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AN IMPLICIT ITERATION PROCESS FOR I-NONEXPANSIVE MAPPINGS IN KOHLENBACH HYPERBOLIC SPACES

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Abstract The first goal of this paper is to propose a composite implicit iteration process for a finite family of I-nonexpansive mappings in hyperbolic spaces. Next, some strong and Δ-convergence theorems are established using the proposed iteration process. New results are obtained as corollaries to the convergence theorems. Finally, we exhibit two finite families of the mappings under consideration.

1 Introduction and Preliminaries

Let K be a nonempty subset of a metric space X. The mapping $T : K \to K$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$.

In [1], Shahzad defines I-nonexpansive mappings in Banach spaces essentially as follows: given two mappings $T, I : K \to K$, T is called I-nonexpansive if $d(Tx, Ty) \leq d(Ix, Iy)$ for all $x, y \in K$.

In what follows, we set $J = \{1, 2, ..., N\}$ for the set of first N natural numbers and take $\{\alpha_n\}$, $\{\beta_n\}$ sequences in $(0, 1)$.

Given x_0 in K (a subset of Banach space), the Mann iteration process defined for a nonexpansive mapping as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T x_{n-1},\ n \geq 1. \tag{1.1}$$

Xu and Ori [19] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in J\}$.

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n,\ n \geq 1, \tag{1.2}$$

where $T_n = T_{n(modN)}$ and the $modN$ function takes values in J.

In 2007, Su and Li [20] introduced the composite implicit iteration process for finite family of strictly pseudocontractive maps defined as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n \left[\beta_n x_{n-1} + (1 - \beta_n) T_n x_n\right],\ n \geq 1, \tag{1.3}$$

where $T_n = T_{n(modN)}$.

In [2], Rhoades and Temir showed that the Mann iteration process converges weakly to a common fixed point of T and I in a Banach space by taking the map T to be I-nonexpansive. Actually, they proved the following theorems.

Theorem 1.1. (Rhoades and Temir [2]) *Let K be a closed convex bounded subset of a uniformly convex Banach space X which satisfies Opial's condition, and let T, I be self-mappings of K with T be an I-nonexpansive mapping, I be a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of Mann iterates converges weakly to common fixed point of $F(T) \cap F(I)$.*

There are numerous papers dealing with the convergence of different iterative techniques for these mappings and generalization of the class of I-nonexpansive mappings in Banach spaces (see, for example, [3, 4, 5, 6, 7] and the references therein).

Motivated by the iteration process (1.3) of Su and Li [20], in this paper we define a new modified composite implicit iteration process for a finite family of I_i-nonexpansive mappings $\{T_i : i \in J\}$ and a finite family of nonexpansive mappings $\{I_i : i \in J\}$ in hyperbolic spaces as follows:

$$\begin{aligned} x_n &= W\left(x_{n-1}, T_n y_n, \alpha_n\right), \\ y_n &= W\left(x_{n-1}, I_n x_n, \beta_n\right), \; n \geq 1 \end{aligned} \tag{1.4}$$

where $T_n = T_{n(modN)}$ and $I_n = I_{n(modN)}$.

Different notions of *hyperbolic space* [12, 13, 14, 15] can be found in the literature. We work in the setting of hyperbolic spaces as introduced by Kohlenbach [11], which are slightly more restrictive than the spaces of hyperbolic type [12] by (W4), but more general then the concept of hyperbolic space from [15].

Definition 1.2. (Kohlenbach [11]) A hyperbolic space is a triple (X, d, W) where (X, d) is a metric space and $W : X^2 \times [0, 1] \to X$ is a mapping such that

W1. $d\left(u, W\left(x, y, \alpha\right)\right) \leq \left(1 - \alpha\right) d\left(u, x\right) + \alpha d\left(u, y\right)$

W2. $d\left(W\left(x, y, \alpha\right), W\left(x, y, \beta\right)\right) = \left|\alpha - \beta\right| d\left(x, y\right)$

W3. $W\left(x, y, \alpha\right) = W\left(y, x, \left(1 - \alpha\right)\right)$

W4. $d\left(W\left(x, z, \alpha\right), W\left(y, w, \alpha\right)\right) \leq \left(1 - \alpha\right) d\left(x, y\right) + \alpha d\left(z, w\right)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

If (X, d, W) satisfies only (W1), then it coincides with the convex metric space introduced by Takahashi [16]. A subset K of a hyperbolic space X is convex if $W\left(x, y, \alpha\right) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

Definition 1.3. A hyperbolic space (X, d, W) is said to be *uniformly convex* [17] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{aligned} d\left(x, u\right) &\leq r \\ d\left(y, u\right) &\leq r \\ d\left(x, y\right) &\geq \varepsilon r \end{aligned}\right\} \Rightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq \left(1 - \delta\right) r.$$

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta\left(r, \varepsilon\right)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called *modulus of uniform convexity*. We call η *monotone* if it decreases with r (for a fixed ε).

The notion of Δ-convergence in general metric spaces was introduced by Lim [8] in 1976. Kirk and Panyanak [9] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting.

To give the definition of Δ-*convergence*, we first recall some notations.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r\left(., \{x_n\}\right) : X \to [0, \infty)$ by $r\left(x, \{x_n\}\right) = \limsup_{n \to \infty} d\left(x, x_n\right)$.

Then the *asymptotic radius* $\rho = r\left(\{x_n\}\right)$ of $\{x_n\}$ is defined by $\rho = \inf\left\{r\left(x, \{x_n\}\right) : x \in X\right\}$ and the *asymptotic center* of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined by $A_K\left(\{x_n\}\right) = \left\{x \in X : r\left(x, \{x_n\}\right) \leq r\left(y, \{x_n\}\right) \text{ for any } y \in K\right\}$. If the asymptotic center is taken with respect to X, then it is simply denoted by $A\left(\{x_n\}\right)$.

A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write Δ -$\lim_n x_n = x$ and call x as Δ -limit of $\{x_n\}$.

The proofs of the following lemmas can be found in Leustean [18] and Khan et al. [10].

Lemma 1.4. [18] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Lemma 1.5. [10] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \le r$, $\limsup_{n\to\infty} d(y_n, x) \le r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \ge 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 1.6. [10] *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{n\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{n\to\infty} y_m = y$.*

2 Main Results

Denote by F the set of common fixed points of the finite families of mappings $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$.

Let X be a hyperbolic space, K be a nonempty closed convex subset of X. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings. Let $\{x_n\}$ be defined by (1.4). Then $x_1 = W(x_0, T_1W(x_0, I_1x_1, \beta_1), \alpha_1)$. Define a mapping $G_1 : K \to K$ by: $G_1x = W(x_0, T_1W(x_0, I_1x, \beta_1), \alpha_1)$ for all $x \in K$. Existence of x_1 is guaranteed if G_1 has a fixed point. Now for any $u, v \in K$, we have

$$\begin{aligned}
d(G_1u, G_1v) &= d(W(x_0, T_1W(x_0, I_1u, \beta_1), \alpha_1), W(x_0, T_1W(x_0, I_1v, \beta_1), \alpha_1)) \\
&\le \alpha_1 d(T_1W(x_0, I_1u, \beta_1), T_1W(x_0, I_1v, \beta_1)) \\
&\le \alpha_1 d(I_1W(x_0, I_1u, \beta_1), I_1W(x_0, I_1v, \beta_1)) \\
&\le \alpha_1 d(W(x_0, I_1u, \beta_1), W(x_0, I_1v, \beta_1)) \\
&\le \alpha_1\beta_1 d(I_1u, I_1v) \\
&\le \alpha_1\beta_1 d(u, v)
\end{aligned}$$

Since $\alpha_1\beta_1 < 1$, G_1 is a contraction. By Banach contraction principle, G_1 has a unique fixed point. Thus the existence of x_1 is established. Similarly, the existence of $x_2, x_3, \ldots$ is established. Thus the implicit iteration process (1.4) is well defined.

We need the following lemma in order to prove our main theorems.

Lemma 2.1. *Let K be a nonempty closed convex subset of a hyperbolic space X. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \ne \emptyset$. Then for the sequence $\{x_n\}$ defined in (1.4), we have $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in F$.*

Proof. Let $p \in F$. From (1.4), we have

$$\begin{aligned}
d(y_n, p) &= d(W(x_{n-1}, I_nx_n, \beta_n), p) \\
&\le (1-\beta_n) d(x_{n-1}, p) + \beta_n d(I_nx_n, p) \\
&\le (1-\beta_n) d(x_{n-1}, p) + \beta_n d(x_n, p). \qquad (2.1)
\end{aligned}$$

By (2.1) and (1.4), we obtain

$$\begin{aligned}
d(x_n, p) &= d(W(x_{n-1}, T_ny_n, \alpha_n), p) \\
&\le (1-\alpha_n) d(x_{n-1}, p) + \alpha_n d(T_ny_n, p) \\
&\le (1-\alpha_n) d(x_{n-1}, p) + \alpha_n d(I_ny_n, p) \\
&\le (1-\alpha_n) d(x_{n-1}, p) + \alpha_n d(y_n, p) \\
&\le (1-\alpha_n) d(x_{n-1}, p) + \alpha_n [(1-\beta_n) d(x_{n-1}, p) + \beta_n d(x_n, p)] \\
&\le (1-\alpha_n\beta_n) d(x_{n-1}, p) + \alpha_n\beta_n d(x_n, p).
\end{aligned}$$

Consequently, we have

$$d(x_n, p) \le d(x_{n-1}, p). \qquad (2.2)$$

Thus $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. □

Lemma 2.2. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ defined in (1.4), we have*

$$\lim_{n\to\infty} d(x_n, T_l x_n) = \lim_{n\to\infty} d(x_n, I_l x_n) = 0 \text{ for each } l = 1, 2, \cdots, N.$$

Proof. In view of Lemma 2.1, we obtain that the limit of the sequence $\{d(x_n, p)\}$ exits for each $p \in F$. Next, we assume that $\lim_{n\to\infty} d(x_n, p) = c$, for some $c > 0$. It follows from (1.4) that

$$\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(W(x_{n-1}, T_n y_n, \alpha_n), p) = c. \tag{2.3}$$

By means of $\lim_{n\to\infty} d(x_n, p) = c$ and nonexpansivity of T_i, we get

$$\begin{aligned}
\limsup_{n\to\infty} d(T_n y_n, p) &\leq \limsup_{n\to\infty} d(I_n y_n, p) \leq \limsup_{n\to\infty} d(y_n, p) \\
&= \limsup_{n\to\infty} d(W(x_{n-1}, I_n x_n, \beta_n), p) \\
&\leq \limsup_{n\to\infty} [(1-\beta_n) d(x_{n-1}, p) + \beta_n d(I_n x_n, p)] \\
&\leq \limsup_{n\to\infty} [(1-\beta_n) d(x_{n-1}, p) + \beta_n d(x_n, p)] \\
&\leq c.
\end{aligned} \tag{2.4}$$

Now using (2.4) with $\lim_{n\to\infty} d(x_n, p) = c$ and applying Lemma 1.5 to (2.3), we get

$$\lim_{n\to\infty} d(x_{n-1}, T_n y_n) = 0. \tag{2.5}$$

From (1.4) and (2.5) we obtain

$$\begin{aligned}
d(x_n, x_{n-1}) &= d(W(x_{n-1}, T_n y_n, \alpha_n), x_{n-1}) \\
&\leq (1-\alpha_n) d(x_{n-1}, x_{n-1}) + \alpha_n d(T_n y_n, x_{n-1}) \\
&\to 0 \ (n \to \infty),
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(x_n, x_{n+l}) = 0, \ \forall l = 1, 2, \ldots, N. \tag{2.6}$$

Note that

$$d(x_n, T_n y_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n y_n).$$

Next, taking limit on both sides in the above inequality we get

$$\lim_{n\to\infty} d(x_n, T_n y_n) = 0. \tag{2.7}$$

Clearly,

$$\begin{aligned}
d(x_n, p) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n y_n) + d(T_n y_n, p) \\
&\leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n y_n) + d(I_n y_n, p) \\
&\leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n y_n) + d(y_n, p).
\end{aligned}$$

Taking lim inf on both sides in the above estimate, from (2.5) and (2.6) we have

$$c \leq \liminf_{n\to\infty} d(y_n, p). \tag{2.8}$$

Also, we get from (2.1)

$$\limsup_{n\to\infty} d(y_n, p) \leq c$$

so that (2.8) gives

$$\lim_{n\to\infty} d(y_n, p) = c. \tag{2.9}$$

Thus $c = \lim_{n\to\infty} d\left(y_n, p\right) = \lim_{n\to\infty} d\left(W\left(x_{n-1}, I_n x_n, \beta_n\right), p\right)$ gives by

$$\limsup_{n\to\infty} d\left(I_n x_n, p\right) \leq c$$

and Lemma 1.5 that

$$\lim_{n\to\infty} d\left(x_{n-1}, I_n x_n\right) = 0 \tag{2.10}$$

On the other hand,

$$d\left(x_n, I_n x_n\right) \leq d\left(x_n, x_{n-1}\right) + d\left(x_{n-1}, I_n x_n\right).$$

Thus we have

$$\lim_{n\to\infty} d\left(x_n, I_n x_n\right) = 0. \tag{2.11}$$

Further, observe that

$$\begin{aligned} d\left(y_n, x_{n-1}\right) &= d\left(W\left(x_{n-1}, I_n x_n, \beta_n\right), x_{n-1}\right) \\ &\leq \beta_n d\left(I_n x_n, x_{n-1}\right). \end{aligned}$$

By (2.10), we have

$$\lim_{n\to\infty} d\left(y_n, x_{n-1}\right) = 0. \tag{2.12}$$

Thus

$$\begin{aligned} d\left(x_n, T_n x_n\right) &\leq d\left(x_n, T_n y_n\right) + d\left(T_n y_n, T_n x_{n-1}\right) + d\left(T_n x_{n-1}, T_n x_n\right) \\ &\leq d\left(W\left(x_{n-1}, T_n y_n, \alpha_n\right), T_n y_n\right) + d\left(y_n, x_{n-1}\right) + d\left(x_{n-1}, x_n\right) \\ &\leq \left(1-\alpha_n\right) d\left(x_{n-1}, T_n y_n\right) + d\left(y_n, x_{n-1}\right) + d\left(x_{n-1}, x_n\right) \end{aligned}$$

together with (2.5), (2.6) and (2.12) implies that

$$\lim_{n\to\infty} d\left(x_n, T_n x_n\right) = 0. \tag{2.13}$$

Since, for each $l = 1, 2, \cdots, N$, we have

$$\begin{aligned} d\left(x_n, T_{n+l} x_n\right) &\leq d\left(x_n, x_{n+l}\right) + d\left(x_{n+l}, T_{n+l} x_{n+l}\right) + d\left(T_{n+l} x_{n+l}, T_{n+l} x_n\right) \\ &\leq d\left(x_n, x_{n+l}\right) + d\left(x_{n+l}, T_{n+l} x_{n+l}\right) + d\left(I_{n+l} x_{n+l}, I_{n+l} x_n\right) \\ &\leq 2d\left(x_n, x_{n+l}\right) + d\left(x_{n+l}, T_{n+l} x_{n+l}\right), \end{aligned} \tag{2.14}$$

it follows from (2.6) and (2.13) that

$$\lim_{n\to\infty} d\left(x_n, T_{n+l} x_n\right) = 0$$

for all $l \in J$. Thus we get

$$\lim_{n\to\infty} d\left(x_n, T_l x_n\right) = 0 \quad \text{for any} \quad l \in J. \tag{2.15}$$

Replacing T_{n+l} by I_{n+l} in the inequality (2.14), we get

$$\lim_{n\to\infty} d\left(x_n, I_l x_n\right) = 0 \tag{2.16}$$

for all $l \in J$. □

For further developments, we need the following concepts and technical result.

A sequence $\{x_n\}$ in a metric space X is said to be *Fejér monotone* with respect to K (a subset of X) if $d\left(x_{n+1}, p\right) \leq d\left(x_n, p\right)$ for all $p \in K$ and for all $n \geq 1$. A map $T : K \to K$ is semi-compact if any bounded sequence $\{x_n\}$ satisfying $d\left(x_n, Tx_n\right) \to 0$ as $n \to \infty$ has a convergent subsequence.

Lemma 2.3. [21] *Let K be a nonempty closed subset of a complete metric space (X, d) and $\{x_n\}$ be Fejér monotone with respect to K. Then $\{x_n\}$ converges to some $p \in K$ if and only if* $\lim_{n\to\infty} d\left(x_n, K\right) = 0$.

Lemma 2.4. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (1.4) converges strongly to $p \in F$ if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

Proof. It follows from (2.2) that $\{x_n\}$ is Fejér monotone with respect to F and $\lim_{n\to\infty} d(x_n, F)$ exists. Now applying the Lemma 2.3, we obtain the result. □

A mappings $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy the *Condition (A)* [24] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in K$.

Khan and Fukhar-ud-din [22], introduced the so-called *Condition (A')* and gave a slightly improved version of it in [23] as follows:

Two mappings $T, I : K \to K$ with $F(T) \cap F(I) \neq \emptyset$ are said to satisfy the *Condition (A')* if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that either $d(x, Tx) \geq f(d(x, F(T) \cap F(I)))$ or $d(x, Ix) \geq f(d(x, F(T) \cap F(I)))$ for all $x \in K$.

We can modify this definition for two finite families of mappings as follows. Let $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$ be two finite families of nonexpansive mappings of K with nonempty fixed points set F. These families are said to satisfy *Condition (B)* on K if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that either $\max\limits_{i\in J} d(x, T_i x) \geq f(d(x, F))$ or $\max\limits_{i\in J} d(x, I_i x) \geq f(d(x, F))$ for all $x \in K$. The Condition (B) reduces to the Condition (A') when $T_1 = T_2 = \cdots = T_N = T$ and $I_1 = I_2 = \cdots = I_N = I$.

Note that the Condition (A) is weaker than both the semicompactness of the mapping $T : K \to K$ and the compactness of its domain K, see [24]. Thus the Condition (A') is weaker than both the semicompactness of the mappings $T, I : K \to K$ and the compactness of their domain K. In this direction Condition (B) is weaker than both the semicompactness of $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$ and the compactness of their domain K.

We are now ready to state and prove our strong convergence theorems.

Theorem 2.5. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Suppose that $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$ satisfy condition (B). Then the sequence $\{x_n\}$ defined in (1.4) converges strongly to $p \in F$.*

Proof. Let $p \in F$. As proved in Lemma 2.1, $d(x_n, p) \leq d(x_{n-1}, p)$ for all $n \in \mathbb{N}$. This implies that

$$d(x_n, F) \leq d(x_{n-1}, F).$$

Thus $\lim_{n\to\infty} d(x_n, F)$ exists. Since $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$ satisfy Condition (B), therefore

$$\text{either } \max_{i\in J} d(x_n, T_i x_n) \geq f(d(x_n, F)) \text{ or } \max_{i\in J} d(x_n, I_i x_n) \geq f(d(x_n, F)).$$

It follows from (2.15) and (2.16) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, so it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Lemma 2.4 implies that $\{x_n\}$ converges strongly to a point p in F. □

Theorem 2.6. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Suppose that either K is compact or one of the map in $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.4) converges strongly to $p \in F$.*

Proof. For any $i \in J$, we first suppose that T_i and I_i are semicompact. By (2.15) and (2.16), we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, I_i x_n) = 0.$$

From the semicompactness of T_i and I_i, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a $q \in K$. Using (2.15) and (2.16), we have

$$\lim_{i\to\infty} d(x_{n_i}, T_i x_{n_i}) = d(q, T_i q) = 0 \text{ and } \lim_{i\to\infty} d(x_{n_i}, I_i x_{n_i}) = d(q, I_i q) = 0$$

for all $i \in J$. This implies that $q \in F$. Since $\lim_{n\to\infty} d(x_{n_i}, q) = 0$ and $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F$ by Lemma 2.1, therefore

$$\lim_{n\to\infty} d(x_n, q) = 0.$$

Next, assume the compactness of K, then again there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a $q \in K$ and the proof follows the above lines. □

Next, we give and prove our $\Delta-$convergence theorem.

Theorem 2.7. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (1.4) $\Delta-$converges to a common fixed point of $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$.*

Proof. It follows from Lemma 2.1 that $\{x_n\}$ is bounded. Since $\{x_n\}$ bounded sequence in a nonempty closed convex subset of a complete uniformly convex hyperbolic space, then $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x_n\}$. Assume that $\{u_n\}$ is any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u_n\}$. Then by (2.15) and (2.16), we have $\lim_{n\to\infty} d(u_n, T_l u_n) = \lim_{n\to\infty} d(u_n, I_l u_n) = 0$ for each $l = 1, 2, \cdots, N$. Now we prove that u is the common fixed point of $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$.
Define a sequence $\{v_n\}$ in K by $v_m = T_m u$, where $T_m = T_{m(modN)}$.
Clearly,

$$\begin{aligned} d(v_n, u_n) &\leq d(T_m u, T_m u_n) + d(T_m u_n, T_{m-1} u_n) + \cdots + d(T u_n, u_n) \\ &\leq d(u, u_n) + \sum_{i=1}^{m-1} d(u_n, T_i u_n). \end{aligned}$$

Thus, we have

$$r(v_m, \{u_n\}) = \limsup_{n\to\infty} d(v_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(v_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. By Lemma 1.6, we obtain $T_{m(modN)} u = u$, which implies that u is the common fixed point of $\{T_i : i \in J\}$. Similarly, we can show that u is the common fixed point of $\{I_i : i \in J\}$. Therefore $u \in F$. Moreover, $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 2.1.
Assume $x \neq u$. By the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u) \end{aligned}$$

a contradiction. Thus $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\Delta-$converges to a common fixed point of $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$. □

Although the followings are corollaries of our main theorems, yet they are new in themselves.

Theorem 2.8. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let T be a I-nonexpansive mapping and I be a nonexpansive mapping on K such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose T and I satisfy the condition (A'). Then the sequence $\{x_n\}$ defined by*

$$\begin{aligned} x_n &= W(x_{n-1}, Ty_n, \alpha_n), \\ y_n &= W(x_{n-1}, Ix_n, \beta_n), \ n \geq 1 \end{aligned}$$

converges strongly to $p \in F$.

Proof. Choose $T_i = T$ and $I_i = I$ for all $i \in J$ in Theorem 2.5. □

Theorem 2.9. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let T be a I-nonexpansive mapping and I be a nonexpansive mapping on K such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose that either K is compact or one of the map T and I is semi-compact. Then the sequence $\{x_n\}$ defined by*

$$\begin{aligned} x_n &= W(x_{n-1}, Ty_n, \alpha_n), \\ y_n &= W(x_{n-1}, Ix_n, \beta_n), \ n \geq 1 \end{aligned}$$

converges strongly to $p \in F$.

Proof. Choose $T_i = T$ and $I_i = I$ for all $i \in J$ in Theorem 2.6. □

Theorem 2.10. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let T be a I-nonexpansive mapping and I be a nonexpansive mapping on K such that $F = F(T) \cap F(I) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by*

$$\begin{aligned} x_n &= W(x_{n-1}, Ty_n, \alpha_n), \\ y_n &= W(x_{n-1}, Ix_n, \beta_n), \ n \geq 1, \end{aligned}$$

$\Delta-$*converges to a common fixed point of T and I.*

Proof. Choose $T_i = T$ and $I_i = I$ for all $i \in J$ in Theorem 2.7. □

Finally, we give an example to show that there do exist two finite families of mentioned mappings with a nonempty common fixed point set.

Example 2.11. Let $X = \mathbb{R}$. Define $T_n : X \to X$ and $I_n : X \to X$ as $T_n x = \frac{n^2-2x+1}{2n^2}$ and $I_n x = \frac{2x+n-1}{2n}$ for all $n \in \mathbb{N}$. Then $\{I_i : i \in J\}$ is a finite family of nonexpansive mappings and $\{T_i : i \in J\}$ is a finite family of I_i-nonexpansive mappings on X with common fixed point set $F = \{\frac{1}{2}\}$.

Remark 2.12. Our result generalize, extend and improve resuls of Gunduz and Akbulut [25, 26, 27, 28] and Khan et al. [10] in view of more general class of mappings.

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