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GENERALIZED SOME HERMITE-HADAMARD-TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract In this paper, several inequalities of Hermite-Hadamard type for functions convex on the co-ordinates are given. Obtained results in this work are the generalization of the some Hermite-Hadamard type inequalities for co-ordinated convex functions.

1 Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. Then the following double inequality is known in the literature as the Hermite-Hadamard's inequality for convex functions:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Let us consider a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on Δ if for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$, it satisfies the following inequality:

$$f(tx + (1-t)\,z, ty + (1-t)\,w) \leq t\,f(x, y) + (1-t)\,f(z, w).$$

A modification for convex function on Δ was defined by Dragomir [6], as follows:

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$ and $y \in [c, d]$.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.1. A function $f : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, if for all $(x, u), (y, v) \in \Delta$ and $t, s \in [0, 1]$, it satisfies the following inequality:

$$\begin{aligned} & f(tx + (1-t)\,y, su + (1-s)\,v) \\ \leq \quad & ts\,f(x, u) + t(1-s)f(x, v) + s(1-t)f(y, u) + (1-t)(1-s)f(y, v). \end{aligned}$$

Note that every convex function $f : \Delta \to \mathbb{R}$ is co-ordinated convex but the converse is not generally true (see, [6]).

In [6], Dragomir proved the following inequality which is Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the following* *inequalities;*

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx\right.\\
&\quad\left.+\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy\right]\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

The above inequalities are sharp.

Some new Hermite-Hadamard type inequalities for co-ordinated convex functions are proved by many authors. In [2], Alomari and Darus defined co-ordinated s-convex functions and proved some inequalities based on this definition. In [8], Latif and Alomari proved similar results for h-convex functions on the co-ordinates. In [5], inequalities of Hadamard type for co-ordinated log-convex functions are defined in rectangle from the plane by Alomari and Darus. In [9], Sarikaya et. al. proved Hermite-Hadamard type inequalities for differentiable co-ordinated convex functions. In [15], Özdemir et. al. proved Hadamard's type inequalities for co-ordinated m convex and (α,m) convex functions.

For recent developments about Hermite-Hadamard's inequality for some convex functions on the coordinates, please refer to ([1],[2], [5]-[11], [14]-[18] and [20]). Also several inequalities for convex functions on the co-ordinates see the references [3], [4], [12], [13], and [19].

The aim of the this paper is to obtain generalized new Hermite-Hadamard type inequalities of co-ordinated convex functions of 2-variables.

2 Main Results

We start with the following Lemma which is important our main results.

Lemma 2.1. *Suppose that $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is a partial differentiable mapping on Δ and $m_1,m_2,n_1,n_2\in\mathbb{R}^+$. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, then we have the following equality;*

$$
\begin{aligned}
&\frac{n_1n_2\,f(a,c)+n_1m_2\,f(b,c)+m_1n_2\,f(a,d)+m_1m_2\,f(b,d)}{(m_1+n_1)(m_2+n_2)}\\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-\frac{1}{(m_2+n_2)(d-c)}\int_c^d(n_2\,f(a,y)+m_2\,f(b,y))dy\\
&-\frac{1}{(m_1+n_1)(b-a)}\int_a^b(n_1\,f(x,c)+m_1\,f(x,d))dx
\end{aligned}
\tag{2.1}
$$

$$= \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}$$

$$\times \int_0^1 \int_0^1 [m_1-(m_1+n_1)s]\,[m_2-(m_2+n_2)t]\,\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d)dtds.$$

Proof. Taking partial integration, we have

$$\int_0^1 \int_0^1 [m_1-(m_1+n_1)s]\,[m_2-(m_2+n_2)t]\,\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d)dtds \tag{2.2}$$

$$= \int_0^1 [m_1-(m_1+n_1)s]\left\{[m_2-(m_2+n_2)t]\ \frac{1}{a-b}\ \frac{\partial f}{\partial s}(ta+(1-t)b, sc+(1-s)d)\Big|_0^1\right.$$

$$\left. +\frac{(m_2+n_2)}{a-b}\int_0^1 \frac{\partial f}{\partial s}(ta+(1-t)b, sc+(1-s)d)\ dt\right\} ds$$

$$= \int_0^1 [m_1-(m_1+n_1)s]\left\{-\frac{n_2}{a-b}\ \frac{\partial f}{\partial s}(a, sc+(1-s)d) - \frac{m_2}{a-b}\frac{\partial f}{\partial s}(b, sc+(1-s)d)\right.$$

$$\left. +\frac{(m_2+n_2)}{a-b}\int_0^1 \frac{\partial f}{\partial s}(ta+(1-t)b, sc+(1-s)d)\ dt\right\} ds$$

$$= \frac{1}{b-a}\left\{\int_0^1 [m_1-(m_1+n_1)s]\left(\ n_2\ \frac{\partial f}{\partial s}(a, sc+(1-s)d) + m_2\ \frac{\partial f}{\partial s}(b, sc+(1-s)d)\right)\ ds\right.$$

$$\left. -(m_2+n_2)\int_0^1\int_0^1 [m_1-(m_1+n_1)s]\ \ \frac{\partial f}{\partial s}(ta+(1-t)b, sc+(1-s)d)\ dtds\right\}.$$

Again taking partial integration in the final equality of (2.2), it follows that;

$$\int_0^1 [m_1-(m_1+n_1)s]\ \left(n_2\ \frac{\partial f}{\partial s}(a, sc+(1-s)d) + m_2\ \frac{\partial f}{\partial s}(b, sc+(1-s)d)\right)\ ds \tag{2.3}$$

$$-(m_2+n_2)\int_0^1\int_0^1 [m_1-(m_1+n_1)s]\ \ \frac{\partial f}{\partial s}(ta+(1-t)b, sc+(1-s)d)\ dtds$$

$$
\begin{aligned}
&= [m_1-(m_1+n_1)s]\ \frac{(n_2\ f(a,sc+(1-s)d)+m_2\ f(b,sc+(1-s)d))}{c-d}\Big|_0^1 \\
&\quad+\frac{(m_1+n_1)}{c-d}\int_0^1 (n_2\ f(a,sc+(1-s)d)+m_2\ f(b,sc+(1-s)d))\ ds \\
&\quad-(m_2+n_2)\int_0^1 \Bigg\{[m_1-(m_1+n_1)s]\ \frac{f(ta+(1-t)b,sc+(1-s)d)}{c-d}\Big|_0^1 \\
&\quad+\frac{(m_1+n_1)}{c-d}\int_0^1 f(ta+(1-t)b,sc+(1-s)d)\ ds\Bigg\}\ dt \\
&= -n_1\frac{n_2\ f(a,c)+m_2\ f(b,c)}{c-d}-m_1\frac{n_2\ f(a,d)+m_2\ f(b,d)}{c-d} \\
&\quad+\frac{(m_1+n_1)}{c-d}\int_0^1 (n_2\ f(a,sc+(1-s)d)+m_2\ f(b,sc+(1-s)d))\ ds \\
&\quad-(m_2+n_2)\int_0^1 \Bigg\{-n_1\ \frac{f(ta+(1-t)b,c)}{c-d}-m_1\frac{f(ta+(1-t)b,d)}{c-d} \\
&\quad+\frac{(m_1+n_1)}{c-d}\int_0^1 f(ta+(1-t)b,sc+(1-s)d)\ ds\Bigg\}\ dt
\end{aligned}
$$

Writing (2.3) in (2.2) and then using change of variable $x=ta+(1-t)b$ and $y=sc+(1-s)d$ for $t,s\in[0,1]$ and finally multiplying the both sides by $\frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}$, we get (2.1). This completes the proof. □

Corollary 2.2. *If we choose $m_1=m_2=m$ and $n_1=n_2=n$ in Lemma 2.1, it follows that;*

$$
\begin{aligned}
&\frac{n^2\ f(a,c)+nm\ f(b,c)+mn\ f(a,d)+m^2\ f(b,d)}{(m+n)^2} \\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-\frac{1}{(m+n)(d-c)}\int_c^d (n\ f(a,y)+m\ f(b,y))dy \\
&-\frac{1}{(m+n)(b-a)}\int_a^b (n\ f(x,c)+m\ f(x,d))dx \\
&=\frac{(b-a)(d-c)}{(m+n)^2}\int_0^1\int_0^1 [m-(m+n)s]\,[m-(m+n)t]\,\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dtds.
\end{aligned}
\tag{2.4}
$$

Remark 2.3. If we take $m = n$ in (2.4), we have;

$$\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$

$$-\frac{1}{2}\left[\frac{1}{(d-c)}\int_c^d (f(a,y)+f(b,y))dy+\frac{1}{(b-a)}\int_a^b (f(x,c)+f(x,d))dx\right]$$

$$=\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 (1-2s)(1-2t)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)dtds.$$

which is proved by Sarikaya et.al. in [9].

Theorem 2.4. *Suppose that $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is a partial differentiable mapping on Δ and $m_1, m_2, n_1, n_2 \in \mathbb{R}^+$. If $|\frac{\partial^2 f}{\partial t\partial s}|$ is convex function on the co-ordinates on Δ, then we have the following inequality;*

$$\Bigg|\frac{n_1n_2\, f(a,c)+n_1m_2\, f(b,c)+m_1n_2\, f(a,d)+m_1m_2\, f(b,d)}{(m_1+n_1)(m_2+n_2)}$$

$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx - A\Bigg|$$

$$\leq \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}$$

$$\times\left(A_1B_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+A_1B_2\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+A_2B_1\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+A_2B_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right)$$

where

$$A=\frac{1}{(m_2+n_2)(d-c)}\int_c^d (n_2\, f(a,y)+m_2\, f(b,y))dy+\frac{1}{(m_1+n_1)(b-a)}\int_a^b (n_1\, f(x,c)+m_1\, f(x,d))dx,$$

$$A_1=\frac{m_2^3+3m_2n_2^2+2n_2^3}{6(m_2+n_2)^2},\ A_2=\frac{n_2^3+3m_2^2n_2+2m_2^3}{6(m_2+n_2)^2}$$

$$B_1=\frac{m_1^3+3m_1n_1^2+2n_1^3}{6(m_2+n_2)^2},\ B_2=\frac{n_1^3+3m_1^2n_1+2m_1^3}{6(m_2+n_2)^2}.$$

Proof. From Lemma 2.1, we know

$$\Bigg|\frac{n_1n_2\, f(a,c)+n_1m_2\, f(b,c)+m_1n_2\, f(a,d)+m_1m_2\, f(b,d)}{(m_1+n_1)(m_2+n_2)}$$

$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx - A\Bigg|$$

$$\leq \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}$$

$$\times\int_0^1\int_0^1 |m_1-(m_1+n_1)s||\, m_2-(m_2+n_2)t|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\right|dtds.$$

Since $|\frac{\partial^2 f}{\partial t \partial s}|$ is co-ordinated convex on Δ, we can write

$$\left|\frac{n_1 n_2\ f(a,c) + n_1 m_2\ f(b,c) + m_1 n_2\ f(a,d) + m_1 m_2\ f(b,d)}{(m_1+n_1)(m_2+n_2)}\right. \tag{2.5}$$

$$\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx - A\right|$$

$$\leq \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}$$

$$\times \int_0^1 \left[\int_0^1 |(m_1-(m_1+n_1)s)(m_2-(m_2+n_2)t)| \left\{t\left|\frac{\partial^2 f}{\partial t \partial s}(a, sc+(1-s)d)\right|\right.\right.$$

$$\left.\left. +(1-t)\left|\frac{\partial^2 f}{\partial t \partial s}(b, sc+(1-s)d)\right|\right\} dt\right] ds.$$

Firstly, we calculate the right-side integral of (2.5), then we have ;

$$\int_0^1 |(m_2-(m_2+n_2)t)| \left\{t\left|\frac{\partial^2 f}{\partial t \partial s}(a, sc+(1-s)d)\right| + (1-t)\left|\frac{\partial^2 f}{\partial t \partial s}(b, sc+(1-s)d)\right|\right\} dt$$

$$= \int_0^{\frac{m_2}{m_2+n_2}} |(m_2-(m_2+n_2)t)| \left\{t\left|\frac{\partial^2 f}{\partial t \partial s}(a, sc+(1-s)d)\right| + (1-t)\left|\frac{\partial^2 f}{\partial t \partial s}(b, sc+(1-s)d)\right|\right\} dt$$

$$+ \int_{\frac{m_2}{m_2+n_2}}^1 |((m_2+n_2)t-m_2)| \left\{t\left|\frac{\partial^2 f}{\partial t \partial s}(a, sc+(1-s)d)\right| + (1-t)\left|\frac{\partial^2 f}{\partial t \partial s}(b, sc+(1-s)d)\right|\right\} dt$$

$$= A_1\left|\frac{\partial^2 f}{\partial t \partial s}(a, sc+(1-s)d)\right| + A_2\left|\frac{\partial^2 f}{\partial t \partial s}(b, sc+(1-s)d)\right|.$$

Therefore we get;

$$\left|\frac{n_1 n_2\ f(a,c) + n_1 m_2\ f(b,c) + m_1 n_2\ f(a,d) + m_1 m_2\ f(b,d)}{(m_1+n_1)(m_2+n_2)}\right. \tag{2.6}$$

$$\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx - A\right|$$

$$\leq \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}$$

$$\times \int_0^1 |(m_1-(m_1+n_1)s)| \left\{A_1\left|\frac{\partial^2 f}{\partial t \partial s}(a, sc+(1-s)d)\right| + A_2\left|\frac{\partial^2 f}{\partial t \partial s}(b, sc+(1-s)d)\right|\right\} ds.$$

Now, we calculate similar way for other integral. Since $|\frac{\partial^2 f}{\partial t \partial s}|$ is co-ordinated convex on Δ, we can write;

$$\begin{aligned}
&\int_0^1 |(m_1-(m_1+n_1)s)| \left\{ A_1 \left| \frac{\partial^2 f}{\partial t \partial s}(a, sc+(1-s)d) \right| + A_2 \left| \frac{\partial^2 f}{\partial t \partial s}(b, sc+(1-s)d) \right| \right\} ds \\
\le\ & A_1 \int_0^1 |(m_1-(m_1+n_1)s)| \left\{ s \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right| \right\} ds \\
&+A_2 \int_0^1 |(m_1-(m_1+n_1)s)| \left\{ s \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| \right\} ds \\
=\ & A_1 \left[\int_0^{\frac{m_1}{m_1+n_1}} (m_1-(m_1+n_1)s) \left\{ s \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right| \right\} ds \right. \\
&\left. + \int_{\frac{m_1}{m_1+n_1}}^1 ((m_1+n_1)s-m_1) \left\{ s \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right| \right\} ds \right] \\
&+A_2 \left[\int_0^{\frac{m_1}{m_1+n_1}} (m_1-(m_1+n_1)s) \left\{ s \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| \right\} ds \right. \\
&\left. + \int_{\frac{m_1}{m_1+n_1}}^1 ((m_1+n_1)s-m_1) \left\{ s \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| + (1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| \right\} ds \right] \\
=\ & A_1 B_1 \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + A_1 B_2 \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right| + A_2 B_1 \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| + A_2 B_2 \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|.
\end{aligned} \tag{2.7}$$

Writing (2.7) in (2.6), we obtain the required inequality. □

Corollary 2.5. *If we choose $m_1 = m_2 = m$ and $n_1 = n_2 = n$ in Theorem 2.4, it follows that;*

$$\begin{aligned}
&\left| \frac{n^2 f(a,c) + nm\, f(b,c) + mn\, f(a,d) + m^2\, f(b,d)}{(m+n)^2} \right. \\
&\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx - A \right| \\
\le\ & \frac{(b-a)(d-c)}{(m+n)^2} \\
&\times \left(A_1 B_1 \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + A_1 B_2 \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right| + A_2 B_1 \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| + A_2 B_2 \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| \right)
\end{aligned} \tag{2.8}$$

where

$$A=\frac{1}{(m+n)(d-c)}\int_c^d (n\ f(a,y)+m\ f(b,y))dy+\frac{1}{(m+n)(b-a)}\int_a^b (n\ f(x,c)+m\ f(x,d))dx,$$

$$A_1=\frac{m^3+3mn^2+2n^3}{6(m+n)^2},\ A_2=\frac{n^3+3m^2n+2m^3}{6(m+n)^2}$$

$$B_1=\frac{m^3+3mn^2+2n^3}{6(m+n)^2},\ B_2=\frac{n^3+3m^2n+2m^3}{6(m+n)^2}.$$

Remark 2.6. If we take $m=n$ in (2.8), we have;

$$\left|\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\right.$$

$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\right|$$

$$\leq \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\right)$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b (f(x,c)+f(x,d))dx\frac{1}{d-c}\int_c^d (f(a,y)+f(b,y))dy\right]$$

which is proved by Sarikaya et.al. in [9].

Theorem 2.7. *Suppose that $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is a partial differentiable mapping on Δ and $m_1,m_2,n_1,n_2\in\mathbb{R}^+$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$ is convex function on the co-ordinates on Δ, then we have the following inequality;*

$$\left|\frac{n_1n_2\ f(a,c)+n_1m_2\ f(b,c)+m_1n_2\ f(a,d)+m_1m_2\ f(b,d)}{(m_1+n_1)(m_2+n_2)}\right.$$

$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\right|$$

$$\leq \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}B$$

$$\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}$$

where

$$A=\frac{1}{(m_2+n_2)(d-c)}\int_c^d (n_2\ f(a,y)+m_2\ f(b,y))dy+\frac{1}{(m_1+n_1)(b-a)}\int_a^b (n_1\ f(x,c)+m_1\ f(x,d))dx,$$

$$B=\frac{\left((m_1m_2)^{p+1}+(m_2n_1)^{p+1}+(m_1n_2)^{p+1}+(n_1n_2)^{p+1}\right)^{\frac{1}{p}}}{(m_1+n_1)^{\frac{1}{p}}(m_2+n_2)^{\frac{1}{p}}(p+1)^{\frac{2}{p}}}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and using Hölder inequality for double integrals, we get;

$$\begin{aligned}
&\left|\frac{n_1 n_2\, f(a,c)+n_1 m_2\, f(b,c)+m_1 n_2\, f(a,d)+m_1 m_2\, f(b,d)}{(m_1+n_1)(m_2+n_2)}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\right| \\
\leq\ & \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)} \\
&\times\int_0^1\int_0^1 |(m_1-(m_1+n_1)s)(m_2-(m_2+n_2)t)|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dtds \\
\leq\ & \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}\left(\int_0^1\int_0^1 |(m_1-(m_1+n_1)s)(m_2-(m_2+n_2)t)|^p dtds\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q dtds\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|\frac{\partial^2 f}{\partial t\partial s}|^q$ is co-ordinated convex on Δ, we can write the following inequalities for $t,s\in[0,1]$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q \\
\leq\ & t\left|\frac{\partial^2 f}{\partial t\partial s}(a,sc+(1-s)d)\right|^q+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,sc+(1-s)d)\right|^q
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q \\
\leq\ & ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \\
&+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q.
\end{aligned}$$

Therefore, it follows that;

$$\Bigg|\frac{n_1n_2\ f(a,c)+n_1m_2\ f(b,c)+m_1n_2\ f(a,d)+m_1m_2\ f(b,d)}{(m_1+n_1)(m_2+n_2)}$$

$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\Bigg|$$

$$\leq \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}B$$

$$\times\left(\int_0^1\int_0^1\left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right.\right.$$

$$\left.\left.+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}dtds\right)$$

$$\leq \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}B$$

$$\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.$$

□

Corollary 2.8. *If we choose $m_1=m_2=m$ and $n_1=n_2=n$ in Theorem 2.7, it follows that;*

$$\Bigg|\frac{n^2\ f(a,c)+nm\ f(b,c)+mn\ f(a,d)+m^2\ f(b,d)}{(m+n)^2} \tag{2.9}$$

$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\Bigg|$$

$$\leq \frac{(b-a)(d-c)}{(m+n)^2}\ B$$

$$\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}$$

where

$$A=\frac{1}{(m+n)(d-c)}\int_c^d(n\ f(a,y)+m\ f(b,y))dy+\frac{1}{(m+n)(b-a)}\int_a^b(n\ f(x,c)+m\ f(x,d))dx,$$

$$B=\frac{\left(m^{2(p+1)}+2(mn)^{p+1}+n^{2(p+1)}\right)^{\frac{1}{p}}}{(m+n)^{\frac{2}{p}}(p+1)^{\frac{2}{p}}}.$$

Remark 2.9. If we take $m = n$ in (2.9), we have;

$$\left| \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx - A \right|$$
$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \times \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{4} \right)^{\frac{1}{q}}$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a} \int_a^b (f(x,c)+f(x,d))dx \frac{1}{d-c} \int_c^d (f(a,y)+f(b,y))dy \right]$$

which is proved by Sarıkaya et. al. in [9].

Theorem 2.10. *Suppose that $f : \Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ is a partial differentiable mapping on Δ and $m_1, m_2, n_1, n_2 \in \mathbb{R}^+$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q, q \geq 1$ is convex function on the co-ordinates on Δ, then we have the following inequality;*

$$\left| \frac{n_1 n_2\, f(a,c) + n_1 m_2\, f(b,c) + m_1 n_2\, f(a,d) + m_1 m_2\, f(b,d)}{(m_1+n_1)(m_2+n_2)} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx - A \right|$$
$$\leq \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)} C \times \left(A_1 B_1 \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + A_1 B_2 \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + A_2 B_1 \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + A_2 B_2 \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q \right)^{\frac{1}{q}}$$

where

$$A = \frac{1}{(m_2+n_2)(d-c)} \int_c^d (n_2\, f(a,y) + m_2\, f(b,y))dy + \frac{1}{(m_1+n_1)(b-a)} \int_a^b (n_1\, f(x,c) + m_1\, f(x,d))dx,$$

$$A_1 = \frac{m_2^3 + 3m_2 n_2^2 + 2n_2^3}{6(m_2+n_2)^2},\ A_2 = \frac{n_2^3 + 3m_2^2 n_2 + 2m_2^3}{6(m_2+n_2)^2}$$

$$B_1 = \frac{m_1^3 + 3m_1 n_1^2 + 2n_1^3}{6(m_2+n_2)^2},\ B_2 = \frac{n_1^3 + 3m_1^2 n_1 + 2m_1^3}{6(m_2+n_2)^2}$$

$$C = \left(\frac{(m_1^2+n_1^2)(m_2^2+n_2^2)}{4(m_1+n_1)(m_2+n_2)} \right)^{1-\frac{1}{q}}.$$

Proof. By Lemma 2.1, and power mean inequality for double integrals, we get;

$$\begin{aligned}
&\left|\frac{n_1n_2\, f(a,c)+n_1m_2\, f(b,c)+m_1n_2\, f(a,d)+m_1m_2\, f(b,d)}{(m_1+n_1)(m_2+n_2)}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\right| \\
\leq\ & \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)} \\
&\times\int_0^1\int_0^1 |(m_1-(m_1+n_1)s)(m_2-(m_2+n_2)t)|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right| dtds
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}\left(\int_0^1\int_0^1 |(m_1-(m_1+n_1)s)(m_2-(m_2+n_2)t)|dtds\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\int_0^1 |(m_1-(m_1+n_1)s)(m_2-(m_2+n_2)t)|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q dtds\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is co-ordinated convex on Δ, it follows that;

$$\begin{aligned}
&\left|\frac{n_1n_2\, f(a,c)+n_1m_2\, f(b,c)+m_1n_2\, f(a,d)+m_1m_2\, f(b,d)}{(m_1+n_1)(m_2+n_2)}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right| \\
\leq\ & \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}C \\
&\times\left(\int_0^1\int_0^1 |(m_1-(m_1+n_1)s)(m_2-(m_2+n_2)t)|\left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right.\right. \\
&\left.\left.+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.10}$$

Firstly, we calculate the right-side integral of (2.10), then we have ;

$$\int_0^1 |(m_2-(m_2+n_2)t)| \left\{ ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \right.$$
$$\left. +(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q \right\} dt$$
$$= \int_0^{\frac{m_2}{m_2+n_2}} (m_2-(m_2+n_2)t) \left\{ ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \right.$$
$$\left. +(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q \right\} dt$$
$$+ \int_{\frac{m_2}{m_2+n_2}}^1 ((m_2+n_2)t-m_2) \left\{ ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \right.$$

$$\left. +(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q \right\} dt$$
$$= s\frac{m_2^3}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + (1-s)\frac{m_2^3}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q$$
$$+s\frac{2m_2^3+3m_2^2n_2}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + (1-s)\frac{2m_2^3+3m_2^2n_2}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q$$
$$+s\frac{2n_2^3+3m_2n_2^2}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + (1-s)\frac{2n_2^3+3m_2n_2^2}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q$$
$$+s\frac{n_2^3}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + (1-s)\frac{n_2^3}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q$$
$$= s\frac{m_2^3+3m_2n_2^2+2n_2^3}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + (1-s)\frac{m_2^3+3m_2n_2^2+2n_2^3}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q$$
$$+s\frac{n_2^3+3m_2^2n_2+2m_2^3}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + (1-s)\frac{n_2^3+3m_2^2n_2+2m_2^3}{6(m_2+n_2)^2}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q$$
$$= sA_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + (1-s)A_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q + sA_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + (1-s)A_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q.$$

Thus, we obtain;

$$\left|\frac{n_1n_2\ f(a,c)+n_1m_2\ f(b,c)+m_1n_2\ f(a,d)+m_1m_2\ f(b,d)}{(m_1+n_1)(m_2+n_2)}\right. \tag{2.11}$$

$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\right|$$

$$\leq \frac{(b-a)(d-c)}{(m_1+n_1)(m_2+n_2)}C$$

$$\times\left(\int_0^1 |(m_1-(m_1+n_1)s)|\left\{sA_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+(1-s)A_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right.\right.$$

$$\left.\left.+sA_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-s)A_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}ds\right)^{\frac{1}{q}}.$$

Now, we calculate similar way for other integral. Since $|\frac{\partial^2 f}{\partial t\partial s}|$ is co-ordinated convex on Δ, we can write;

$$\tag{2.12}$$

$$\int_0^1 |(m_1-(m_1+n_1)s)|\left\{sA_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+(1-s)A_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right.$$

$$\left.+sA_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-s)A_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}ds$$

$$=\int_0^{\frac{m_1}{m_1+n_1}} (m_1-(m_1+n_1)s)\left\{sA_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+(1-s)A_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right.$$

$$\left.+sA_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-s)A_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}ds$$

$$+\int_{\frac{m_1}{m_1+n1}}^1 ((m_1+n_1)s-m_1)\left\{sA_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+(1-s)A_1\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right.$$

$$\left.+sA_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-s)A_2\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}ds$$

$$\begin{aligned}
&= A_1 \frac{m_1^3}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + A_1 \frac{2m_1^3+3m_1^2 n_1}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \\
&\quad + A_2 \frac{m_1^3}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + A_2 \frac{2m_1^3+3m_1^2 n_1}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q \\
&\quad + A_1 \frac{2n_1^3+3m_1 n_1^2}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + A_1 \frac{n_1^3}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \\
&\quad + A_2 \frac{2n_1^3+3m_1 n_1^2}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + A_2 \frac{n_1^3}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q \\
&= A_1 \frac{m_1^3+3m_1 n_1^2+2n_1^3}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + A_1 \frac{n_1^3+3m_1^2 n_1+2m_1^3}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \\
&\quad + A_2 \frac{m_1^3+3m_1 n_1^2+2n_1^3}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + A_2 \frac{n_1^3+3m_1^2 n_1+2m_1^3}{6(m_1+n_1)^2}\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q \\
&= A_1 B_1 \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + A_1 B_2 \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + A_2 B_1 \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + A_2 B_2 \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q .
\end{aligned}$$

Writing (2.12) in (2.11) we obtain the required inequality. □

Corollary 2.11. *If we choose* $m_1 = m_2 = m$ *and* $n_1 = n_2 = n$ *in Theorem 2.10, it follows that;*

$$\begin{aligned}
&\left| \frac{n^2 f(a,c) + nm\, f(b,c) + mn\, f(a,d) + m^2 f(b,d)}{(m+n)^2} \right. \\
&\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx - A \right| \\
\leq\ & \frac{(b-a)(d-c)}{(m+n)^2} C \\
&\times \left(A_1 B_1 \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + A_1 B_2 \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + A_2 B_1 \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + A_2 B_2 \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q \right)^{\frac{1}{q}}
\end{aligned} \tag{2.13}$$

where

$$A = \frac{1}{(m+n)(d-c)} \int_c^d (n\, f(a,y) + m\, f(b,y))\,dy + \frac{1}{(m+n)(b-a)} \int_a^b (n\, f(x,c) + m\, f(x,d))\,dx,$$

$$A_1 = \frac{m^3+3mn^2+2n^3}{6(m+n)^2},\ A_2 = \frac{n^3+3m^2 n+2m^3}{6(m+n)^2}$$

$$B_1 = \frac{m^3+3mn^2+2n^3}{6(m+n)^2},\ B_2 = \frac{n^3+3m^2 n+2m^3}{6(m+n)^2}$$

$$C = \left(\frac{(m^2+n^2)^2}{4(m+n)^2} \right)^{1-\frac{1}{q}}.$$

Remark 2.12. If we take $m = n$ in (2.13), we have;

$$\begin{aligned}
&\left| \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} \right. \\
&\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx - A \right| \\
\leq\ & \frac{(b-a)(d-c)}{16} \\
&\times \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{4} \right)^{\frac{1}{q}}
\end{aligned}$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b (f(x,c)+f(x,d))dx \frac{1}{d-c}\int_c^d (f(a,y)+f(b,y))dy \right]$$

which is proved by Sarıkaya et.al in [9].

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