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Some Properties of Q-Fuzzy Ideals in po-Γ-Semigroups

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Abstract We introduce Q-fuzzy ideals in po-Γ-semigroups. A characterization of regular po-Γ-semigroups in terms of Q-fuzzy ideals has been studied. We prove that in po-Γ-groupoids, the Q-fuzzy right (resp. left) ideals are Q-fuzzy quasi-ideals and in po-Γ-semigroups, the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals. Finally, we prove that if S is a po-Γ-semigroup, then a Q-fuzzy subset f is a Q-fuzzy quasi-ideal of S if and only if there exists a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.

1 Introduction and Basic Definitions

A fuzzy subset μ of a nonempty set X is a function $\mu : X \to [0, 1]$. L. A. Zadeh[4] gave the notion of a fuzzy set. Zadeh's work is a paradigm shift that earned popularity among researchers of various subjects of study and its successful deployment has assured its acceptance all around the world. Fuzzy set theory was inspired by the assumption that classical sets were not precise concepts in dealing with the concrete problems, since every object experienced in the physical situation brings some degree of fuzziness. Fuzzy set theoretic approach was given by A. Rosenfeld[1] for groups. Rosenfeld's work opened up a new branch of study: fuzzy abstract algebra. Many authors like N. Kuroki [8], [9], [10], J. Ahsan and several others [2], [3], [5] applied the notion of fuzzy sets in groups, semigroups, ordered groupoids, rings, semirings, fuzzy ideals, fuzzy quasi-ideals in semigroups, fuzzy bi-ideals in semigroups. Fuzzy ideals in ordered groupoids and fuzzy bi-ideals in ordered semigroups have been introduced by N. Kehayopulu and M. Tsingelis [6] and [7]. In an endeavor to investigate how similar is the theory of po-Γ-semigroups based on Q-fuzzy-ideals (Q-fuzzy right, Q-fuzzy-left, Q-fuzzy quasi- etc.) with the theory of semigroups based on ideals, using the concepts of Q-fuzzy right ideal, Q-fuzzy left ideal,Q-fuzzy quasi-ideal and Q-fuzzy bi-ideal, we show that in po-Γ-groupoids the Q-fuzzy right (resp. Q-fuzzy left) ideals are Q-fuzzy quasi-ideals, in po-Γ-semigroups, the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals, and in regular po-Γ-semigroups the Q-fuzzy quasi-ideals and the Q-fuzzy bi-ideals are the same concepts. Moreover, we show that if S is a po-Γ-semigroup, then a Q-fuzzy subset f is a Q-fuzzy quasi-ideal of S if and only if there exist a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.

The concept of po-Γ-semigroup was introduced by Y. I. Kwon and S. K. Lee[11]. A po-Γ-semigroup is an ordered set $(S, \Gamma, \cdot)$ at the same time a Γ-semigroup $(S, \Gamma, \cdot)$ such that $a \leq b \Rightarrow a \cdot \alpha \cdot x \leq b \cdot \alpha \cdot x$ and $x \cdot \beta \cdot a \leq x \cdot \beta \cdot b$ for all $a, b, x \in S$ and $\alpha, \beta \in \Gamma$. A function μ from $S \times Q$ to the real closed interval [0, 1] is called Q-fuzzy subset of S, where Q is a nonempty set. The po-Γ-semigroup S itself is a Q-fuzzy subset of S. Its characteristic function, also denoted by S, is given as follows:

$$S : S \times Q \to [0, 1] | (s, q) \mapsto S(s, q) := 1,$$

for all $s \in S$ and $q \in Q$.

Let f and g be two Q-fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ implies that

$f(s,q) \leq g(s,q)$, for all $s \in S$ and $q \in Q$, $f \cap g$ and $f \cup g$ are defined by:

$$(f \cap g)(s,q) = min\{f(s,q), g(s,q)\} = f(s,q) \wedge g(s,q),$$

$$(f \cup g)(s,q) := max\{f(s,q), g(s,q)\} = f(s,q) \vee g(x,q),$$

for all $s \in S$ and $q \in Q$.
Let S and Q be a po-Γ-semigroup and a nonempty set, respectively. For $s \in S$, we define $A_s := \{(y,z) \in S \times S : s \leq y\gamma z\}$. The product $f \circ g$ of f and g is defined as follows:

$$(\forall s \in S, \forall q \in Q)(f \circ g)(s,q) = \begin{cases} \bigvee_{(u,v)\in A_s} \{min\{f(u,q), g(v,q)\}, if A_s \neq \emptyset \\ 0, if A_s = \emptyset \end{cases}.$$

The characteristic function of $A \times Q$ is denoted by $f_{A\times Q}$, that is, the mapping from $S \times Q$ into [0,1] defined as follows:

$$A_{A\times Q}(s,q) := \begin{cases} 1, if s \in A \times Q, \\ 0, if\ s \notin A \times Q, \end{cases}$$

for all $(s,q) \in A \times Q$.

Definition 1.1. A nonempty Q-fuzzy subset μ of a po-Γ-groupoid S is called a Q-fuzzy left(resp. right) ideal of S if
(i) $x \leq y \Rightarrow \mu(x,q) \geq \mu(y,q)$, and
(ii) $\mu(x\gamma y,q) \geq \mu(x,q)(resp. \mu(x\gamma y,q) \geq \mu(y,q))$,
for all $x,y \in S$, for all $q \in Q$, and for all $\gamma \in \Gamma$.

Definition 1.2. A nonempty Q-fuzzy subset μ of a po-Γ-groupoid S is called a Q-fuzzy quasi-ideal of S if
(i) $x \leq y \Rightarrow \mu(x,q) \geq \mu(y,q)$, and
(ii) $(\mu \circ S) \cap (S \circ \mu) \subseteq \mu$,
for all $x,y \in S$ and for all $q \in Q$.

Definition 1.3. A nonempty Q-fuzzy sub-Γ-semigroup μ of a po-Γ-semigroup S is called a Q-fuzzy bi-ideal of S if
(i) $x \leq y \Rightarrow \mu(x,q) \geq \mu(y,q)$, and
(ii) $\mu(x\gamma_1 y\gamma_2 z,q) \geq min\{\mu(x,q), \mu(z,q)\}$,
for all $x,y \in S$, for all $q \in Q$, and $\gamma_1, \gamma_2 \in \Gamma$.

The next section provides the elaborated notions of Q-fuzzy ideals, Q-fuzzy quasi-ideals, and Q-fuzzy bi-ideals over a po-Γ-semigroup. Many interesting results are obtained through the theory. It is tried to give a complete investigation of Q-fuzzy ideal theory in po-Γ-semigroups.

2 Main Results

Theorem 2.1. *If S is a po-Γ-groupoid and Q is a nonempty set, then the Q-fuzzy right (resp. left) ideals of S are Q-fuzzy quasi-ideals of S.*

Proof. Let μ be a Q-fuzzy right ideal of S and $s \in S$, $q \in Q$. We have the following:

$$((\mu \circ S) \cap (S \circ \mu))(s,q) = min\{(\mu \circ S)(s,q), (S \circ \mu)(s,q)\}.$$

If $A_s = \emptyset$, then we obtain $(\mu \circ S)(s,q) = 0 = (S \circ \mu)(s,q)$ and, since μ is a Q-fuzzy right ideal of S, we have min $\{(\mu \circ S)(s,q), (S \circ \mu)(s,q)\} = 0 \leq \mu(s,q)$.
If $A_s \neq \emptyset$, then we obtain

$$(\mu \circ S)(s,q) = \bigvee_{(x,y)\in A_s} \{min\{\mu(x,q), S(y,q)\}\}.$$

If $(x, y) \in A_s$, then $s \leq x\gamma y$ and $\mu(s, q) \geq \mu(x\gamma y, q) \geq \mu(x, q)$ =min $\{\mu(x, q), S(y, q)\}$ for $\gamma \in \Gamma$. Hence, we obtain the following:

$$\begin{aligned}\mu(s,q) &\geq \bigvee_{(x,y)\in A_s} \{min\{\mu(x,q)\}\}\\ &\geq min\{(\mu\circ S)(s,q),(S\circ\mu)(s,q)\}\\ &= ((\mu\circ S)\cap(S\circ\mu))(s,q)\end{aligned}$$

Therefore, μ is a Q-fuzzy quasi-ideal of S. This completes the proof. □

Theorem 2.2. *Let S be a po-Γ-semigroup and Q be a nonempty set, then the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals of S.*

Proof. Let μ be a Q-fuzzy quasi-ideal of S, $x, y, z \in S$, $q \in Q$ and $\gamma_1, \gamma_2 \in \Gamma$. Then we have the following: $\mu(x\gamma_1 y\gamma_2 z, q) \geq ((\mu \circ S) \cap (S \circ \mu))(x\gamma_1 y\gamma_2 z, q) = min\{(\mu \circ S)(x\gamma_1 y\gamma_2 z, q), (S \circ \mu)(x\gamma_1 y\gamma_2 z, q)\}$. As $(x, y\gamma z) \in A_{x\gamma_1 y\gamma_2 z}$, we have for $\gamma, \gamma_1, \gamma_2 \in \Gamma$.

$$\begin{aligned}(\mu\circ S)(x\gamma_1y\gamma_2z,q) &= \bigvee_{(x,y)\in A_{x\gamma_1y\gamma_2z}} \{min\{\mu(u,q),S(v,q)\}\}\\ &\geq min\{\mu(x,q),S(y\gamma z,q)\}\\ &= \mu(x,q)\end{aligned}$$

As $(x\gamma y, z) \in A_{x\gamma_1 y\gamma_2 z}$, we have the following:

$$\begin{aligned}(S\circ\mu)(x\gamma_1y\gamma_2z,q) &= \bigvee_{(u,v)\in A_{x\gamma_1y\gamma_2z}} \{min\{S(u,q),\mu(v,q)\}\}\\ &\geq min\{S(x\gamma y,q),\mu(z,q)\}\\ &= \mu(z,q)\end{aligned}$$

Therefore, we have $\mu(x\gamma_1 y\gamma_2 z, q) \geq min\{(\mu\circ S)(x\gamma_1 y\gamma_2 z, q), (S\circ\mu)(x\gamma_1 y\gamma_2 z, q)\} \geq min\{\mu(x, q), \mu(z, q)\}$.
Hence, μ is a Q-fuzzy bi-ideal of S. This completes the proof. □
A po-Γ-semigroup S is called regular if for any $s \in S$, there exists $x \in S$ such that $s \leq s\gamma_1 x\gamma_2 s$ for $\gamma_1, \gamma_2 \in \Gamma$.

Theorem 2.3. *Let S be a regular po-Γ-semigroup and Q a nonempty set, then the Q-fuzzy quasi-ideals and Q-fuzzy bi-ideals coincide.*

Proof. Let μ be a Q-fuzzy bi-ideal of S, and $s \in S, q \in Q$. We will show the following:

$$((\mu\circ S)\cap(S\circ\mu))(s,q) \leq \mu(s,q) \tag{2.1}$$

We have the following:

$$((\mu\circ S)\cap(S\circ\mu))(s,q) = min\{(\mu\circ S)(s,q),(S\circ\mu)(s,q)\}.$$

As $A_s = \emptyset$, then as by Theorem 2.1, condition (2.1) is satisfied.
If $A_s \neq \emptyset$, then

$$(\mu\circ S)(s,q) = \bigvee_{(z,w)\in A_s} \{min\{\mu(z,q),S(w,q)\}\} \tag{2.2}$$

$$(S\circ\mu)(s,q) = \bigvee_{(u,v)\in A_s} \{min\{S(u,q),\mu(v,q)\}\} \tag{2.3}$$

Let $(\mu \circ S)(s, q) \leq \mu(s, q)$. Therefore, we have the following:

$$\begin{aligned}\mu(s,q) &\geq (\mu\circ S)(s,q)\\ &\geq min\{(\mu\circ S)(s,q),(S\circ\mu)(s,q)\}\\ &= ((\mu\circ S)\cap(S\circ\mu))(s,q),\end{aligned}$$

and so the condition (2.1) is satisfied. Let $(\mu \circ S)(s,q) > \mu(s,q)$. Then, by (2.2), there exists $(z,w) \in A_s$, such that

$$min\{\mu(z,q), S(w,q)\} > \mu(s,q) \tag{2.4}$$

Else, $\mu(s,q) \leq (\mu \circ S)(s,q)$, which is impossible. As $(z,w) \in A_s$, we have $z,w \in S$ and $s \leq z\gamma w$. In a similar fashion, from min$\{\mu(z,q), S(w,q)\} = \mu(z,q)$, by (2.4), we obtain the following:

$$\mu(z,q) > \mu(s,q) \tag{2.5}$$

We will show that $(S \circ \mu)(s,q) \leq \mu(s,q)$, then

$$min\{(\mu \circ S)(s,q), (S \circ \mu)(s,q)\} \leq (S \circ \mu)(s,q) \leq \mu(s,q).$$

therefore, $((\mu \circ S) \cap (S \circ \mu))(s,q) \leq \mu(s,q)$, and so the condition (2.1) is satisfied. By (2.3), it is enough to show that

$$min\{S(u,q), \mu(v,q)\} \leq \mu(s,q), \forall (u,v) \in A_s.$$

Suppose $(u,v) \in A_s$. Then $s \leq u\gamma v$ for some $u,v \in S$ and $\gamma \in \Gamma$. As S is regular, there exists $s \in S$ such that $s \leq x\gamma_1 w\gamma_2 s\gamma_3 u\gamma_4 v$. As μ is a Q-fuzzy bi-ideal of S, we have the following:

$$min\{S(u,q), \mu(v,q)\} \leq \mu(s,q), \forall (u,v) \in A_s,$$

and, we have the following:

$$\mu(s,q) \geq \mu(z\gamma_1 w\gamma_2 s\gamma_3 u\gamma_4 v, q) \geq min\{\mu(z,q), \mu(v,q)\}.$$

If min$\{\mu(z,q), \mu(v,q)\} = \mu(z,q)$, then $\mu(z,q) \leq \mu(s,q)$ which is impossible by (2.5). Therefore, we have min$\{\mu(z,q), \mu(v,q)\} = \mu(v,q)$, then $\mu(s,q) \geq \mu(v,q) = min\{S(u,q), \mu(v,q)\}$. This completes the proof. □

Next, we prove that the Q-fuzzy quasi-ideals of a po-Γ-semigroup are intersections of Q-fuzzy right and Q-fuzzy left ideals.

Lemma 2.4. *Let S and Q be a po-Γ-semigroup and a nonempty set, respectively. Let μ be a Q-fuzzy subset of S. Then the following assertions are true:*
(i) $(S \circ \mu)(x\gamma y, q) \geq \mu(y,q)$ for all $x,y \in S$, $q \in Q$ and $\gamma \in \Gamma$,
(ii) $(S \circ \mu)(x\gamma y, q) \geq (S \circ \mu)(y,q)$ for all $x,y \in S$, $q \in Q$ and $\gamma \in \Gamma$.

Proof. (i) Let $x,y \in S$, $q \in Q$ and $\gamma \in \Gamma$. As $(x,y) \in A_{x\gamma y}$, we have the following:

$$(S \circ \mu)(x\gamma y, q) = \bigvee_{(x\gamma y) \in A_{x\gamma y}} \{min\{S(w,q), \mu(z,q)\}\} \geq min\{S(x,q), \mu(y,q)\} = \mu(y,q).$$

(ii) Let $x,y \in S$ and $q \in Q$. If $A_y = \emptyset$, then $(S \circ \mu)(y,q) = 0$. As $(S \circ \mu)$ is a Q-fuzzy subset of S, we have $(S \circ \mu)(x\gamma y, q) \geq 0 = (S \circ \mu)(y,q)$. If $A_x \neq \emptyset$, then

$$(S \circ \mu)(y,q) = \bigvee_{(w,z) \in A_y} \{min\{S(w,q), \mu(z,q)\}\}.$$

Also,

$$(S \circ \mu)(x\gamma y, q) \geq min\{S(w,q), \mu(z,q)\}, \forall (w,z) \in A_y \tag{2.6}$$

Suppose $(w,z) \in A_y$. As $(x,y) \in A_{x\gamma y}$, we obtain the following:

$$(S \circ \mu)(x\gamma y, q) = \bigvee_{(s,t) \in A_{x\gamma y}} \{min\{S(s,q), \mu(t,q)\}\}.$$

As $(w,z) \in A_y$, we obtain $y \leq w\gamma z$, then $x\gamma_1 y \leq x\gamma_2 w\gamma_3 z$, and $(x\gamma w, z) \in A_{x\gamma y}$ for $\gamma, \gamma_1, \gamma_2 \in \Gamma$. Hence, we have the following:

$$(S \circ \mu)(x\gamma y, q) \geq min\{S(x\gamma w, q), \mu(z,q)\} = \mu(z,q) = min\{S(w,q), \mu(z,q)\}.$$

By (2.6), we have the following:

$$(S \circ \mu)(x\gamma y, q) \geq \bigvee_{(w,z) \in A_y} \{min\{S(w,q), \mu(z,q)\}\} = (S \circ \mu)(y,q).$$

The proof is complete. □
In a similar fashion, we can prove the following:

Lemma 2.5. *Suppose S and Q are po-Γ-semigroup and a nonempty set, respectively. Let μ be a Q-fuzzy subset of S. Then the following assertions are true:*
(i) $(S \circ \mu)(x\gamma y, q) \geq \mu(x,q)$ for all $x, y \in S$, $q \in Q$ and $\gamma \in \Gamma$,
(ii) $(S \circ \mu)(x\gamma y, q) \geq (S \circ \mu)(x,q)$ for all $x, y \in S$, $q \in Q$ and $\gamma \in \Gamma$.

Lemma 2.6. *Suppose S and Q are po-Γ-semigroup and a nonempty set, respectively. Let μ be a Q-fuzzy subset of S and $x \leq y$. Then we have $(S \circ \mu)(x,q) \geq (S \circ \mu)(y,q)$ for all $q \in Q$.*

Proof. Let $x, y \in S$ and $q \in Q$. Then, if $A_y = \emptyset$, then $(S \circ \mu(y,q)) = 0$. As $S \circ \mu$ is a Q-fuzzy subset of S, we have $(S \circ \mu)(x,q) \geq 0$, then $(S \circ \mu)(x,q) \geq (S \circ \mu)(y,q)$. If $A_y \neq \emptyset$, then

$$(S \circ \mu)(y,q) = \bigvee_{(w,z) \in A_y} \{min\{S(w,q), \mu(z,q)\}\} = \bigvee_{(w,z) \in A_y} \{\mu(z,q)\}.$$

Also,

$$(S \circ \mu)(x,q) \geq \mu(z,q), \forall (w,z) \in A_y. \tag{2.7}$$

Let $(w,z) \in A_y$. As $x \leq y \leq w\gamma z$ for $\gamma \in \Gamma$, we obtain $(w,z) \in A_x$. Then, we have the following:

$$(S \circ \mu)(x\gamma y, q) = \bigvee_{(s,t) \in A_{x\gamma y}} \{min\{S(s,q), \mu(t,q)\}\} \geq min\{S(w,q), \mu(z,q)\} = \mu(z,q).$$

Therefore, by (2.7), we obtain the following:

$$(S \circ \mu)(x,q) \geq \bigvee_{(w,z) \in A_y} \{\mu(z,q)\} = (S \circ \mu)(y,q).$$

Lemma 2.7. *Suppose S and Q are po-Γ-semigroup and a nonempty set, respectively. Suppose μ is a Q-fuzzy subset of S and $x \leq y$. Then we have $(\mu \circ S)(x,q) \geq (f \circ S)(y,q)$ for all $q \in Q$.*

Lemma 2.8. *Suppose S and Q are po-Γ-semigroup and a nonempty set, respectively. Let μ be a Q-fuzzy subset of S such that $x \leq y$, we have $\mu(x,q) \geq \mu(y,q)$ for all $x, y \in S$, $q \in Q$. Then the Q-fuzzy subset $\mu \cup (S \circ \mu)$ is a Q-fuzzy left ideal of S.*

Proof. Let $x, y \in S$ and $q \in Q$. By Theorem 2.3, it follows that $(\mu \cup (S \circ \mu))(x\gamma y, q) \geq (\mu \cup (S \circ \mu))(y,q)$ for $\gamma \in \Gamma$. Suppose $x \leq y$. Then $(\mu \cup (S \circ \mu))(x,q) \geq (\mu \cup (S \circ \mu))(y,q)$. As μ is a Q-fuzzy subset of S and $x \leq y$, by Lemma 2.4, we obtain $(S \circ \mu)(x,q) \geq (S \circ \mu)(y,q)$ and, by the assumption, $\mu(x,q) \geq \mu(y,q)$. Then

$$\begin{aligned}(\mu \cup (S \circ \mu))(x,q) &= max\{\mu(x,q), (S \circ \mu)(x,q)\} \\ &\geq max\{\mu(y,q), (S \circ \mu)(y,q)\} \\ &= (\mu \cup (S \circ \mu))(y,q).\end{aligned}$$

In a similar fashion, we can prove the following:

Lemma 2.9. *Suppose S and Q are a po-Γ-semigroup and a nonempty set, respectively. Let μ be a Q-fuzzy subset of S such that $x \leq y$, we have $\mu(x,q) \geq \mu(y,q)$ for all $x, y \in S$, $q \in Q$. Then the Q-fuzzy subset $\mu \cup (\mu \circ S)$ is a Q-fuzzy right ideal of S.*

Lemma 2.10. *Suppose S and f, g, h are po-Γ-semigroup and Q-fuzzy subsets of S, respectively. Then*

$$f \cap (g \cup h) = (f \cap g) \cup (f \cap h).$$

Proof. Suppose $s \in S$ and $q \in Q$. Then we obtain the following:

$$\begin{aligned}(f \cap (g \cup h))(s,q) &= min\{f(s,q), (g \cup h)(s,q)\} \\ &= min\{f(s,q), max\{g(s,q), h(s,q)\}\} \\ &= max\{min\{f(s,q), g(s,q)\}, min\{f(s,q), h(s,q)\} \\ &= max\{(f \cap g)(s,q), (f \cap h)(s,q)\} \\ &= ((f \cap g) \cup (f \cap h))(s,q).\end{aligned}$$

□

Corollary 2.11. *Suppose S and Q are po-Γ-semigroups. Then the set of all Q-fuzzy subsets of S is a distributive lattice.*

Theorem 2.12. *Suppose S and Q are po-Γ-semigroup and a nonempty set, respectively. Then a Q-fuzzy subset f of S is a Q-fuzzy quasi-ideal of S if and only if there exists a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.*

Proof. By Lemma 2.8 and Lemma 2.9, $f \cup (S \circ f)$ is a Q-fuzzy left ideal and $f \cup (f \circ S)$ is a Q-fuzzy right ideal of S. Moreover, we have the following:

$$f = (f \cup (S \circ f)) \cap (f \cup (f \circ S)).$$

By Corollary 2.11, we obtain the following:

$$\begin{aligned}(f \cup (S \circ f)) \cap (f \cup (f \circ S)) &= ((f \cup (S \circ f)) \cap f) \cup ((f \cup (S \circ f)) \cap (f \circ S)) \\ &= (f \cap f) \cup ((S \circ f) \cap f) \cup (f \cap (f \circ S)) \cup ((S \circ f) \cap (f \circ S)) \\ &= f \cup ((S \circ f) \cap f) \cup (f \cap (f \circ S)) \cup ((S \circ f) \cap (f \circ S))\end{aligned}$$

As f is a Q-fuzzy quasi-ideal of S, we have $(f \circ S) \cap (S \circ f) \subseteq f$. Also, $(S \circ f) \cap f \subseteq f$ and $f \cap (f \circ S) \subseteq f$. Hence

$$(f \cup (S \circ f)) \cap (f \cup (f \circ S)) = f.$$

Conversely, suppose $s \in S$ and $q \in Q$. Then

$$((f \circ S) \cap (S \circ f))(s,q) \leq f(s,q). \tag{2.8}$$

Indeed, $((f \circ S) \cap (S \circ f))(s,q) = min\{(f \circ S)(s,q), (S \circ f)(s,q)\}$. As $A_s = \emptyset$, then $(f \circ S)(s,q) = 0 = (S \circ f)(s,q)$. Therefore, condition (2.8) is satisfied. If $A_s \neq \emptyset$, then we have:

$$(f \circ S)(s,q) = \bigvee_{(y,z) \in A_s} \{min\{f(y,q), S(z,q)\}\} = \bigvee_{(y,z) \in A_s} \{f(y,q)\}. \tag{2.9}$$

We have the following:

$$f(y,q) \leq h(s,q), \forall (y,z) \in A_s. \tag{2.10}$$

In fact, for $(y,z) \in A_s$, we have $s \leq y\gamma z$ and $h(s,q) \geq h(y,q)$ as h is a Q-fuzzy left ideal of S. Therefore, by (2.9) and (2.10), we get

$$(f \circ S)(s,q) = \bigvee_{(y,z) \in A_s} \{f(y,q)\} \leq h(s,q).$$

In a similar fashion, we obtain $(S \circ f)(s,q) \leq g(s,q)$. Hence

$$\begin{aligned}((f \circ S) \cap (S \circ f))(s,q) &= min\{(f \circ S)(s,q), (S \circ f)(s,q)\} \\ &\leq min\{h(s,q), g(s,q)\} \\ &= (h \cap g)(s,q) \\ &= f(s,q)\end{aligned}$$

Hence (2.8) is established.

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