AN IMPLICIT ITERATION PROCESS FOR I-NONEXPANSIVE MAPPINGS IN KOHLENBACH HYPERBOLIC SPACES

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Abstract The first goal of this paper is to propose a composite implicit iteration process for a finite family of I-nonexpansive mappings in hyperbolic spaces. Next, some strong and $\Delta$-convergence theorems are established using the proposed iteration process. New results are obtained as corollaries to the convergence theorems. Finally, we exhibit two finite families of the mappings under consideration.

1 Introduction and Preliminaries

Let $K$ be a nonempty subset of a metric space $X$. The mapping $T: K \to K$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$.

In [1], Shahzad defines $I$-nonexpansive mappings in Banach spaces essentially as follows: given two mappings $T, I: K \to K$, $T$ is called $I$-nonexpansive if $d(Tx, Ty) \leq d(Ix, Iy)$ for all $x, y \in K$.

In what follows, we set $J = \{1, 2, \ldots, N\}$ for the set of first $N$ natural numbers and take $\{\alpha_n\}, \{\beta_n\}$ sequences in $(0, 1)$.

Given $x_0$ in $K$ (a subset of Banach space), the Mann iteration process defined for a nonexpansive mapping as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) Tx_{n-1}, \quad n \geq 1.$$  \hspace{1cm} (1.1)

Xu and Ori [19] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i: i \in J\}$.

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1,$$  \hspace{1cm} (1.2)

where $T_n = T_{n(modN)}$ and the $modN$ function takes values in $J$.

In 2007, Su and Li [20] introduced the composite implicit iteration process for finite family of strictly pseudocontractive maps defined as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n [\beta_n x_{n-1} + (1 - \beta_n) T_n x_{n-1}] , \quad n \geq 1,$$  \hspace{1cm} (1.3)

where $T_n = T_{n(modN)}$.

In [2], Rhoades and Temir showed that the Mann iteration process converges weakly to a common fixed point of $T$ and $I$ in a Banach space by taking the map $T$ to be $I$-nonexpansive. Actually, they proved the following theorems.

Theorem 1.1. (Rhoades and Temir [2]) Let $K$ be a closed convex bounded subset of a uniformly convex Banach space $X$ which satisfies Opial’s condition, and let $T, I$ be self-mappings of $K$ with $T$ be an $I$-nonexpansive mapping, $I$ be a nonexpansive on $K$. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of Mann iterates converges weakly to common fixed point of $F(T) \cap F(I)$.

There are numerous papers dealing with the convergence of different iterative techniques for these mappings and generalization of the class of $I$-nonexpansive mappings in Banach spaces (see, for example, [3, 4, 5, 6, 7] and the references therein).
Motivated by the iteration process (1.3) of Su and Li [20], in this paper we define a new modified composite implicit iteration process for a finite family of \( I_i \)-nonexpansive mappings \( \{T_i: i \in J\} \) and a finite family of nonexpansive mappings \( \{I_i: i \in J\} \) in hyperbolic spaces as follows:

\[
\begin{align*}
x_n &= W(x_{n-1}, T_n y_n, \alpha_n), \\
y_n &= W(x_{n-1}, I_n x_n, \beta_n), \quad n \geq 1
\end{align*}
\]

where \( T_n = T_n^{(\text{mod}N)} \) and \( I_n = I_n^{(\text{mod}N)} \).

Different notions of hyperbolic space \([12, 13, 14, 15]\) can be found in the literature. We work in the setting of hyperbolic spaces as introduced by Kohlenbach [11], which are slightly more restrictive than the spaces of hyperbolic type \([12]\) by (W4), but more general than the concept of hyperbolic space from \([15]\).

**Definition 1.2.** (Kohlenbach [11]) A hyperbolic space is a triple \((X, d, W)\) where \((X, d)\) is a metric space and \(W: X^2 \times [0, 1] \to X\) is a mapping such that

\[
\begin{align*}
&W1. \quad d(u, W(x, y, \alpha)) \leq (1 - \alpha) d(u, x) + \alpha d(u, y) \\
&W2. \quad d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y) \\
&W3. \quad W(x, y, \alpha) = W(y, x, (1 - \alpha)) \\
&W4. \quad d(W(x, z, \alpha), W(y, w, \beta)) \leq (1 - \alpha) d(x, y) + \alpha d(z, w)
\end{align*}
\]

for all \(x, y, z, w \in X\) and \(\alpha, \beta \in [0, 1]\).

If \((X, d, W)\) satisfies only (W1), then it coincides with the convex metric space introduced by Takahashi [16]. A subset \(K\) of a hyperbolic space \(X\) is convex if \(W(x, y, \alpha) \in K\) for all \(x, y \in K\) and \(\alpha \in [0, 1]\).

**Definition 1.3.** A hyperbolic space \((X, d, W)\) is said to be uniform convex [17] if for all \(u, x, y \in X, r > 0\) and \(\varepsilon \in (0, 2]\), there exists a \(\delta \in (0, 1]\) such that

\[
\begin{align*}
&d(x, u) \leq r \\
&d(y, u) \leq r \\
&d(x, y) \geq \varepsilon r
\end{align*}
\]

\(\Rightarrow\)

\[
\begin{align*}
&W \left( x, y, \frac{1}{2} \right), u \leq (1 - \delta) r.
\end{align*}
\]

A map \(\eta: (0, \infty) \times (0, 2] \to (0, 1]\) which provides such a \(\delta = \eta(r, \varepsilon)\) for given \(r > 0\) and \(\varepsilon \in (0, 2]\) is called modulus of uniform convexity. We call \(\eta\) monotone if it decreases with \(r\) (for a fixed \(\varepsilon\)).

The notion of \(\Delta\)-convergence in general metric spaces was introduced by Lim [8] in 1976. Kirk and Panyanak [9] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting.

To give the definition of \(\Delta\)-convergence, we first recall some notations.

Let \(\{x_n\}\) be a bounded sequence in a hyperbolic space \(X\). For \(x \in X\), define a continuous functional \(r(\cdot, \{x_n\}): X \to [0, \infty)\) by \(r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n)\).

Then the asymptotic radius \(\rho = r(\{x_n\})\) of \(\{x_n\}\) is defined by \(\rho = \inf \{r(x, \{x_n\}): x \in X\}\) and the asymptotic center of a bounded sequence \(\{x_n\}\) with respect to a subset \(K\) of \(X\) is defined by \(A_K(\{x_n\}) = \{x \in X: r(x, \{x_n\}) \leq r(y, \{x_n\})\} \text{ for any } y \in K\). If the asymptotic center is taken with respect to \(X\), then it is simply denoted by \(A(\{x_n\})\).

A sequence \(\{x_n\}\) in \(X\) is said to \(\Delta\)-converge to \(x \in X\) if \(x\) is the unique asymptotic center of \(\{u_n\}\) for every subsequence \(\{u_n\}\) of \(\{x_n\}\). In this case, we write \(\Delta\)-lim_{\{x_n\}} x = x and call \(x\) as \(\Delta\)-limit of \(\{x_n\}\).

The proofs of the following lemmas can be found in Leustean [18] and Khan et al. [10].

**Lemma 1.4.** [18] Let \((X, d, W)\) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence \(\{x_n\}\) in \(X\) has a unique asymptotic center with respect to any nonempty closed convex subset \(K\) of \(X\).
Thus the implicit iteration process (point. Thus the existence of $y$ is established. Similarly, the existence of $x_2, x_3, \ldots$ is established. Thus the implicit iteration process (1.4) is well defined.

We need the following lemma in order to prove our main theorems.

**Lemma 2.1.** Let $K$ be a nonempty closed convex subset of a hyperbolic space $X$. Let $\{T_i : i \in I\}$ be a finite family of nonexpansive mappings on $K$ such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ defined in (1.4), we have $\lim_{n \to \infty} d(x_n, p) \geq 0$ exists for $p \in F$.

**Proof.** Let $p \in F$. From (1.4), we have

$$d(y_n, p) = d(W(x_{n-1}, I_n x_n, \beta_n), p) \leq (1 - \beta_n) d(x_{n-1}, p) + \beta_n d(I_n x_n, p) \leq (1 - \beta_n) d(x_{n-1}, p) + \beta_n d(x_n, p). \quad (2.1)$$

By (2.1) and (1.4), we obtain

$$d(x_n, p) = d(W(x_{n-1}, T_n y_n, \alpha_n), p) \leq (1 - \alpha_n) d(x_{n-1}, p) + \alpha_n d(T_n y_n, p) \leq (1 - \alpha_n) d(x_{n-1}, p) + \alpha_n d(I_n y_n, p) \leq (1 - \alpha_n) d(x_{n-1}, p) + \alpha_n d(y_n, p) \leq (1 - \alpha_n) d(x_{n-1}, p) + \alpha_n [d(I_n y_n, p)] \leq (1 - \alpha_n) d(x_{n-1}, p) + \alpha_n \beta_n d(x_n, p).$$

Consequently, we have

$$d(x_n, p) \leq d(x_{n-1}, p). \quad (2.2)$$

Thus $\lim_{n \to \infty} d(x_n, p)$ exists for each $p \in F$. \hfill $\Box$

**2 Main Results**

Denote by $F$ the set of common fixed points of the finite families of mappings $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$.

Let $X$ be a hyperbolic space, $K$ be a nonempty closed convex subset of $X$. Let $\{I_i : i \in J\}$ be a finite family of $I_i$-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings. Let $\{x_n\}$ be defined by (1.4). Then $x_1 = W(x_0, T_1 W(x_0, I_1 x_1, \beta_1), \alpha_1)$. Define a mapping $G_1 : K \to K$ by: $G_1 x = W(x_0, T_1 W(x_0, I_1 x, \beta_1), \alpha_1)$ for all $x \in K$. Existence of $x_1$ is guaranteed if $G_1$ has a fixed point. Now for any $u, v \in K$, we have

$$d(G_1 u, G_1 v) = d(W(x_0, T_1 W(x_0, I_1 u, \beta_1), \alpha_1), W(x_0, T_1 W(x_0, I_1 v, \beta_1), \alpha_1)) \leq \alpha_1 d(T_1 W(x_0, I_1 u, \beta_1), T_1 W(x_0, I_1 v, \beta_1)) \leq \alpha_1 d(I_1 W(x_0, I_1 u, \beta_1), I_1 W(x_0, I_1 v, \beta_1)) \leq \alpha_1 \beta_1 d(I_1 u, I_1 v) \leq \alpha_1 \beta_1 d(u, v)$$

Since $\alpha_1 \beta_1 < 1$, $G_1$ is a contraction. By Banach contraction principle, $G_1$ has a unique fixed point. Thus the existence of $x_1$ is established. Similarly, the existence of $x_2, x_3, \ldots$ is established. Thus the implicit iteration process (1.4) is well defined.

We need the following lemma in order to prove our main theorems.

**Lemma 1.5.** Let $(X, d, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\limsup_{n \to \infty} d(x_n, x) \leq r$, $\limsup_{n \to \infty} d(y_n, x) \leq r$ and $\lim_{n \to \infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n \to \infty} d(x_n, y_n) = 0$.

**Lemma 1.6.** Let $K$ be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in $K$ such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_n\}$ is another sequence in $K$ such that $\lim_{n \to \infty} r(y_n, \{x_n\}) = \rho$, then $\lim_{n \to \infty} y_n = y$.
Lemma 2.2. Let \( K \) be a nonempty closed convex subset of a uniformly convex hyperbolic space \( X \) with monotone modulus of uniform convexity \( \eta \). Let \( \{ T_i : i \in I \} \) be a finite family of \( I \)-nonexpansive mappings and \( \{ I_i : i \in J \} \) be a finite family of nonexpansive mappings on \( K \) such that \( F \neq \emptyset \). Then for the sequence \( \{ x_n \} \) defined in (1.4), we have

\[
\lim_{n \to \infty} d(x_n, T_l x_n) = \lim_{n \to \infty} d(x_n, I_l x_n) = 0 \quad \text{for each} \quad l = 1, 2, \ldots, N.
\]

Proof. In view of Lemma 2.1, we obtain that the limit of the sequence \( \{ d(x_n, p) \} \) exits for each \( p \in F \). Next, we assume that \( \lim_{n \to \infty} d(x_n, p) = c \), for some \( c > 0 \). It follows from (1.4) that

\[
\lim_{n \to \infty} d(x_n, p) = \lim_{n \to \infty} d(W(x_{n-1}, T_n y_n, \alpha_n), p) = c.
\]

(2.3)

By means of \( \lim_{n \to \infty} d(x_n, p) = c \) and nonexpansivity of \( T_i \), we get

\[
\limsup_{n \to \infty} d(T_n y_n, p) \leq \limsup_{n \to \infty} d(I_n y_n, p) \leq \limsup_{n \to \infty} d(y_n, p)
\]

\[
= \limsup_{n \to \infty} d(W(x_{n-1}, I_n x_n, \beta_n), p)
\]

\[
\leq \limsup_{n \to \infty} [(1 - \beta_n) d(x_{n-1}, p) + \beta_n d(I_n x_n, p)]
\]

\[
\leq \limsup_{n \to \infty} [(1 - \beta_n) d(x_{n-1}, p) + \beta_n d(x_n, p)]
\]

\[
\leq c.
\]

(2.4)

Now using (2.4) with \( \lim_{n \to \infty} d(x_n, p) = c \) and applying Lemma 1.5 to (2.3), we get

\[
\lim_{n \to \infty} d(x_{n-1}, T_n y_n) = 0.
\]

(2.5)

From (1.4) and (2.5) we obtain

\[
d(x_n, x_{n-1}) = d(W(x_{n-1}, T_n y_n, \alpha_n), x_{n-1})
\]

\[
\leq (1 - \alpha_n) d(x_{n-1}, x_{n-1}) + \alpha_n d(T_n y_n, x_{n-1})
\]

\[
\to 0 \quad (n \to \infty),
\]

which implies that

\[
\lim_{n \to \infty} d(x_n, x_{n+l}) = 0, \quad \forall l = 1, 2, \ldots, N.
\]

(2.6)

Note that

\[
d(x_n, T_n y_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n y_n).
\]

Next, taking limit on both sides in the above inequality we get

\[
\lim_{n \to \infty} d(x_n, T_n y_n) = 0.
\]

(2.7)

Clearly,

\[
d(x_n, p) \leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n y_n) + d(T_n y_n, p)
\]

\[
\leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n y_n) + d(I_n y_n, p)
\]

\[
\leq d(x_n, x_{n-1}) + d(x_{n-1}, T_n y_n) + d(y_n, p).
\]

Taking \( \liminf \) on both sides in the above estimate, from (2.5) and (2.6) we have

\[
c \leq \liminf_{n \to \infty} d(y_n, p).
\]

(2.8)

Also, we get from (2.1)

\[
\limsup_{n \to \infty} d(y_n, p) \leq c
\]

so that (2.8) gives

\[
\lim_{n \to \infty} d(y_n, p) = c.
\]

(2.9)
Thus \( c = \lim_{n \to \infty} d(y_n, p) = \lim_{n \to \infty} d(W(x_{n-1}, I_n x_n, \beta_n), p) \) gives by

\[
\lim_{n \to \infty} d(I_n x_n, p) \leq c
\]

and Lemma 1.5 that

\[
\lim_{n \to \infty} d(x_{n-1}, I_n x_n) = 0
\tag{2.10}
\]

On the other hand,

\[
d(x_n, I_n x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, I_n x_n).
\]

Thus we have

\[
\lim_{n \to \infty} d(x_n, I_n x_n) = 0.
\tag{2.11}
\]

Further, observe that

\[
d(y_n, x_{n-1}) = d(W(x_{n-1}, I_n x_n, \beta_n), x_{n-1}) \\
\leq \beta_n d(I_n x_n, x_{n-1}).
\]

By (2.10), we have

\[
\lim_{n \to \infty} d(y_n, x_{n-1}) = 0.
\tag{2.12}
\]

Thus

\[
d(x_n, T_n x_n) \leq d(x_n, T_n y_n) + d(T_n y_n, T_n x_{n-1}) + d(T_n x_{n-1}, T_n x_n) \\
\leq d(W(x_{n-1}, T_n y_n, \alpha_n), T_n x_{n-1}) + d(y_n, x_{n-1}) + d(x_{n-1}, x_n) \\
\leq (1 - \alpha_n) d(x_{n-1}, T_n y_n) + d(y_n, x_{n-1}) + d(x_{n-1}, x_n)
\]

together with (2.5), (2.6) and (2.12) implies that

\[
\lim_{n \to \infty} d(x_n, T_n x_n) = 0.
\tag{2.13}
\]

Since, for each \( l = 1, 2, \cdots, N \), we have

\[
d(x_n, T_{n+l} x_n) \leq d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) + d(T_{n+l} x_{n+l}, T_{n+l} x_n) \\
\leq d(x_{n+l}, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) + d(I_{n+l} x_{n+l}, I_{n+l} x_n) \\
\leq 2d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l})
\tag{2.14}
\]

it follows from (2.6) and (2.13) that

\[
\lim_{n \to \infty} d(x_n, T_{n+l} x_n) = 0
\]

for all \( l \in J \). Thus we get

\[
\lim_{n \to \infty} d(x_n, I_l x_n) = 0 \quad \text{for any} \quad l \in J.
\tag{2.15}
\]

Replacing \( T_{n+l} \) by \( I_{n+l} \) in the inequality (2.14), we get

\[
\lim_{n \to \infty} d(x_n, I_l x_n) = 0
\tag{2.16}
\]

for all \( l \in J \).

For further developments, we need the following concepts and technical result.

A sequence \( \{x_n\} \) in a metric space \( X \) is said to be \textit{Fejér monotone} with respect to \( K \) (a subset of \( X \)) if \( d(x_{n+1}, p) \leq d(x_n, p) \) for all \( p \in K \) and for all \( n \geq 1 \). A map \( T : K \to K \) is semi-compact if any bounded sequence \( \{x_n\} \) satisfying \( d(x_n, T x_n) \to 0 \) as \( n \to \infty \) has a convergent subsequence.

\textbf{Lemma 2.3.} [21] \textit{Let} \( K \) \textit{be a nonempty closed subset of a complete metric space} \( (X, d) \) \textit{and} \( \{x_n\} \) \textit{be Fejér monotone with respect to} \( K \). \textit{Then} \( \{x_n\} \) \textit{converges to some} \( p \in K \) \textit{if and only if} \( \lim_{n \to \infty} d(x_n, K) = 0 \).
Lemma 2.4. Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $\{T_i : i \in I\}$ be a finite family of $I_i$-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on $K$ such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (1.4) converges strongly to $p \in F$ if and only if $\lim_{n \to \infty} d(x_n, F) = 0$.

Proof. It follows from (2.2) that $\{x_n\}$ is Fejér monotone with respect to $F$ and $\lim_{n \to \infty} d(x_n, F)$ exists. Now applying the Lemma 2.3, we obtain the result.\qed

A mappings $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy the Condition (A) [24] if there exists a nondecreasing function $f : (0, \infty) \to [0, \infty]$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in K$.

Khan and Fukhar-ud-din [22], introduced the so-called Condition (A') and gave a slightly improved version of it in [23] as follows:

Two mappings $T, I : K \to K$ with $F(T) \cap F(I) \neq \emptyset$ are said to satisfy the Condition (A') if there exists a nondecreasing function $f : (0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that either $d(x, Tx) \geq f(d(x, F(T) \cap F(I)))$ or $d(x, Tx) \geq f(d(x, F(T) \cap F(I)))$ for all $x \in K$.

We can modify this definition for two finite families of mappings as follows. Let $\{T_i : i \in I\}$ and $\{I_i : i \in J\}$ be two finite families of nonexpansive mappings of $K$ with nonempty fixed points set $F$. These families are said to satisfy Condition (B) on $K$ if there exists a nondecreasing function $f : (0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that either $\max_{i \in J} d(x, T_ix) \geq f(d(x, F(T)))$ or $\max_{i \in J} d(x, I_ix) \geq f(d(x, F(I)))$ for all $x \in K$.

Note that the Condition (A') is weaker than both the semicompactness of the mapping $T : K \to K$ and the compactness of its domain $K$, see [24]. Thus the Condition (A') is weaker than both the semicompactness of the mappings $T, I : K \to K$ and the compactness of their domain $K$. In this direction Condition (B) is weaker than both the semicompactness of $\{T_i : i \in I\}$ and $\{I_i : i \in J\}$ and the compactness of their domain $K$.

We are now ready to state and prove our strong convergence theorems.

Theorem 2.5. Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $\{T_i : i \in I\}$ be a finite family of $I_i$-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on $K$ such that $F \neq \emptyset$. Suppose that $\{T_i : i \in I\}$ and $\{I_i : i \in J\}$ satisfy condition (B). Then the sequence $\{x_n\}$ defined in (1.4) converges strongly to $p \in F$.

Proof. Let $p \in F$. As proved in Lemma 2.1, $d(x_n, p) \leq d(x_{n-1}, p)$ for all $n \in \mathbb{N}$. This implies that $d(x_n, F) \leq d(x_{n-1}, F)$. Thus $\lim_{n \to \infty} d(x_n, F)$ exists. Since $\{T_i : i \in I\}$ and $\{I_i : i \in J\}$ satisfy Condition (B), therefore

\[\max_{i \in J} d(x_n, T_ix) \geq f(d(x_n, F)) \quad \text{or} \quad \max_{i \in J} d(x_n, I_ix) \geq f(d(x_n, F)).\]

It follows from (2.15) and (2.16) that $\lim_{n \to \infty} f(d(x_n, F)) = 0$. Since $f$ is a nondecreasing function and $f(0) = 0$, so it follows that $\lim_{n \to \infty} d(x_n, F) = 0$. Therefore, Lemma 2.4 implies that $\{x_n\}$ converges strongly to a point $p \in F$.

Theorem 2.6. Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $\{T_i : i \in I\}$ be a finite family of $I_i$-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on $K$ such that $F \neq \emptyset$. Suppose that either $K$ is compact or one of the map in $\{T_i : i \in I\}$ and $\{I_i : i \in J\}$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.4) converges strongly to $p \in F$.

Proof. For any $i \in J$, we first suppose that $T_i$ and $I_i$ are semicompact. By (2.15) and (2.16), we have

\[\lim_{n \to \infty} d(x_n, T_ix) = \lim_{n \to \infty} d(x_n, I_ix) = 0.\]
From the semicompactness of $T_i$ and $I_i$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a $q \in K$. Using (2.15) and (2.16), we have
\[
\lim_{i \to \infty} d(x_{n_i}, T_i x_{n_i}) = d(q, T_i q) = 0 \quad \text{and} \quad \lim_{i \to \infty} d(x_{n_i}, I_i x_{n_i}) = d(q, I_i q) = 0
\]
for all $i \in J$. This implies that $q \in F$. Since $\lim_{n \to \infty} d(x_n, q) = 0$ and $\lim_{n \to \infty} d(x_n, q)$ exists for all $q \in F$ by Lemma 2.1, therefore
\[
\lim_{n \to \infty} d(x_n, q) = 0.
\]

Next, assume the compactness of $K$, then again there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a $q \in K$ and the proof follows the above lines. \(\square\)

We have proved that $\{x_n\}$ is bounded. Since $\{x_n\}$ bounded sequence in a nonempty closed convex subset of a complete uniformly convex hyperbolic space, then $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Assume that $\{u_n\}$ is any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by (2.15) and (2.16), we have $\lim_{n \to \infty} d(u_n, T_i u_n) = \lim_{n \to \infty} d(u_n, I_i u_n) = 0$ for each $i = 1, 2, \cdots, N$. Now we prove that $u$ is the common fixed point of $T_i : i \in J$ and $I_i : i \in J$.

Define a sequence $\{v_n\}$ in $K$ by $v_m = T_m u$, where $T_m = T_{(m \mod N)}$. Clearly,
\[
d(v_n, u_n) \leq d(T_m u, T_m u_n) + d(T_m u_n, T_{m-1} u_n) + \cdots + d(T u_n, u_n)
\leq d(u, u_n) + \sum_{i=1}^{m-1} d(u_n, T_i u_n).
\]

Thus, we have
\[
r(v_m, \{u_n\}) = \limsup_{n \to \infty} d(v_m, u_n) \leq \limsup_{n \to \infty} d(u, u_n) = r(u, \{u_n\})
\]
This implies that $|r(v_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. By Lemma 2.1, we obtain $T_{(m \mod N)} u = u$, which implies that $u$ is the common fixed point of $\{T_i : i \in J\}$. Similarly, we can show that $u$ is the common fixed point of $\{I_i : i \in J\}$. Therefore $u \in F$. Moreover, $\lim_{n \to \infty} d(x_n, u)$ exists by Lemma 2.1.

Assume $x \neq u$. By the uniqueness of asymptotic centers,
\[
\limsup_{n \to \infty} d(u_n, u) < \limsup_{n \to \infty} d(u_n, x) \leq \limsup_{n \to \infty} d(x_n, x) < \limsup_{n \to \infty} d(x_n, u) = \limsup_{n \to \infty} d(u_n, u)
\]
a contradiction. Thus $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\Delta$-converges to a common fixed point of $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$. \(\square\)

Although the followings are corollaries of our main theorems, yet they are new in themselves.
Theorem 2.8. Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T$ be a $I$-nonexpansive mapping and $I$ be a nonexpansive mapping on $K$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose $T$ and $I$ satisfy the condition $(A')$. Then the sequence $\{x_n\}$ defined by

\[
x_n = W(x_{n-1}, Ty_n, \alpha_n), \\
y_n = W(x_{n-1}, Ix_n, \beta_n), \quad n \geq 1
\]

converges strongly to $p \in F$.

Proof. Choose $T_i = T$ and $I_i = I$ for all $i \in J$ in Theorem 2.5. \hfill \Box

Theorem 2.9. Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T$ be a $I$-nonexpansive mapping and $I$ be a nonexpansive mapping on $K$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose that either $K$ is compact or one of the map $T$ and $I$ is semi-compact. Then the sequence $\{x_n\}$ defined by

\[
x_n = W(x_{n-1}, Ty_n, \alpha_n), \\
y_n = W(x_{n-1}, Ix_n, \beta_n), \quad n \geq 1
\]

converges strongly to $p \in F$.

Proof. Choose $T_i = T$ and $I_i = I$ for all $i \in J$ in Theorem 2.6. \hfill \Box

Theorem 2.10. Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$. Let $T$ be a $I$-nonexpansive mapping and $I$ be a nonexpansive mapping on $K$ such that $F = F(T) \cap F(I) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by

\[
x_n = W(x_{n-1}, Ty_n, \alpha_n), \\
y_n = W(x_{n-1}, Ix_n, \beta_n), \quad n \geq 1
\]

$\Delta$--converges to a common fixed point of $T$ and $I$.

Proof. Choose $T_i = T$ and $I_i = I$ for all $i \in J$ in Theorem 2.7. \hfill \Box

Finally, we give an example to show that there do exist two finite families of mentioned mappings with a nonempty common fixed point set.

Example 2.11. Let $X = \mathbb{R}$. Define $T_n : X \to X$ and $I_n : X \to X$ as $T_n x = \frac{n^2 + 2x + 1}{2n^2}$ and $I_n x = \frac{2x + n - 1}{2n}$ for all $n \in \mathbb{N}$. Then $\{T_i : i \in J\}$ is a finite family of nonexpansive mappings and $\{I_i : i \in J\}$ is a finite family of $I_i$-nonexpansive mappings on $X$ with common fixed point set $F = \{1\}$.

Remark 2.12. Our result generalize, extend and improve results of Gunduz and Akbulut [25, 26, 27, 28] and Khan et al. [10] in view of more general class of mappings.

References

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