Palestine Journal of Mathematics

Vol. 7(2)(2018) , 521–526

SOME QUOTIENTS OF THE MODULAR GROUP

Hacer Ozden and Ismail Naci Cangul

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 20F10, 11F06; Secondary 20F05, 20H05.

Keywords and phrases: modular group, Hecke groups, one-relator quotient, quotient group, cyclically reduced word

Abstract The purpose of this paper is to obtain a formula giving the number of all possible one relator quotients of the modular group. By thinking of an element w of the modular group as a word in terms of its generators a and b, one can find, by means of the formula given here, the total number of all possible words having a given number of a's and b's in some order. A formula for the number of cyclically reduced words is also given.

1 Introduction

The modular group Γ is the discrete subgroup of $PSL(2,\mathbb{R})$ generated by

$$a(z) = -\frac{1}{z} \qquad \text{ve} \qquad b(z) = -\frac{1}{z+\lambda}.$$

a is of order 2 and b is of order 3, respectively, [6]. A Fuchsian group is a finitely generated discrete subgroup of $PSL(2,\mathbb{R})$. Therefore Γ is a Fuchsian group. A Fuchsian group has the following well-known represention:

Generators:

$$\begin{array}{ll} a_1,\ a_2,\ ...,\ a_g,\ b_g & (hyperbolic) \\ x_1,\ ...,\ x_r & (elliptic) \\ p_1,\ ...,\ p_t & (parabolic) \\ h_1,\ ...,\ h_u & (hyperbolic\ boundary\ elements) \end{array}$$

Relations:

$$x_j^{m_j} = \prod [a_i, b_i] \prod x_j \prod p_k \prod h_l = 1.$$

Such a group is associated with a signature $(g;\ m_1,\ \cdots,\ m_r;\ t;\ u)$; where $m_1,\ \cdots,\ m_r$ are integers $\geq\ 2$, and are called the periods of the Fuchsian group. In modular group case, there is no hyperbolic boundary elements and hence we omit u in the signature. Therefore Γ has a presentation

$$a^2 = b^3 = (ab)^\infty = 1$$

with signature $(0;\ 2,\ 3;\ 1)$. Usually, in such a presentation, the last relation $(ab)^\infty = 1$ is omitted.

As Fuchsian groups act on the Riemann surfaces, all their normal subgroups also act on Riemann surfaces and this fact combines algebra with analysis. In fact, the Hecke groups can be considered as special triangle groups with an infinity, in the following sense: Recall that a triangle group $T(l,\ m,\ n)$ is a two generated group with representation

$$< a,\ b\ :\ a^l = b^m = (ab)^n = 1 > .$$

It therefore has the signature $(g;\ l,\ m,\ n)$. It is known that $T(l,\ m,\ n)$ is finite precisely when $(1/l)+(1/m)+(1/n)>1$. The finite triangle groups occuring as the quotients of Γ are

$$\begin{array}{ll} T(1,n,n)\cong C_n, & \text{the cyclic group of order n,}\\ T(2,n,2)\cong D_n, & \text{the dihedral group of order 2n,}\\ T(2,3,3)\cong A_4, & \text{the tetahedral group of order 12,}\\ T(2,4,3)\cong S_4, & \text{the octahedral group of order 24,}\\ T(2,5,3)\cong A_5, & \text{the icosahedral group of order 60.} \end{array}$$

In this paper, we study the normal subgroups of finite index in the modular group corresponding to one-relator quotients obtained in [9]. As a result, all such subgroups are determined to be either power subgroups, commutator subgroups, or congruence subgroups.

Theorem 1.1. *[6] The commutator subgroup Γ' of Γ is of index 6 in Γ and is a free group of rank 2, freely generated by*

$$A=abab^2 \quad \textit{and} \quad B=ab^2ab.$$

It has signature $(1;-;1)$ and

$$\Gamma/\Gamma'\cong C_6.$$

Let $m\in\mathbb{N}$. The m-th power subgroup Γ^m of Γ is defined as the subgroup generated by the m-th powers of the elements of Γ. It is well-known that Γ^m are normal in Γ.

We first have some specific cases:

Theorem 1.2. *[6]* ***a)*** *Γ^2 is isomorphic to the free product of two cyclic groups of order 3, and*

$$(\Gamma\ :\ \Gamma^2)=2,\ \ \Gamma=\Gamma^2\cup a\Gamma^2,\ \ \Gamma^2=<b,aba>.$$

The elements of Γ^2 may be characterized by the reqirement that the sum of the exponents of a is divisible by 2. Γ^2 has signature $(0;\ 3;\ 3;\ 1)$.

b) *The group Γ^3 is the free product of three cyclic groups of order 2, and*

$$(\Gamma\ :\ \Gamma^3)=3,\quad \Gamma=\Gamma^3\cup b\Gamma^3\cup b^2\Gamma^3,\quad \Gamma^3=<a,\ bab^2,\ b^2ab>.$$

The elements of Γ^3 may be characterized by the reqirement that the sum of the exponents of a is divisible by 3. Γ^3 has signature (0;2,2,2;1).

The principal congruence subgroup of level n of Γ, denoted by $\Gamma(n)$, is defined by

$$\Gamma(n)=\{T\in\Gamma\ :\ T\ \equiv\pm I\ (n)\}.$$

When one relator small quotients of Γ are considered, the only congruence subgroups appearing are $\Gamma(2)$, $\Gamma(3)$, $\Gamma(4)$ and $\Gamma(5)$, with signatures (0;-;3), (0;-;4), (0;-;6) and (0;-;12), respectively.

In the calculation of the normal subgroups of the normal subgroups of Γ, we use a method known as permutation method given by Singerman, [8].

2 Normal subgroups of Γ corresponding to one-relator quotients

We now determine the normal subgroups of Γ corresponding to small one relator quotients of Γ, by means of the permutation method. We think of the quotient groups of Γ as images, in the following sense, of Γ by the second isomorphism theorem.

Let the quotient group be G. Then we have a natural epimorpism from Γ onto G. The kernel of this epimorphism is a normal subgroup N of Γ, where $G\cong\Gamma/N$. First we consider cyclic quotients. All cyclic quotients of Γ are found to be C_1, C_2, C_3 and C_6, see [8, 9].

1. If we map $\Gamma = (2, 3, \infty)$ onto $C_1 \cong (1, 1, 1)$, we get $N = \Gamma$.

2. If we map Γ onto $C_2 \cong (2, 1, 2)$, then a goes to the generator while b goes to the identity of the cyclic group:

$$\begin{array}{lcl} a & \rightarrow & (1\ 2) \\ b & \rightarrow & (1)(2) \\ ab & \rightarrow & (1\ 2). \end{array}$$

Hence we have a permutation representation of $\Gamma/N \cong C_2$. By permutation method, N has the signature $(g;\ 3,\ 3;\ 1)$. Here g is the genus of the underlying Riemann surface on which the group acts, which is 0 by the Riemann-Hurwitz formula. As a is of order 2, and as there is one cycle of length 2, 2/2=1 and we omit it. As b is of order 5, and as there are two cycles of length 1 each, we have two 3/1=3's in the signature. Finally, ab is of infinite order, and there is only one cycle for ab giving $\infty/1 = \infty$, denoted by 1 in the signature. Finally the corresponding normal subgroup has the signature $(0;\ 3,\ 3;\ 1)$ which is Γ^2, by Theorem 1.2 (a).

3. Let us map Γ onto C_3. As

$$\begin{array}{lcl} a & \rightarrow & (1)(2)(3) \\ b & \rightarrow & (1\ 2\ 3) \\ ab & \rightarrow & (1\ 2\ 3). \end{array}$$

we have,

$$N = (0;\ 2^{(3)};\ 1) = \Gamma^3,$$

by Theorem 1.2 (b).

4. We now map Γ onto a finite quotient C_6. We use the finite quotient of $(2,\ 3,\ 6)$ of order 6 to obtain the natural epimorphism:

$$C_6 \cong C_2 \times C_3 \cong (2,\ 3,\ 6)/N.$$

Then,

$$\begin{array}{lcl} a & \rightarrow & (1\ 2)(3\ 4)(5\ 6) \\ b & \rightarrow & (1\ 3\ 5)(2\ 4\ 6) \\ ab & \rightarrow & (1\ 4\ 5\ 2\ 3\ 6), \end{array}$$

and we get $N = (g; -, 1)$. By the Riemann-Hurwitz formula, one finds $g = 1$. This subgroup is the commutator subgroup Γ' of the modular group. Secondly, we consider dihedral quotients of the modular group. In fact there is only one:

5. If we map Γ onto the dihedral quotient $D_3 \cong (2,\ 3,\ 2)$, we have

$$\begin{array}{lcl} a & \rightarrow & (1\ 2)(3\ 4)(5\ 6) \\ b & \rightarrow & (1\ 5\ 3)(2\ 4\ 6) \\ ab & \rightarrow & (1\ 4)(2\ 5)(3\ 6), \end{array}$$

and we get $N = (0; -, 3) = \Gamma(2)$.

Finally we consider the other finite quotients of the modular group:

6. If we map Γ onto $A_4 \cong (2,\ 3,\ 3)$, then

$$\begin{array}{lcl} a & \rightarrow & (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12) \\ b & \rightarrow & (1\ 7\ 6)(2\ 3\ 10)(4\ 5\ 12)(8\ 9\ 11) \\ ab & \rightarrow & (1\ 3\ 5)(2\ 7\ 9)(4\ 10\ 11)(6\ 12\ 8), \end{array}$$

giving $N = (0; -, 4) = \Gamma(3)$.

7. If we map Γ onto $S_4 \cong (2,\ 3,\ 4)$, we get, similarly to the above cases, that $N = (0; -, 6) = \Gamma(4)$.

8. Finally mapping Γ onto $PSL(2,7)$ which is isomorphic to a finite quotient of $(2,\ 3,\ 7)$ of order 168, we get $N = (g; -, 24)$. Here g can be found by the Riemann-Hurwitz formula as $g = 3$. This is a free subgroup of Γ of rank 29, which is denoted by $\Gamma(7)$.

3 Table

Let $w \in \Gamma$ be a word. Then

$$w = ab^{\varepsilon_1} ab^{\varepsilon_2} \cdots ab^{\varepsilon_n}$$

with $1 \leq \varepsilon_i \leq 2$. We denote the number of a's and b's in w by $e_a(w)$ and $e_b(w)$, respectively, [7]. Here note that $e_b(w) = \sum \varepsilon_i$. For the sake of brevity, we let $s = e_a(w)$ and $t = e_b(w)$. In the table at the end of the paper, we summarize all our results. In the first two columns we see the values of s and t. Third column denotes the abstract group structure of the quotient group. The latter columns respectively denote the signature, abstract group structure and the given name of the normal subgroup N.

4 Number of one-relator quotients of the modular group

Now we find the number of non-reduced words and then, by means of this number, we calculate the number of cyclically reduced words, which helps us in finding the number of one-relator quotients of the modular group. Γ has the relations

$$a^2 = b^3 = (ab)^\infty = 1.$$

We now add an extra relation $w = R(a,b) = 1$ where $R(a,b)$ is a cylically reduced word of the form $R(a,b) = ab^{\varepsilon_1} ab^{\varepsilon_2} \cdots ab^{\varepsilon_n}$ with $1 \leq \varepsilon_i \leq 2$, [5]. Then it is easy to see the following result:

Theorem 4.1. *[9] If $e_a(w) = 0$, then $1 \leq e_b(w) \leq 2$ and if $e_a(w) = n$, then $n \leq e_b(w) \leq 2n$.*

5 Number of non-reduced words

We denote the number of all possible words w having $e_a(w) = s$ and $e_b(w) = t$ by $n_{s,t}(w)$. Then we have the following result for $n_{s,t}(w)$:

Theorem 5.1. *Let $t \geq s$. Then*

$$n_{s,t}(w) = \binom{s}{t\text{-}s}.$$

Proof. Let $w = ab^{\varepsilon_1} ab^{\varepsilon_2} \cdots ab^{\varepsilon_n}$ where $\varepsilon_i = 1$ or 2. As $e_a(w) = s$, there are s times a in w, and similarly as $e_b(w) = t$, we have $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = t$. As there are s a's and a is of order 2, we must have b's exactly in s places between a's and there is a power b at the end, i.e. the word must finish by a power of b, as w is cyclically reduced. If we put s b's in this way, there remain $t - s$ b's. Therefore our problem turns into the one in which we want to put $t - s$ black beads to a necklace of s beads. This number is well known as the binomial coefficient $\binom{\text{s}}{\text{t-s}}$ given above. □

Corollary 5.2. *Let $m \geq 1$ be an integer. Then*

$$n_{m,m}(w) + n_{m,m+1}(w) = n_{m+1,m+2}(w),$$

and for $0 \leq k \leq 2n$,

$$n_{m,m+k}(w) + n_{m,m+k+1}(w) = n_{m+1,m+k+2}(w).$$

Proof. It is obvious since $\begin{pmatrix} \text{n} \\ \text{k} \end{pmatrix} + \begin{pmatrix} \text{n} \\ \text{k+1} \end{pmatrix} = \begin{pmatrix} \text{n+1} \\ \text{k+1} \end{pmatrix}$. □

Example 5.3. Let $s = 3$. Then $w = ab^{\varepsilon_1}ab^{\varepsilon_2}ab^{\varepsilon_3}$ with $\varepsilon_i = 1$ or 2 for $i = 1,\ 2,\ 3$. If $t = 3$, then $w = ababab$. If $t = 4$, then $w = ab^2abab$, $w = abab^2ab$ or $w = ababab^2$. If $t = 5$, then $w = ab^2ab^2ab$, $w = abab^2ab^2$ or $w = ab^2abab^2$. Finally if $t = 6$, then $w = ab^2ab^2ab^2$. Note that the total number of words in each case is 1, 3, 3, 1, which are $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$, respectively.

6 Number of non-equivalent cyclically reduced words

Let $w \in \Gamma$ be a word. We denote the number of non-equivalent cyclically reduced words w having $e_a(w) = s$ and $e_b(w) = t$ by $N_{s,t}$.

Example 6.1. In Example 5.3, we have found all non-reduced words. When $s = t = 3$, there is only one word. For $t = 4$, there are three words and they give the same cyclically reduced word. Similarly when $t = 5$, the three words obtained are cyclically the same. So $N_{3,3}(w) = N_{3,4}(w) = N_{3,5}(w) = N_{3,6}(w) = 1$.

For the higher values of s, it is not easy to calculate $N_{s,t}$. We may think of this problem in the following way:

There are s black beads and t white beads to be put onto a necklace, where $t \geq s$. We think of a necklace because of cyclic reduction. When $t = s$, there is only one possible situation with every white bead is between two blacks. If $s = t + 1$, then after putting t white beads between s black beads, we need a place to put the excess one. Thinking of cyclic reduction, there is, again, one possible case.

In general, as we have s black and t white beads to be ordered on a necklace, so that no two black beads are next to each other, we need to express the number t as the ordered sum of s positive integers, counting only ones those ordered sums that can be transformed onto each other by a cyclic permutation. This number, we denoted by $N_{s,t}$ earlier, which is given in [4], is also valid for the number of cyclically reduced words.

Theorem 6.2. *Let* $t \geq s$. *Then the number of non-equivalent cyclically reduced words in* Γ *with* $e_a(w) = s$ *and* $e_b(w) = t$ *is*

$$N_{s,t} = \frac{1}{s} \sum_{d|(s,t-s)} \varphi(d) \begin{pmatrix} s/d \\ (t\text{-}s)/d \end{pmatrix}.$$

Knowing the total number of possible words w in the modular group Γ one can easily find the normal subgroups of Γ corresponding to one relator quotients.

References

[1] I. N. Cangul, Normal Subgroups of Hecke Groups, PhD Thesis. Un. of Southampton, Southampton, UK 1994

[2] I. N. Cangul, Determining Isomorphism Class of a Fuchsian Group from its Signature, Bull. Acad. Sinica 29 (2001) 313-316

[3] I. N. Cangul, D. Singerman, Normal Subgroups of Hecke Groups and Regular Maps, Math. Proc. Cambridge Phil. Soc. 123 (1998) 59-74

[4] D. E. Knuth, Computer Science and its Relation to Mathematics, Amer. Math. Monthly 81 (1974) 323-343

[5] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory, Dover Pub. Inc., New York, USA 1976

[6] M. Newman, The Structure of Some Subgroups of the Modular Group, Illionis J. Math. 6 (1962) 480-487

[7] M. Newman, Classification of Normal Subgroups of the Modular Group, Trans. A.M.S. 126 (1967) 267-277

[8] D. Singerman, Subgroups of Fuchsian Groups and Finite Permutation Groups, Bull. L. M. S. 2 (1970) 319-323

[9] Y. T. Ulutas, I. N. Cangul, One Relator Quotients of the Hecke Group $H(((1+5)/2))$, Bull. Acad. Sinica 31 (2003) 59-74

s	t	Γ/N	triangle group	abstract structure of N	N
0	1	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
0	2	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
1	0	C_3	(0;2âĄ¡ÂşâĄ¿,1)	$C_2 * C_2 * C_2$	Γ^3
1	1	C_1	(0;2,3;1)	$C_2 * C_3$	Γ
1	2	C_1	(0;2,3;1)	$C_2 * C_3$	Γ
2	2	D_3	(0;-;3)	F_2	$\Gamma(2)$
2	3	C_6	(1;-;1)	F_2	Γ'
2	4	D_3	(0;-;3)	F_2	$\Gamma(2)$
3	3	A_4	(0;-;4)	F_3	$\Gamma(3)$
3	4	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
3	5	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
3	6	A_4	(0;-,4)	F_3	$\Gamma(3)$
4	4	S_4	(0;-,6)	F_5	$\Gamma(4)$
4	5	D_3	(0;-,3)	F_2	$\Gamma(2)$
4	7	D_3	(0;-,3)	F_2	$\Gamma(2)$
4	8	S_4	(0;-,6)	F_5	$\Gamma(4)$
5	5	A_5	(0;-,12)	F_{11}	$\Gamma(5)$
5	6	A_4	(0;-;4)	F_3	$\Gamma(3)$
5	7	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
5	8	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
5	9	A_4	(0;-;4)	F_3	$\Gamma(3)$
5	10	A_5	(0;-;12)	F_{11}	$\Gamma(5)$
6	7	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
6	8	D_3	(0;-;4)	F_2	$\Gamma(2)$
6	10	D_3	(0;-;4)	F_2	$\Gamma(2)$
6	11	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
7	7	$PSL(2,7)$	(3;-;24)	F_{29}	$\Gamma(7)$
7	8	C_1	(0;2,3;1)	$C_2 * C_3$	Γ
7	9	D_3	(0;-;3)	F_2	$\Gamma(2)$
7	10	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
7	11	C_2	(0;3,3;1)	$C_3 * C_3$	Γ^2
7	12	D_3	(0;-;4)	F_2	$\Gamma(2)$
7	13	C_1	(0;2,3;1)	$C_2 * C_3$	Γ
7	14	$PSL(2,7)$	(3;-;24)	F_{29}	$\Gamma(7)$

Author information

Hacer Ozden and Ismail Naci Cangul, Department of Mathematics, Uludag University, Bursa 16059, Turkey.
E-mail: `hozden@uludag.edu.tr, ncangul@gmail.com`

Received: November 23, 2016.

Accepted: April 17, 2017.