.

Corollary 4.4. *(see [5]) Let $f(z) \in \Sigma$ given by (1.1) be in the class $\mathcal{C}_\Sigma(1; \phi)$. Then,*

$$|a_2| \leq min\left\{ \frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2 - B_1|}{2}}, \frac{B_1\sqrt{B_1}}{\sqrt{2\,|B_1^2 + 2(B_1 - B_2)|}} \right\}$$

and

$$|a_3| \leq min\left\{ \frac{B_1}{6} + \frac{B_1^2}{4}, \frac{B_1 + |B_2 - B_1|}{2} \right\}.$$

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran, S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.* **25**, 344–351 (2012).

[2] D. A. Brannan and J. G. Clunie (Editors), *Aspects of Contemporary Complex Analysis*, Proceedings of the NATO Advanced Study Institute (University of Durham, Durham; July 1âĂŞ20, 1979), Academic Press, New York and London (1980).

[3] D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, *Studia Univ. Babes-Bolyai Math.* **31** (2), 70–77 (1986).

[4] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer, New York (1983).

[5] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, *J. Classical Anal.* **2** (1), 49–60 (2013).

[6] A. W. Goodman, *Univalent Functions*, Vol. I, Polygonal Publishing House, Washington, New Jersey (1983).

[7] S. P. Goyal, O. Singh, R. Mukherjee, Certain results on a subclass of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, *Palestine J. Math.* **5** (1), 79–85 (2016).

[8] Yu. E. Hohlov, Hadamard convolutions, hypergeometric functions and linear operators in the class of univalent functions, *Dokl. Akad. Nauk Ukrain. SSR Ser. A* **7**, 25–27 (1984).

[9] Yu. E. Hohlov, Convolution operators that preserve univalent functions, *Ukrain. Mat. Zh.* **37**, 220–226 (1985).

[10] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* **18**, 63–68 (1967).

[11] U. H. Naik, A. B. Patil, On initial coefficient inequalities for certain new subclasses of bi-univalent functions, *J. Egyptian Math. Soc.* **25**, 291–293 (2017).

[12] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Rational Mech. Anal.* **32**, 100–112 (1969).

[13] A. B. Patil, U. H. Naik, Initial coefficient bounds for a general subclass of bi-univalent functions defined by Al-Oboudi differential operator, *J. Analysis* **23**, 111–120 (2015).

[14] A. B. Patil, U. H. Naik, Estimates on initial coefficients of certain subclasses of bi-univalent functions associated with quasi-subordination, *Global J. Mathematical Anal.* **5** (1), 6–10 (2017).

[15] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Rupercht, Göttingen (1975).

[16] H. M. Srivastava, D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, *J. Egyptian Math. Soc.* **23** (2), 242–246 (2015).

[17] H. M. Srivastava, S. Bulut, M. Căglar, N. Yăgmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat* **27** (5), 831–842 (2013).

[18] H. M. Srivastava, S. S. Eker, R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, *Filomat* **29** (8), 1839–1845 (2015).

[19] H. M. Srivastava, S. B. Joshi, S. S. Joshi, H. Pawar, Coefficient estimates for certain subclasses of meromorphically bi-univalent functions, *Palest. J. Math.* **5** (Special Issue 1), 250–258 (2016).

[20] H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* **23**, 1188–1192 (2010).

[21] H. M. Srivastava, G. Murugusundaramoorthy, N. Magesh, Certain subclasses of bi-univalent functions associated with the Hohlov operator, *Global J. Math. Anal.* **1** (2), 67–73 (2013).

[22] T. S. Taha, *Topics in Univalent Function Theory*, Ph.D. Thesis, University of London (1981).

[23] H. Tang, G-T. Deng, N. Magesh, S-H. Li, Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, *J. Ineq. and Appl.* 2013:317 (2013).

[24] Q.-H. Xu, Y.-C. Gui, H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* **25**, 990–994 (2012).

[25] Q.-H. Xu, H.-G. Xiao, H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.* **218**, 11461–11465 (2012).

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