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ON CONVEXITY FOR ENERGY DECAY RATES OF A VISCOELASTIC EQUATION WITH A DYNAMIC BOUNDARY AND NONLINEAR DELAY TERM IN THE NONLINEAR INTERNAL FEEDBACK

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Abstract. In this paper we consider the weak viscoelastic wave equation with a delay term in the nonlinear internal feedback

$$u_{tt}(x,t) - \Delta_x u(x,t) + \alpha(t)\int_0^t h(t-s)\Delta_x u(x,s)ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0$$

in a bounded domain, and prove a global existence result which depends on the behavior of both α and h using the energy method combined with the Faedo-Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we study the asymptotic behavior of solutions using a perturbed energy method.

1 Introduction

In this work, we investigate the existence and decay properties of solutions for the initial boundary value problem of the nonlinear weak viscoelastic wave equation of the type

$$(P)\qquad \begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \alpha(t)\int_0^t h(t-s)\Delta_x u(x,s)\,ds & \\ \quad +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0 & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times]0,\tau[, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, α and h are positive non-increasing function defined on $\mathbb{R}^+$, g_1 and g_2 are two functions, $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers, and the initial data (u_0, u_1, f_0) belong to a suitable function space.This type of problems arise in viscoelasticity and, for $\alpha = 1$, the problem has been discussed by many researchers.

In the absence of the viscoelastic term (that is, if $h = 0$), problem (P) has been studied by many mathematicians. It is well known that in the further absence of a damping mechanism, the delay term causes instability of system (see, for instance **[13]**). In the contrast, in the absence of the delay term, the damping term assures global existence for arbitrary initial data and energy decay estimates depending on the rate of growth of g_1 (see **[2]**,**[4]**, **[15]**, **[16]** and **[19]**). In recent years, the PDEs with time delay effects have become an active area of research and arise in many pratical real world problems (see for example **[1]**, **[32]**). To stabilize a hyperbolic system involving delay terms, additional control terms are necessary (see **[27]**, **[28]** and **[33]**). In **[27]** the authors examined problem (P) in the linear situation (that is, if $g_1(s) = g_2(s) = s\ \forall s \in \mathbb{R}$) and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if

$\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in **[27]** is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay acting in the boundary domain. We also recall the result by Xu, Yung and Li **[33]**, where the authors proved the same result as in **[27]** for the one space dimension by adopting the spectral analysis approach. Very recently, Benaissa and Louhibi **[5]** extended the result of **[27]** to the nonlinear case.

In the presence of the viscoelastic term $(h \neq 0)$, Cavalcanti et al. **[8]** studied (P) for $g_2 \equiv 0$ and in the presence of a linear localized frictional damping $(a(x)u_t)$. They obtained an exponential rate of decay by assuming that the kernel h is of exponential decay. This work was later improved by Berrimi and Messaoudi **[7]** by introducing a different functional, which allowed them to weaken the conditions on h. In **[23]**, Messaoudi investigated the decay rate to (P) under a more general condition on h and improved earlier results in which only the exponential and polynomial rates were considered. Kirane and Said Houari **[5]** extended the result in **[23]** to the case when g_1, g_2 are linear and $\mu_1 \geq \mu_2$.

Motivated by the works, we investigate in this paper system (P) and prove a global solvability and energy decay estimates of the solutions of problem (P) which depends on the behavior of both α and h and g_1, g_2 are nonlinear. To obtain global solutions of problem (P), we use the Galerkin approximation scheme (see **[20]**) together with the energy estimate method. The technic based on the theory of nonlinear semigroups used in **[27]** does not seem to be applicable in the nonlinear case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Lasiecka et al. **[9]**, **[12]**, **[17]**, **[18]** and **[19]**, and used by Liu and Zuazua **[21]**, Eller et al **[14]** and Alabau-Boussouira **[2]**.

2 Preliminaries and main results

For the relaxation function g and the potential α, we assume that (see **[24]**):
(H1) $(*)$ $h, \alpha : \mathbb{R}_+ \to \mathbb{R}_+$ are non-increasing differentiable function satisfying

$$h(0) = h_0 > 0, \qquad \int_0^{+\infty} h(s)ds < +\infty \qquad \alpha(t) > 0, \qquad 1 - \alpha(t)\int_0^t h(s)ds \geq l > 0.$$

$(**)$ There exists a non-increasing differentiable function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\zeta(s) > 0 \qquad h'(s) \leq -\zeta(s)h(s), \qquad \forall s \geq 0, \qquad \lim_{s\to+\infty} \frac{-\alpha'(s)}{\zeta(s)\alpha(s)} = 0$$

Remark 2.1. Note $(*)$ and $(**)$ imply $\lim_{s\to+\infty} \frac{-\alpha'(s)}{\alpha(s)} = 0$

Remark 2.2. Condition $1 - \alpha(t)\int_0^t h(s)ds \geq l > 0$ is made so that (P) is hyperbolic and the energy functional (2.11) below is nonnegative.

(H2) $g_1 : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of class $C(\mathbb{R})$ such that there exist $\epsilon', c_1, c_2, \alpha_1, \alpha_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0,\infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon']$ or $(H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon'])$, such that

$$|g_1(s)| \leq c_2|s|, \quad \text{if } |s| \geq \epsilon'. \tag{2.1}$$

$$g_1^2(s) \leq H^{-1}(sg_1(s)), \quad \text{if } |s| \leq \epsilon'. \tag{2.2}$$

$g_2 : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|g_2'(s)| \leq c_3 \tag{2.3}$$

$$\alpha_1\ sg_2(s) \le G_2(s) \le \alpha_2\ sg_1(s), \tag{2.4}$$

where

$$G_2(s) = \int_0^s g_2(r)\, dr.$$

$$\alpha_2\mu_2 < \alpha_1\mu_1. \tag{2.5}$$

Remark 2.3. 1.By the mean value Theorem for integrals and the monotonicity of g_2, we find that

$$G_2(s) = \int_0^s g_2(r)\, dr \le sg_2(s).$$

Then, $\alpha_1 \le 1$.

2. We need condition (2.3) only to prove global existence. For the energy decay, we can replace the linear growth of the function $g_2(s)$, for large $|s|$, by nonlinear polynomial growth.

We also state a Lemma which will be needed later.

Lemma 2.4 (Sobolev-Poincaré's inequality). *Let q be a number with $2 \le q < +\infty$ $(n = 1, 2)$ or $2 \le q \le 2n/(n-2)$ $(n \ge 3)$. Then there is a constant $c_* = c_*(\Omega, q)$ such that*

$$\|u\|_q \le c_* \|\nabla u\|_2 \quad \textit{for} \quad u \in H_0^1(\Omega).$$

We introduce as in **[27]** the new variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho),\ x \in \Omega,\ \rho \in]0, 1[, \quad t > 0. \tag{2.6}$$

Then, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \ \text{ in } \Omega \times]0, 1[\times]0, +\infty[. \tag{2.7}$$

Therefore, problem (P) takes the form:

$$\begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + \alpha(t)\int_0^t h(t-s)\Delta_x u(x,s)ds & \\ \qquad +\mu_1 g_1(u_t(x,t)) + \mu_2 g_2(z(x,1,t)) = 0, & \text{in } \Omega\times]0,+\infty[, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & \text{in } \Omega\times]0,1[\times]0,+\infty[, \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,+\infty[, \\ z(x,0,t) = u_t(x,t), & \text{on } \Omega\times]0,+\infty[, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau), & \text{in } \Omega\times]0,1[. \end{cases} \tag{2.8}$$

Let ξ be a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1} < \xi < \tau\frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}. \tag{2.9}$$

We define the energy associated to the solution of problem (2.8) by the following formula:

$$\begin{aligned} E(t) = \tfrac{1}{2}\|u'(t)\|_2^2 + \tfrac{1}{2}\left(1 - \alpha(t)\int_0^t h(s)\, ds\right)\|\nabla_x u(t)\|_2^2 + \alpha(t)\tfrac{1}{2}(h\circ\nabla u)(t) \\ +\xi\int_\Omega\int_0^1 G_2(z(x,\rho,t))\, d\rho\, dx. \end{aligned} \tag{2.10}$$

Where

$$(h\circ\nabla u)(t) = \int_0^t h(t-s)\ \|\ \nabla u(t) - \nabla u(t-s)\ \|_2^2\ ds$$

We have the following theorem.

Theorem 2.5. *Let* $(u_0, u_1, f_0) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega; H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(.,0) = u_1.$$

Assume that the hypotheses $(H1) - (H2)$ *hold. Then problem* (P) *admits a unique weak solution*

$$u \in L^\infty_{loc}((-\tau, +\infty); H^2(\Omega) \cap H_0^1(\Omega)), u' \in L^\infty_{loc}((-\tau, +\infty); H_0^1(\Omega)), u'' \in L^\infty_{loc}((-\tau, +\infty); L^2(\Omega))$$

and, for some constants ω, ϵ_0*, we obtain the following decay property:*

$$E(t) \leq H_1^{-1}\left(\omega \int_0^t \alpha(s)\zeta(s)\, ds\right), \quad \forall t > 0, \tag{2.11}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds \tag{2.12}$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \epsilon'], \\ tH'(\epsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \epsilon']. \end{cases}$$

Example. Let g be given by $g(s) = s^p(-\ln s)^q$, where $0 \leq p \leq 1$ and $q \in \mathbb{R}$ on $(0, \epsilon_1]$. Then $g'(s) = s^{p-1}(-\ln s)^{q-1}(p(-\ln s) - q)$ which is an increasing function in a right neighborhood of 0 (if $q = 0$ we can take $\epsilon_1 = 1$). The function H is defined in the neighborhood of 0 by

$$H(s) = \begin{cases} cs^{\frac{p+1}{2p}}(-\ln s)^{-\frac{q}{p}} & \text{if } 0 < p < 1, \quad q \in \mathbb{R} \\ cs(-\ln s)^{-q} & \text{if } p = 1, \quad q > 0 \\ c\sqrt{s}\, e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases}$$

and we have

$$H'(s) = \begin{cases} cs^{\frac{1-p}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right) & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R} \\ c\frac{1}{\sqrt{s}}\left(1 - \frac{1}{q}\, s^{\frac{1}{2q}}\right) e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases} \quad \text{when } s \text{ is near } 0.$$

Thus

$$\varphi(s) = \begin{cases} cs^{\frac{p+1}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right) & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R} \\ c\sqrt{s}\left(1 - \frac{1}{q}\, s^{\frac{1}{2q}}\right) e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases} \quad \text{when } s \text{ is near } 0.$$

and

$$\begin{aligned} \psi(t) &= c\int_t^1 \frac{1}{s^{\frac{p+1}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right)}\, ds \\ &= c\int_1^{\frac{1}{t}} \frac{z^{\frac{1-3p}{2p}}}{(\ln z)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}\ln z + \frac{q}{p}\right)}\, dz \text{ when } t \text{ is near } 0. \end{aligned}$$

and

$$\begin{aligned} \psi(t) &= c\int_t^1 \frac{1}{\sqrt{s}\left(1 - \frac{1}{q}\, s^{\frac{1}{2q}}\right) e^{-s^{\frac{1}{2q}}}}\, ds \\ &= c\int_1^{\frac{1}{t}} \frac{e^{\left(\frac{1}{z}\right)^{\frac{1}{2q}}}}{z^{\frac{3}{2}}\left(1 - \frac{1}{q}\left(\frac{1}{z}\right)^{\frac{1}{2q}}\right)}\, dz, \quad p = 0, q < 0, \text{ when } t \text{ is near } 0 \end{aligned}$$

We obtain in a neighborhood of 0

$$\psi(t) \equiv \begin{cases} ct^{\frac{p-1}{2p}}(-\ln t)^{\frac{q}{p}} & \text{if } 0 < p < 1, \quad q \in \mathbb{R} \\ c(-\ln t)^{1+q} & \text{if } p = 1, \, q > 0, \\ ct^{\frac{q-2}{2q}} e^{t^{\frac{1}{2q}}} & \text{if } p = 0, \, q < 0 \end{cases}$$

and then in a neighborhood of $+\infty$ (see Appendix)

$$\psi^{-1}(t) \equiv \begin{cases} ct^{\frac{2p}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R}, \\ ce^{-t^{\frac{1}{1+q}}} & \text{if } p = 1,\ q > 0, \\ c(\ln t)^{2q} & \text{if } p = 0,\ q < 0. \end{cases}$$

Using the fact that $h(t) = t$ as t goes to infinity, then

$$E(t) \leq \begin{cases} c\sigma(t)^{-\frac{2}{p-1}}(\ln \tilde{\xi}(t))^{-\frac{2q}{p-1}} & \text{if } 0 < p \leq 1, \quad q \in \mathbb{R}, \\ ce^{-\sigma(t)^{\frac{1}{1+q}}} & \text{if } p = 1,\ q < 1, \\ c(\ln \sigma(t))^{2q} & \text{if } p = 0,\ q < 0, \\ ce^{-\sigma(t)} & \text{if } p > 1 \text{ or } p = 1 \text{ and } q \leq 0. \end{cases}$$

where

$$\sigma(t) = \int_0^t \alpha(s)\zeta(s)\, ds$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 2.6. *Let (u, z) be a solution to the problem (2.6). Then, the energy functional defined by (2.10) satisfies*

$$\begin{aligned} E'(t) \quad \leq \quad & -\left(\mu_1 - \tfrac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega u'g_1(u')\,dx \\ & -\left(\tfrac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \\ & -\tfrac{1}{2}\alpha(t)h(t)\|\nabla u\|_2^2 + \tfrac{1}{2}\alpha(t)(h' \circ \nabla u)(t) \\ & -\tfrac{1}{2}\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2 + \tfrac{1}{2}\alpha'(t)(h \circ \nabla u)(t) \\ \leq \quad & -\tfrac{1}{2}\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2 + \tfrac{1}{2}\alpha(t)(h' \circ \nabla u)(t). \end{aligned} \tag{2.13}$$

Proof. Multiplying the first equation in (2.8) by $u_t(x,t)$, and integrating the result over Ω, to obtain:

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}(\|u_t(x,t)\|_2^2 + \|\nabla u(x,t)\|_2^2) + \mu_1 \textstyle\int_\Omega g_1(u_t(x,t))u_t(x,t)\,dx \\ &+\mu_2 \textstyle\int_\Omega g_2(z(x,1,t))u_t(x,t)\,dx = \alpha(t)\int_\Omega\int_0^t h(t-s)\nabla u(x,s)\nabla u_t(x,t)\,ds\,dx. \end{aligned} \tag{2.14}$$

The term in the right-hand side of (2.14) can be rewritten as follows

$$\begin{aligned} &\alpha(t)\textstyle\int_\Omega\int_0^t h(t-s)\nabla u(x,s)\nabla u_t(x,t)\;ds\;dx + \tfrac{1}{2}\alpha(t)h(t)\;\|\nabla u(x,t)\|_2^2 \\ &= \tfrac{1}{2}\tfrac{d}{dt}\left[\alpha(t)\textstyle\int_0^t h(s)\,ds\|\nabla u(x,t)\|_2^2 - \alpha(t)(h\circ\nabla u)(t)\right] + \tfrac{1}{2}\alpha(t)(h'\circ\nabla u)(t). \end{aligned}$$

Consequently, equality (2.14) becomes

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left[\|u_t(x,t)\|_2^2 + \left(1 - \alpha(t)\textstyle\int_0^t h(s)\,ds\right)\|\nabla u(x,t)\|_2^2 + \alpha(t)(h\circ\nabla u)(t)\right] = -\mu_1\textstyle\int_\Omega g_1(u_t(x,t))u_t(x,t)\,dx \\ &\quad -\mu_2\textstyle\int_\Omega g_2(z(x,1,t))u_t(x,t)\,dx - \tfrac{1}{2}\alpha(t)h(t)\|\nabla u(x,t)\|_2^2 + \tfrac{1}{2}\alpha(t)(h'\circ\nabla u)(t) \\ &\quad -\tfrac{1}{2}\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2 + \tfrac{1}{2}\alpha'(t)(h\circ\nabla u)(t). \end{aligned} \tag{2.15}$$

We multiply the second equation in (2.8) by $\xi g_2(z(x,\rho,t))$ and integrate over $\Omega\times]0,1[$ to obtain:

$$\begin{aligned} \xi\textstyle\int_\Omega\int_0^1 z_t(x,\rho,t))g_2(z(x,\rho,t))\,d\rho\,dx &= -\tfrac{\xi}{\tau}\textstyle\int_\Omega\int_0^1 \tfrac{\partial}{\partial\rho}G_2(z(x,\rho,t))\,d\rho\,dx \\ &= -\tfrac{\xi}{\tau}\textstyle\int_\Omega(G_2(z(x,1,t)) - G_2(z(x,0,t)))\,dx. \end{aligned} \tag{2.16}$$

Hence

$$\xi\frac{d}{dt}\int_\Omega\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx + \frac{\xi}{\tau}\int_\Omega G_2(z(x,1,t))\,dx - \frac{\xi}{\tau}\int_\Omega G_2(u_t(x,t))\,dx = 0. \tag{2.17}$$

From (2.15), (2.17) and use of Young's inequality, we get

$$\begin{aligned}E'(t) &= \tfrac{1}{2}\alpha(t)(h'\circ\nabla u)(t) - \tfrac{1}{2}\alpha(t)h(t)\|\nabla u(x,t)\|_2^2 - \mu_1\int_\Omega g_1(u_t(x,t))u_t(x,t)\,dx\\ &-\mu_2\int_\Omega g_2(z(x,1,t))u_t(x,t)\,dx - \tfrac{\xi}{\tau}\int_\Omega G_2(z(x,1,t))\,dx + \tfrac{\xi}{\tau}\int_\Omega G_2(u_t(x,t))\,dx\\ &-\tfrac{1}{2}\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2 + \tfrac{1}{2}\alpha'(t)(h\circ\nabla u)(t).\end{aligned}$$

By recalling (2.4), we arrive at

$$\begin{aligned}E'(t) \le &-\left(\mu_1 - \tfrac{\xi\alpha_2}{\tau}\right)\int_\Omega g_1(u_t(x,t))u_t(x,t)\,dx - \tfrac{1}{2}\alpha(t)(h'\circ\nabla u)(t) - \tfrac{1}{2}\alpha(t)h(t)\,\|\nabla u(x,t)\|_2^2\\ &-\mu_2\int_\Omega g_2(z(x,1,t))u_t(x,t)\,dx - \tfrac{\xi}{\tau}\int_\Omega G_2(z(x,1,t))\,dx\\ &-\tfrac{1}{2}\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2 + \tfrac{1}{2}\alpha'(t)(h'\circ\nabla u)(t).\end{aligned} \tag{2.18}$$

Let us denote by G_2^* the conjugate function of the convex function G_2, i.e., $G_2^*(s) = \sup_{t\in\mathbb{R}^+}(st - G_2(t))$. Then G_2^* is the Legendre transform of G_2, which is given by

$$G_2^*(s) = s(G_2')^{-1}(s) - G_2[(G_2')^{-1}(s)], \quad \forall s \ge 0 \tag{2.19}$$

and satisfies the following inequality

$$st \le G_2^*(s) + G_2(t), \quad \forall s,t \ge 0. \tag{2.20}$$

(see Arnold **[3]**, p. 61-62, and Lasiecka **[9]**, **[12]**, **[17]**-**[18]** for more information).

Then, from the definition of G_2, we get

$$G_2^*(s) = sg_2^{-1}(s) - G_2(g_2^{-1}(s)).$$

Hence

$$G_2^*(g_2(z(x,1,t))) = z(x,1,t)g_2(z(x,1,t)) - G_2(z(x,1,t)). \tag{2.21}$$

Making use of (2.18), (2.20) and (2.21), we arrive at

$$\begin{aligned}E'(t) \quad \le \quad &-\left(\mu_1 - \tfrac{\xi\alpha_2}{\tau}\right)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx - \tfrac{\xi\alpha_1}{\tau}\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx\\ &+\mu_2\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx \;+\mu_2\int_\Omega G_2(u_t(x,t))dx - \mu_2\int_\Omega G_2(z(x,1,t))dx\\ &+\tfrac{1}{2}\alpha(t)(h'\circ\nabla u)(t) - \tfrac{1}{2}\alpha(t)h(t)\|\nabla u(x,t)\|_2^2 - \tfrac{1}{2}\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2 + \tfrac{1}{2}\alpha'(t)(h\circ\nabla u)(t).\end{aligned} \tag{2.22}$$

Again, use of (2.4) yields

$$\begin{aligned}E'(t) \quad &\le \quad -\left(\mu_1 - \tfrac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,dx\\ &\quad -\left(\tfrac{\xi\alpha_1}{\tau} - \mu_2 + \mu_2\alpha_1)\right)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,dx\\ &\quad +\tfrac{1}{2}\alpha(t)(h'\circ\nabla u)(t) - \tfrac{1}{2}\alpha(t)h(t)\,\|\nabla u(x,t)\|_2^2\\ &\quad -\tfrac{1}{2}\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2 + \tfrac{1}{2}\alpha'(t)(h\circ\nabla u)(t)\\ &\le \quad -\tfrac{1}{2}\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2 + \tfrac{1}{2}\alpha(t)(h'\circ\nabla u)(t).\end{aligned} \tag{2.23}$$

□

Remark 2.7. Since $-\alpha'(t)(\int_0^t h(s)ds)\|\nabla u\|_2^2 \ge 0$, $E(t)$ may not be non-increasing.

3 Global Existence

We are now ready to prove Theorem 2.5 in the next two sections.

Throughout this section we assume $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $f_0 \in H_0^1(\Omega; H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2(\Omega)\cap H_0^1(\Omega)$.

Now, we define, for $1 \leq j \leq k$, the sequence $\phi_j(x, \rho)$ as follows:

$$\phi_j(x, 0) = w_j.$$

Then, we may extend $\phi_j(x, 0)$ by $\phi_j(x, \rho)$ over $L^2(\Omega \times (0, 1))$ such that $(\phi_j)_j$ form a basis of $L^2(\Omega; H^1(0, 1))$ and denote Z_k the space generated by $\{\phi_1, \phi_2, \dots, \phi_k\}$.

We construct approximate solutions (u_k, z_k), $k = 1, 2, 3, \dots$, in the form

$$u_k(t) = \sum_{j=1}^{k} g_{jk}(t) w_j,$$

$$z_k(t) = \sum_{j=1}^{k} h_{jk}(t) \phi_j,$$

where g_{ik} and h_{ik}, $j = 1, 2, \dots, k$, are determined by the following ordinary differential equations:

$$\begin{cases} (u_k''(t), w_j) + (\nabla_x u_k(t), \nabla_x w_j) - \alpha(t) \int_0^t h(t-s)(\nabla_x u_k(s), \nabla_x w_j)\, ds + \mu_1(g_1(u_k'), w_j) \\ \qquad +\mu_2(g_2(z_k(.,1)), w_j) = 0, \quad 1 \leq j \leq k, \\ \qquad z_k(x, 0, t) = u_k'(x, t) \end{cases} \tag{3.1}$$

$$u_k(0) = u_{0k} = \sum_{j=1}^{k} (u_0, w_j) w_j \to u_0 \text{ in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } k \to +\infty, \tag{3.2}$$

$$u_k'(0) = u_{1k} = \sum_{j=1}^{k} (u_1, w_j) w_j \to u_1 \text{ in } H_0^1(\Omega) \text{ as } k \to +\infty \tag{3.3}$$

and

$$(\tau z_{kt} + z_{k\rho}, \phi_j) = 0, \quad 1 \leq j \leq k, \tag{3.4}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } H_0^1(\Omega; H^1(0, 1)) \text{ as } k \to +\infty. \tag{3.5}$$

By virtue of the theory of ordinary differential equations, the system (3.1)-(3.5) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$) by Zorn lemma since the nonlinear terms in (3.1) are locally Lipschitz continuous. Note that $u_k(t)$ is C^2-class.

In the next step, we obtain a priori estimates for the solution, so that it can be extended outside $[0, T_k[$ to obtain one solution defined for all $t > 0$.

In order to use a standard compactness argument with the limiting procedure, it suffices to derive some a priori estimates for (u_k, z_k).

The first estimate. Since the sequences u_{0k}, u_{1k} and z_{0k} converge, then standard calculations, using (3.1)-(3.5), similar to those used to derive (2.13), yield

$$\begin{aligned} E_k(t) + a_1 \int_0^t \int_1^0 u_k'(x,t) g_1(u_k'(x,t))\, dx\, ds + a_2 \int_0^t \int_1^0 z_k(x,1,t) g_2(z_k(x,1,t))\, dx\, ds \\ +\tfrac{1}{2}\alpha(t)h(t)\|\nabla u_k\|_2^2 - \tfrac{1}{2}\alpha(t)(h' \circ \nabla u_k)(t) + \tfrac{1}{2}\alpha'(t)(\textstyle\int_0^t h(s) ds)\|\nabla u_k\|_2^2 - \tfrac{1}{2}\alpha'(t)(h' \circ \nabla u_k)(t) \\ \leq E_k(0) \leq C, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} E(t) = \tfrac{1}{2}\|u_k'(t)\|_2^2 + \tfrac{1}{2}\left(1 - \alpha(t) \int_0^t h(s)\, ds\right) \|\nabla_x u_k(t)\|_2^2 + \alpha(t)\tfrac{1}{2}(h \circ \nabla u_k)(t) \\ +\xi \int_\Omega \int_0^1 G_2(z(x, \rho, t))\, d\rho\, dx. \end{aligned}$$

$$a_1 = \mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2 \text{ and } a_2 = \frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1).$$

for some C independent of k. These estimates imply that the solution (u_k, z_k) exists globally in $[0,+\infty[$.

Estimate (3.6) yields

$$u_k \text{ is bounded in } L^\infty_{loc}(0,\infty; H^1_0(\Omega)), \tag{3.7}$$

$$u'_k \text{ is bounded in } L^\infty_{loc}(0,\infty; L^2(\Omega)), \tag{3.8}$$

$$u'_k(t)g_1(u'_k(t)) \text{ is bounded in } L^1(\Omega\times(0,T)), \tag{3.9}$$

$$G_2(z_k(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0,\infty; L^1(\Omega\times(0,1))), \tag{3.10}$$

$$z_k(x,1,t)g_2(z_k(x,1,t)) \text{ is bounded in } L^1(\Omega\times(0,T)). \tag{3.11}$$

The second estimate. first, we estimate $u''_k(0)$. Testing (3.1) by $g''_{jk}(t)$ and choosing $t=0$, we obtain:

$$\|u''_k(0)\|_2 \le \|\Delta_x u_{0k}\|_2 + \mu_1\|g_1(u_{1k})\|_2 + \mu_2\|g_2(z_{0k})\|_2.$$

Since $g_1(u_{1k}), g_2(z_{0k})$ are bounded in $L^2(\Omega)$ hence, from (3.2), (3.3) and (3.5),

$$\|u''_k(0)\|_2 \le C.$$

Differentiating (3.1) with respect to t, we get

$$\left(u'''_k(t) + \Delta_x u'_k(t) + \frac{d}{dt}\left(\alpha(t)\int_0^t h(t-s)\Delta_x u_k(s)\,ds\right) + \mu_1 u''_k(t)g'_1(u'_k) + \mu_2 z'_k g'_2(z_k), w_j\right) = 0.$$

Multiplying by $g''_{jk}(t)$ and summing over j from 1 to k, it follows that

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\left(\|u''_k(t)\|_2^2 + \|\nabla_x u'_k(t)\|_2^2\right) - \alpha(t)h(0)\tfrac{d}{dt}(\nabla_x u_k(t), \nabla_x u'_k(t)) + \alpha(t)h(0)\|\nabla_x u'_k(t)\|_2^2\\
&-\alpha(t)\tfrac{d}{dt}\textstyle\int_0^t h'(t-s)(\nabla_x u_k(s), \nabla_x u'_k(t))\,ds + \alpha(t)h'(0)(\nabla_x u_k(t), \nabla_x u'_k(t))\\
&+\alpha(t)\textstyle\int_0^t h''(t-s)(\nabla_x u_k(s), \nabla_x u'_k(t))\,ds + \mu_1\int_\Omega u''^2_k(t)g'_1(u'_k(t))\,dx\\
&+\mu_2\textstyle\int_\Omega u''_k(t)z'_k(x,1,t)g'_2(z_k(x,1,t))\,dx = 0.
\end{aligned} \tag{3.12}$$

Differentiating (3.4) with respect to t, we get

$$(\tau z''_k(t) + \frac{\partial}{\partial\rho}z'_k, \phi_j) = 0.$$

Multiplying by $h'_{jk}(t)$ and summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|z'_k(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2 = 0. \tag{3.13}$$

Taking the sum of (3.12) and (3.13), we obtain

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\left(\|u''_k(t)\|_2^2 + \|\nabla_x u'_k(t)\|_2^2 + \tau\|z'_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right) + \alpha(t)h(0)\|\nabla_x u'_k(t)\|_2^2\\
&+\mu_1\textstyle\int_\Omega u''^2_k(t)g'_1(u'_k(t))\,dx + \frac{1}{2}\int_\Omega |z'_k(x,1,t)|^2\,dx = \alpha(t)h(0)\frac{d}{dt}(\nabla_x u_k(t), \nabla_x u'_k(t))\\
&+\alpha(t)\tfrac{d}{dt}\textstyle\int_0^t h'(t-s)(\nabla_x u_k(s), \nabla_x u'_k(t))\,ds - \alpha(t)h'(0)(\nabla_x u_k(t), \nabla_x u'_k(t))\\
&-\alpha(t)\textstyle\int_0^t h''(t-s)(\nabla_x u_k(s), \nabla_x u'_k(t))\,ds - \mu_2\int_\Omega u''_k(t)z'_k(x,1,t)g'_2(z_k(x,1,t))\,dx + \frac{1}{2}\|u''_k(t)\|_2^2.
\end{aligned}$$

Using (2.3), Cauchy-Schwarz and Young's inequalities, we obtain

$$|\alpha(t)h'(0)(\nabla_x u_k(t), \nabla_x u'_k(t))| \le \varepsilon\alpha(t)\|\nabla_x u_k(t)\|_2^2 + \frac{\alpha(t)h'(0)^2}{4\varepsilon}\|\nabla_x u'_k(t)\|_2^2,$$

$$\begin{aligned}\left|\alpha(t)\int_0^t h''(t-s)(\nabla_x u_k(s),\nabla_x u_k'(t))\,ds\right| &\le \alpha(t)\|\nabla_x u_k'(t)\|_2\int_0^t |h''(t-s)|\|\nabla_x u_k(s)\|_2\,ds\\ &\le \frac{\alpha(t)}{4\varepsilon}\|\nabla_x u_k'(t)\|_2^2\\ &\quad+\varepsilon\alpha(t)\|h''\|_{L^1}\int_0^t |h''(t-s)|\|\nabla_x u_k(s)\|_2^2\,ds,\end{aligned}$$

$$\begin{aligned}&\tfrac12\tfrac{d}{dt}\left(\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\tau\|z_k'(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right)+\mu_1\int_\Omega {u''}_k^2(t)g_1'(u_k'(t))\,dx\\ &\quad+c\int_\Omega |z_k'(x,1,t)|^2\,dx+\alpha(t)h(0)\|\nabla_x u_k'(t)\|_2^2\\ &\le c'\|u_k''(t)\|_2^2+\varepsilon\alpha(t)\|\nabla_x u_k(t)\|_2^2+\tfrac{\alpha(t)h'(0)^2}{4\varepsilon}\|\nabla_x u_k'(t)\|_2^2+\tfrac{\alpha(t)}{4\varepsilon}\|\nabla_x u_k'(t)\|_2^2\\ &\quad+\varepsilon\alpha(t)\|h''\|_{L^1}\int_0^t|h''(t-s)|\|\nabla_x u_k(s)\|_2^2\,ds+\alpha(t)h(0)\tfrac{d}{dt}(\nabla_x u_k(t),\nabla_x u_k'(t))\\ &\quad+\alpha(t)\tfrac{d}{dt}\int_0^t h'(t-s)(\nabla_x u_k(s),\nabla_x u_k'(t))\,ds.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's lemma, we obtain

$$\begin{aligned}&\tfrac12\left(\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\tau\|z_k'(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right)+\mu_1\int_0^t\int_\Omega {u''}_k^2(t)g_1'(u_k'(s))\,dx\,ds\\ &\quad+c\int_0^t\int_\Omega|z_k'(x,1,s)|^2\,dx\,ds\le\\ &\tfrac12\left(\|u_k''(0)\|_2^2+\|\nabla_x u_k'(0)\|_2^2+\tau\|z_k'(x,\rho,0)\|^2_{L^2(\Omega\times(0,1))}\right)+\alpha(t)h(0)(\nabla_x u_k(t),\nabla_x u_k'(t))\\ &\quad-\alpha(t)h(0)(\nabla_x u_k(0),\nabla_x u_k'(0))+\alpha(t)\int_0^t h'(t-s)(\nabla_x u_k(s),\nabla_x u_k'(t))\,ds\\ &\quad+\alpha(t)\left(\tfrac{1}{4\varepsilon}+\tfrac{h'(0)^2}{4\varepsilon}-h(0)\right)\int_0^t\|\nabla_x u_k'(s)\|_2^2\,ds+\alpha(t)\left(\varepsilon+\varepsilon\|h''\|_{L^1}^2\right)\int_0^t\|\nabla_x u_k(s)\|_2^2\,ds\\ &\quad+c'\int_0^t\|u_k''(s)\|_2^2\,ds,\end{aligned}\tag{3.14}$$

$$\alpha(t)h(0)(\nabla_x u_k(t),\nabla_x u_k'(t))\le\alpha(t)\varepsilon\|\nabla_x u_k'(t)\|_2^2+\frac{\alpha(t)h(0)^2}{4\varepsilon}\|\nabla_x u_k(t)\|_2^2,$$

$$\alpha(t)\int_0^t h'(t-s)(\nabla_x u_k(s),\nabla_x u_k'(t))\,ds\le\alpha(t)\varepsilon\|\nabla_x u_k'(t)\|_2^2+\frac{\alpha(t)\xi(0)\|h\|_{L^1}\|h\|_{L^\infty}}{4\varepsilon}\int_0^t\|\nabla_x u_k(s)\|_2^2\,ds.$$

Then from (3.14), choosing ε small enough and using Gronwall's lemma, we obtain

$$\begin{aligned}&\|u_k''(t)\|_2^2+\|\nabla_x u_k'(t)\|_2^2+\tau\|z_k'(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}+\mu_1\int_0^t\int_\Omega {u''}_k^2(t)g_1'(u_k'(s))\,dx\,ds\\ &\quad+c\int_0^t\int_\Omega|z_k'(x,1,s)|^2\,dx\,ds\le M\alpha(t),\end{aligned}\tag{3.15}$$

for all $t\in[0,T]$ and M is a positive constant independent of $k\in\mathbb{N}$. Therefore, we conclude that

$$u_k'' \text{ is bounded in } L^\infty_{loc}(0,+\infty;L^2(\Omega)),\tag{3.16}$$

$$u_k' \text{ is bounded in } L^\infty_{loc}(0,+\infty;H_0^1(\Omega)),\tag{3.17}$$

$$z_k' \text{ is bounded in } L^\infty_{loc}(0,+\infty;L^2(\Omega\times(0,1))).\tag{3.18}$$

The third estimate. Replacing w_j by $-\Delta_x w_j$ in (3.1), multiplying by $g_{jm}'(t)$, summing over j from 1 to k, it follows that

$$\begin{aligned}&\tfrac12\tfrac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2+\|\Delta_x u_k(t)\|_2^2\right)-\alpha(t)\int_0^t h(t-s)(\Delta_x u(s),\Delta_x u'(t))\,ds\\ &+\mu_1\int_\Omega|\nabla_x u_k'(t)|^2g_1'(u_k'(t))\,dx+\mu_2\int_\Omega\nabla_x {u'}_k(t)\nabla_x z_k'(x,1,t)g_2'(z_k(x,1,t))\,dx=0,\end{aligned}\tag{3.19}$$

$$\begin{aligned}&\alpha(t)\int_0^t h(t-s)(\Delta_x u(s),\Delta_x u'(t))\,ds+\tfrac12\alpha(t)h(t)\|\Delta_x u(t)\|_2^2\\ &=\tfrac12\tfrac{d}{dt}\left[\alpha(t)\int_0^t h(s)\,ds\|\Delta_x u(t)\|_2^2-\alpha(t)(h\circ\Delta_x u)(t)\right]+\tfrac12\alpha(t)(h'\circ\Delta_x u)(t).\end{aligned}$$

Consequently, equality (3.19) becomes

$$\begin{aligned}&\tfrac12\tfrac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2+\left(1-\alpha(t)\int_0^t h(s)\,ds\right)\|\Delta_x u_k(t)\|_2^2+\alpha(t)(h\circ\Delta_x u)(t)\right)+\alpha(t)h(t)\|\nabla_x u_k(t)\|_2^2\\ &\quad-\tfrac12\alpha(t)(h'\circ\Delta_x u)(t)+\mu_1\int_\Omega|\nabla_x u_k'(t)|^2g_1'(u_k'(t))\,dx\\ &\quad+\mu_2\int_\Omega\nabla_x {u'}_k(t)\nabla_x z_k'(x,1,t)g_2'(z_k(x,1,t))\,dx=0.\end{aligned}\tag{3.20}$$

Replacing ϕ_j by $-\Delta_x\phi_j$ in (3.4), multiplying by $h_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|\nabla_x z_k(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|\nabla_x z_k(t)\|_2^2=0. \tag{3.21}$$

From (3.19) and (3.21), we have

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2+\left(1-\alpha(t)\int_0^t h(s)\,ds\right)\|\Delta_x u_k(t)\|_2^2+\tau\|\nabla_x z_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right)\\&+\alpha(t)h(t)\|\nabla_x u_k(t)\|_2^2-\tfrac{1}{2}\alpha(t)(h'\circ\Delta_x u)(t)+\mu_1\int_\Omega|\nabla_x u_k'(t)|^2 g_1'(u_k'(t))\,dx+\tfrac{1}{2}\int_\Omega|\nabla_x z_k(x,1,t)|^2\,dx\\&=-\mu_2\int_\Omega\nabla_x u'_k(t)\nabla_x z_k'(x,1,t)g_2'(z_k(x,1,t))\,dx+\tfrac{1}{2}\|\nabla_x u_k'(t)\|_2^2.\end{aligned}$$

Using (2.3), Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\|\nabla_x u_k'(t)\|_2^2+\left(1-\alpha(t)\int_0^t h(s)\,ds\right)\|\Delta_x u_k(t)\|_2^2+\tau\|\nabla_x z_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\right)\\&+\mu_1\int_\Omega|\nabla_x u_k'(t)|^2 g_1'(u_k'(t))\,dx+c\int_\Omega|\nabla_x z_k(x,1,t)|^2\,dx\le c'\|\nabla_x u_k'(t)\|_2^2.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's lemma, we obtain

$$\begin{aligned}&\|\nabla_x u_k'(t)\|_2^2+\left(1-\alpha(t)\int_0^t h(s)\,ds\right)\|\Delta_x u_k(t)\|_2^2+\tau\|\nabla_x z_k(x,\rho,t)\|^2_{L^2(\Omega\times(0,1))}\le\\&\qquad e^{cT}\left(\|\nabla_x u_k'(0)\|_2^2+\|\Delta_x u_k'(0)\|_2^2+\tau\|\nabla_x z_k(x,\rho,0)\|^2_{L^2(\Omega\times(0,1))}\right)\end{aligned}$$

for all $t\in\mathbb{R}_+$, therefore, we conclude that

$$u_k \text{ is bounded in } L^\infty_{loc}(0,+\infty;H^2(\Omega)\cap H_0^1(\Omega)), \tag{3.22}$$

$$z_k \text{ is bounded in } L^\infty_{loc}(0,+\infty;H_0^1(\Omega;L^2(0,1))). \tag{3.23}$$

Applying Dunford-Petti's theorem, we conclude from (3.7),(3.8), (3.9), (3.10), (3.16), (3.17), (3.18) (3.22) and (3.23), replacing the sequences u_k and z_k with subsequence if necessary, that

$$u_k\to u \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;H^2(\Omega)\cap H_0^1(\Omega)), \tag{3.24}$$

$$u_k'\to u' \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;H_0^1(\Omega)),$$

$$u''_k\to u'' \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;L^2(\Omega)), \tag{3.25}$$

$$g_1(u_k')\to\chi \ \text{ weak in } \ L^2(\Omega\times(0,T)),$$

$$z_k\to z \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;H_0^1(\Omega;L^2(0,1)),$$

$$z_k'\to z' \ \text{ weak-star in } \ L^\infty_{loc}(0,+\infty;L^2(\Omega\times(0,1))), \tag{3.26}$$

$$g_2(z_k(x,1,t))\to\psi \ \text{ weak in } \ L^2(\Omega\times(0,T))$$

for suitable functions $u\in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, $z\in L^\infty(0,T;L^2(\Omega\times(0,1)))$, $\chi\in L^2(\Omega\times(0,T))$, $\psi\in L^2(\Omega\times(0,T))$ for all $T\ge 0$. We have to show that (u,z) is a solution of (2.8).

From (3.7) and (3.8) we have (u_k') is bounded in $L^\infty(0,T;H_0^1(\Omega))$. Then (u_k') is bounded in $L^2(0,T;H_0^1(\Omega))$. Since (u_k'') is bounded in $L^\infty(0,T;L^2(\Omega))$, then (u_k'') is bounded in $L^2(0,T;L^2(\Omega))$. Consequently, (u_k') is bounded in $H^1(Q)$.
Since the embedding $H^1(Q)\hookrightarrow L^2(Q)$ is compact, using Aubin-Lions' theorem **[20]**, we can extract a subsequence (u_ν) of (u_k) such that

$$u_\nu'\to u' \text{ strongly in } \ L^2(Q).$$

Therefore

$$u_\nu'\to u' \text{ a.e in } Q. \tag{3.27}$$

Similarly we obtain

$$z_\nu\to z \text{ a.e in } Q. \tag{3.28}$$

Lemma 3.1. *For each $T > 0$, $g(u'), g(z(x,1,t)) \in L^1(Q)$ and* $\|g(u')\|_{L^1(Q)}, \|g(z(x,1,t))\|_{L^1(Q)} \le K_1$, *where K_1 is a constant independent of t.*

Proof. By (**H2**) and (3.27) we have

$$g_1(u_k'(x,t)) \to g_1(u'(x,t)) \text{ a.e. in } Q,$$
$$0 \le g_1(u_k'(x,t))u_k'(x,t) \to g_1(u'(x,t))u'(x,t) \text{ a.e. in } Q$$

Hence, by (3.9) and Fatou's lemma we have

$$\int_0^T \int_\Omega u'(x,t)g_1(u'(x,t))\,dx\,dt \le K \text{ for } T > 0. \tag{3.29}$$

By Cauchy-Schwarz inequality, using (3.29), we have

$$\begin{aligned}\int_0^T \int_\Omega |g_1(u'(x,t))|\,dx\,dt &\le c|Q|^{\frac12}\left(\int_0^T \int_\Omega u'g_1(u')\,dx\,dt\right)^{\frac12}\\ &\le c|Q|^{\frac12}K^{\frac12} \equiv K_1\end{aligned}$$

□

Lemma 3.2. $g(u_k') \to g(u')$ *in* $L^1(\Omega\times(0,T))$ *and* $g(z_k) \to g(z)$ *in* $L^1(\Omega\times(0,T))$.

Proof. Let $E \subset \Omega\times[0,T]$ and set

$$E_1 = \left\{(x,t)\in E;\ |g_1(u_k'(x,t))| \le \frac{1}{\sqrt{|E|}}\right\},\quad E_2 = E\setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s\in\mathbb{R} \text{ and } |g_1(s)|\ge r\}$,

$$\int_E |g_1(u_k')|\,dxdt \le c\sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}}\right)\right)^{-1}\int_{E_2}|u_k'g_1(u_k')|\,dxdt.$$

Applying (3.9) we deduce that $\sup_k \int_E |g_1(u_k')|\,dxdt \to 0$ as $|E|\to 0$. From Vitali's convergence theorem we deduce that $g_1(u_k') \to g_1(u')$ in $L^1(\Omega\times(0,T))$, hence

$$g_1(u_k') \to g_1(u') \text{ weak in } L^2(Q).$$

Similarly, we have

$$g_2(z_k') \to g_2(z') \text{ weak in } L^2(Q),$$

and this implies that

$$\int_0^T \int_\Omega g_1(u_k')v\,dx\,dt \to \int_0^T \int_\Omega g_1(u')v\,dx\,dt, \text{ for all } v\in L^2(0,T;H_0^1) \tag{3.30}$$

$$\int_0^T \int_\Omega g_2(z_k)v\,dx\,dt \to \int_0^T \int_\Omega g_2(z)v\,dx\,dt, \text{ for all } v\in L^2(0,T;H_0^1) \tag{3.31}$$

as $k\to+\infty$. □

It follows at once from (3.24), (3.25), (3.30), (3.31) and (3.26) that for each fixed $v \in L^2(0,T;H_0^1)$ and $w\in L^2(0,T;H_0^1(\Omega\times(0,1)))$

$$\begin{aligned}&\int_0^T \int_\Omega (u_k'' - \Delta_x u_k + +\alpha(t)\int_0^t h(t-s)\Delta_x u_k(x,s)ds + \mu_1 g_1(u_k') + \mu_2 g_2(z_k))v\,dx\,dt\\ &\to \int_0^T \int_\Omega (u'' - \Delta_x u + \alpha(t)\int_0^t h(t-s)\Delta_x u(x,s)ds + \mu_1 g_1(u') + \mu_2 g_2(z))v\,dx\,dt\end{aligned}$$

$$\int_0^T \int_0^1 \int_\Omega (\tau z_k' + \frac{\partial}{\partial\rho}z_k)w\,dx\,d\rho\,dt \to \int_0^T \int_0^1 \int_\Omega (\tau z' + \frac{\partial}{\partial\rho}z)w\,dx\,d\rho\,dt$$

as $k\to+\infty$. Hence

$$\int_0^T \int_\Omega (u''+\Delta_x u+\alpha(t)\int_0^t h(t-s)\Delta_x u_k(x,s)ds+\mu_1 g_1(u')+\mu_2 g_2(z))v\,dx\,dt = 0, \qquad \forall v\in L^2(0,T;H_0^1).$$

$$\int_0^T \int_0^1 \int_\Omega (\tau u' + \frac{\partial}{\partial\rho}z)w\,dx\,d\rho\,dt = 0, \qquad \forall w\in L^2(0,T;H_0^1(\Omega\times(0,1))).$$

Thus the problem (P) admits a global weak solution u.

4 Asymptotic behavior

For $M > 0$ and $\varepsilon_1, \varepsilon_2 > 0$, we define the perturbed modified energy by

$$L(t) = ME(t) + \varepsilon_1\alpha(t)\Psi(t) + \varepsilon_2\alpha(t)I(t) + \alpha(t)\chi(t), \tag{4.1}$$

where

$$\Psi(t) = \int_\Omega u_t(x,t)\, u(x,t)\, dx, \tag{4.2}$$

$$I(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t))\, d\rho\, dx, \tag{4.3}$$

$$\chi(t) = -\int_\Omega u_t(x,t) \int_0^t h(t-s)(u(t)-u(s))\, ds\, dx. \tag{4.4}$$

Lemma 4.1. *([24]) For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t h(t-s)(u(t)-u(s))\, ds \right)^2 dx \le c_*^2 (\int_0^t h(s)\, ds)(h \circ \nabla u)(t),$$

Where c_* is the Poincaré constant.

Lemma 4.2. *There exist two positive constants B_1 and B_2 depending on $\varepsilon_1, \varepsilon_2$ and M such that for all $t > 0$*

$$B_1 E(t) \le L(t) \le B_2 E(t).$$

Proof. We consider the functional

$$K(t) = L(t) - ME(t) = \varepsilon_1\alpha(t)\Psi(t) + \varepsilon_2\alpha(t)I(t) + \alpha(t)\chi(t)$$

and show that

$$|K(t)| \le C\alpha(t)E(t), \quad C > 0. \tag{4.5}$$

Using Young's inequality and Poincaré's inequality, we obtain

$$\begin{aligned} |\alpha(t)\chi(t)| &= \left|\alpha(t) \int_\Omega u_t(x,t) \int_0^t h(t-s)(u(t)-u(s))\, ds\, dx\right| \\ &\le \frac{\alpha(t)}{2} \int_\Omega u_t^2\, dx + \frac{\alpha(t)}{2} \int_\Omega \left(\int_0^t h(t-s)(u(t)-u(s))\, ds \right)^2 dx \\ &\le \frac{\alpha(t)}{2} \int_\Omega u_t^2\, dx + \frac{\alpha(t)}{2} \left(\int_0^t h(s)\, ds \right) c_*^2 (h \circ \nabla u)(t). \end{aligned} \tag{4.6}$$

Where c_* is the Poincaré constant.

Similarly, we have

$$\begin{aligned} |\varepsilon_1\alpha(t)\Psi(t) + \varepsilon_2\alpha(t)I(t)| &\le \varepsilon_1\alpha(t) \int_\Omega |u_t||u|\, dx + \varepsilon_2\alpha(t) \int_\Omega \int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t))\, d\rho\, dx \\ &\le \frac{\varepsilon_1\alpha(t)}{2} \int_\Omega u_t^2\, dx + \frac{\varepsilon_1\alpha(t)}{2} c_*^2 \int_\Omega |\nabla u|^2\, dx \\ &\quad +\varepsilon_2\alpha(t) \int_\Omega \int_0^1 G_2(z(x,\rho,t))\, d\rho\, dx. \end{aligned} \tag{4.7}$$

Using $1 - \alpha(t) \int_0^t h(s)\, ds \ge l > 0$, (2.10), (4.6) and (4.7), we get (4.5) for some positive constant C. By choosing M large enough, our result follows from (4.1), (4.5). □

Proposition 4.3. *For each $t_0 > 0$ and sufficiently large $M > 0$ and appropriately small $\varepsilon_1, \varepsilon_2 > 0$, there exist positive constants C_3, C_4, and C_5 such that*

$$\frac{d}{dt} L(t) \le -C_3\alpha(t)E(t) + C_4\alpha(t)(h \circ \nabla u)(t) + C_5\alpha(t)\|g_1(u_t)\|_2^2 \quad \forall t \ge t_0.$$

The proof of this proposition will be carried out through three lemmas.

Lemma 4.4. *Let* (u, z) *be the solution of (2.8), then for any* $\gamma > 0$*, we have*

$$\begin{aligned}\Psi'(t) \quad \leq \quad & \|u_t\|_2^2 - (l - \gamma - \gamma c_*^2((\mu_1 + \mu_2))\|\nabla u\|_2^2 \\ & + \tfrac{\mu_1}{4\gamma}\int_\Omega |g_1(u_t(x,t))|^2\,dx + \tfrac{\mu_2}{4\gamma}\int_\Omega |g_2(z(x,1,t))|^2\,dx + \tfrac{(1-l)\alpha(t)}{4\gamma}(h\circ\nabla u)(t),\end{aligned} \tag{4.8}$$

Proof. Using the first equation in (2.8), a direct computation leads to

$$\begin{aligned}\Psi'(t) &= \int_\Omega u_t^2(x,t)\,dx + \int_\Omega u_{tt}(x,t)u(x,t)\,dx \\ &= \|u_t\|_2^2 + \int_\Omega \Big[\Delta u(x,t) - \alpha(t)\int_0^t h(t-s)\Delta u(x,s)ds \\ &\quad -\mu_1 g_1(u_t(x,t)) - \mu_2 g_2(u_t(x,t-\tau))\Big]\, u(x,t)\,dx \\ &= \|u_t\|_2^2 - \|\nabla u\|_2^2 + \alpha(t)\int_\Omega \nabla u(x,t)\int_0^t h(t-s)\nabla u(x,s)\,ds\,dx \\ &\quad -\mu_1\int_\Omega g_1(u_t(x,t))u(x,t)\,dx - \mu_2\int_\Omega g_2(z(x,1,t))u(x,t)\,dx.\end{aligned} \tag{4.9}$$

Since $\int_0^t h(s)\,ds \leq \int_0^\infty h(s)\,ds \leq \frac{1-l}{\alpha(t)}$
Now, the third term in the right-hand side of (4.9) can be estimated as follows:

$$\begin{aligned}&\alpha(t)\int_\Omega \nabla u(x,t)\int_0^t h(t-s)\nabla u(x,s)\,ds\,dx \\ = \;& \alpha(t)\int_\Omega\int_0^t h(t-s)\Big[\nabla u(x,s) - \nabla u(x,t)\Big]\nabla u(x,t)\,ds\,dx + \alpha(t)\int_\Omega\int_0^t h(t-s)|\nabla u(x,t)|^2\,ds\,dx \\ \leq \;& (1-l)\|\nabla u(x,t)\|_2^2 + \alpha(t)\int_\Omega |\nabla u(x,t)|\int_0^t h(t-s)|\nabla u(x,s) - \nabla u(x,t)|\,ds\,dx \\ \leq \;& (1-l)\|\nabla u(x,t)\|_2^2 + \|\nabla u(x,t)\|_2\left(\alpha^2(t)\int_\Omega\left(\int_0^t h(t-s)|\nabla u(x,s) - \nabla u(x,t)|ds\right)^2 dx\right)^{\frac12} \\ \leq \;& (l-1)\|\nabla u(x,t)\|_2^2 \\ & + \|\nabla u(x,t)\|_2\left[\int_\Omega\left(\alpha^2(t)\int_0^t h(t-s)\,ds\int_0^t h(t-s)|\nabla u(x,s) - \nabla u(x,t)|^2 ds\right)dx\right]^{\frac12} \\ \leq \;& (1-l)\|\nabla u(x,t)\|_2^2 + (1-l)^{\frac12}\|\nabla u(x,t)\|_2\left[\int_\Omega \alpha(t)\left(\int_0^t h(t-s)|\nabla u(x,s) - \nabla u(x,t)|^2 ds\right)dx\right]^{\frac12} \\ \leq \;& (1-l)\|\nabla u(x,t)\|_2^2 + (1-l)^{\frac12}\|\nabla u(x,t)\|_2\alpha^{\frac12}(t)(h\circ\nabla u)^{\frac12}(t) \\ \leq \;& (1-l)\|\nabla u(x,t)\|_2^2 + \gamma\|\nabla u(x,t)\|_2^2 + \tfrac{l}{4\gamma}(h\circ\nabla u)(t) \\ \leq \;& (1-l+\gamma)\|\nabla u(x,t)\|_2^2 + \tfrac{(1-l)\alpha(t)}{4\gamma}(h\circ\nabla u)(t),\end{aligned}$$

then we conclude

$$\begin{aligned}\Psi'(t) \quad\leq\quad & \|u_t\|_2^2 - \|\nabla u\|_2^2 + (1-l+\gamma)\|\nabla u\|_2^2 + \tfrac{(1-l)\alpha(t)}{4\gamma}(h\circ\nabla u)(t) \\ & + \mu_1\int_\Omega |g_1(u_t(x,t))||u(x,t)|\,dx + \mu_2\int_\Omega |g_2(z(x,1,t))||u(x,t)|\,dx.\end{aligned}$$

Since

$$\int_\Omega |g_1(u_t(x,t))||u(x,t)|\,dx \leq \gamma c_*^2\|\nabla u\|_2^2 + \frac{1}{4\gamma}\int_\Omega |g_1(u_t(x,t))|^2\,dx$$

$$\int_\Omega |g_2(z(x,1,t))||u(x,t)|\,dx \leq \gamma c_*^2\|\nabla u\|_2^2 + \frac{1}{4\gamma}\int_\Omega |g_2(z(x,1,t))|^2\,dx$$

we obtain

$$\begin{aligned}\Psi'(t) \quad\leq\quad & \|u_t\|_2^2 - (l - \gamma - \gamma c_*^2(\mu_1 + \mu_2))\|\nabla u\|_2^2 \\ & + \tfrac{\mu_1}{4\gamma}\int_\Omega |g_1(u_t(x,t))|^2\,dx + \tfrac{\mu_2}{4\gamma}\int_\Omega |g_2(z(x,1,t))|^2\,dx + + \tfrac{(1-l)\alpha(t)}{4\gamma}(h\circ\nabla u)(t).\end{aligned}$$

□

Lemma 4.5. *Let* (u, z) *be the solution of (2.8), then we have*

$$\frac{d}{dt}I(t) \leq -2I(t) - \frac{e^{-2\tau}}{\tau}\int_\Omega G_2(z(x,1,t))dx + \frac{1}{\tau}\int_\Omega G_2(u_t(x,t))\,dx. \tag{4.10}$$

Proof. Differentiating (4.3) and using the second equation in (2.8), we have

$$\begin{aligned}
\frac{d}{dt}I(t) &= \int_\Omega \int_0^1 e^{-2\tau\rho} z_t(x,\rho,t) g_2(z(x,\rho,t))\, d\rho\, dx \\
&= -\frac{1}{\tau}\int_\Omega \int_0^1 e^{-2\tau\rho} z_\rho(x,\rho,t) g_2(z(x,\rho,t))\, d\rho\, dx \\
&= -\frac{1}{\tau}\int_\Omega \int_0^1 e^{-2\tau\rho} \frac{d}{d\rho} G_2(z(x,\rho,t))\, d\rho\, dx \\
&= -\frac{1}{\tau}\int_\Omega \int_0^1 \left[\frac{d}{d\rho}\Big(e^{-2\tau\rho} G_2(z(x,\rho,t))\Big) + 2\tau e^{-2\tau\rho} G_2(z(x,\rho,t))\right] d\rho\, dx \\
&= -\frac{1}{\tau}\int_\Omega \left[e^{-2\tau} G_2(z(x,1,t)) - G_2(u_t(x,t))\right] dx \\
&\quad -2\int_\Omega \int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t))\, d\rho\, dx \\
&\le -2\int_\Omega \int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t))\, d\rho\, dx - \frac{1}{\tau}\int_\Omega e^{-2\tau} G_2(z(x,1,t))dx \\
&\quad +\frac{1}{\tau}\int_\Omega G_2(u_t(x,t))dx\, dx \\
&\le -2I(t) - \frac{e^{-2\tau}}{\tau}\int_\Omega G_2(z(x,1,t))dx + \frac{1}{\tau}\int_\Omega G_2(u_t(x,t))\, dx.
\end{aligned}$$

□

Lemma 4.6. *Let* (u,z) *be the solution of (2.8), then we have the estimate*

$$\begin{aligned}
\frac{d}{dt}\chi(t) \le \eta(1+2(1-l)^2)\|\nabla u\|_2^2 - \left(\left(\int_0^t h(s)\, ds\right) - \eta\right)\|u_t\|_2^2 \\
+\left(2\eta\alpha^2(t) + \frac{1}{4\eta} + \frac{c_*^2}{4\eta}(\mu_1+\mu_2) + \frac{1}{4\eta}\right)\left(\int_0^t h(s)\, ds\right)(h\circ\nabla u)(t) - \frac{h_0 c_*^2}{4\eta}(h'\circ\nabla u)(t) \\
+\eta\mu_1\|g_1(u_t)\|_2^2 + \eta\mu_2\|g_2(z(x,1,t))\|_2^2.
\end{aligned} \tag{4.11}$$

for any η a positive constant.

Proof. A differentiation of (4.4) leads to

$$\chi(t) = -\int_\Omega u_t(x,t)\int_0^t h(t-s)\Big(u(t)-u(s)\Big)ds\, dx,$$

we have

$$\begin{aligned}
\chi'(t) &= -\int_\Omega u_{tt}(x,t)\int_0^t h(t-s)(u(t)-u(s))ds\, dx \\
&\quad -\int_\Omega u_t(x,t)\left[u_t(x,t)\int_0^t h(t-s)\, ds + \int_0^t h'(t-s)(u(t)-u(s))ds\right]dx, \\
&= -\int_\Omega\Bigg[\Big(\Delta_x u(x,t) - \alpha(t)\int_0^t h(t-s)\Delta_x u(x,s)ds - \mu_1\, g_1(u_t(x,t)) \\
&\quad -\mu_2\, g_2(z(x,1,t))\Big)\int_0^t h(t-s)(u(t)-u(s))ds\Bigg]dx \\
&\quad -\int_0^t h(s)\, ds\, \|u_t\|_2^2 - \int_\Omega u_t(x,t)\int_0^t h'(t-s)(u(t)-u(s))ds\, dx, \\
&= \int_\Omega \nabla u(x,t)\int_0^t h(t-s)(\nabla u(x,t)-\nabla u(x,s))ds\, dx \\
&\quad -\alpha(t)\int_\Omega\left[\int_0^t h(t-s)\nabla u(x,s)\, ds\right]\left[\int_0^t h(t-s)\,(\nabla u(x,t)-\nabla u(x,s))\, ds\right]dx \\
&\quad +\mu_1\int_\Omega g_1(u_t(x,t))\int_0^t h(t-s)(u(x,t)-u(x,s))ds\, dx \\
&\quad +\mu_2\int_\Omega g_1(z(x,1,t))\int_0^t h(t-s)(u(x,t)-u(x,s))ds\, dx \\
&\quad -\left(\int_0^t h(s)\, ds\right)\|u_t\|_2^2 - \int_\Omega u_t(x,t)\int_0^t h'(t-s)(u(t)-u(s))ds\, dx.
\end{aligned} \tag{4.12}$$

Using Young's inequality and the embedding $H_0^1(\Omega)\to L^2(\Omega)$, we infer

$$\begin{aligned}
&\int_\Omega u_t(x,t)\int_0^t h'(t-s)\Big(u(t)-u(s)\Big)ds\, dx \\
&\le \eta\|u_t\|_2^2 + \frac{1}{4\eta}\left(\int_0^t -h'(t-\tau)\|u(t)-u(\tau)\|_2\, d\tau\right)^2 \\
&\le \eta\|u_t\|_2^2 + \frac{1}{4\eta}\left(\int_0^t h'(\tau)\, d\tau\right)\int_0^t h'(t-\tau)\|u(t)-u(\tau)\|_2^2\, d\tau \\
&\le \eta\|u_t\|_2^2 - \frac{h_0 c_*^2}{4\eta}(h'\circ\nabla u)(t),
\end{aligned}$$

$$\left|\int_\Omega \nabla u(x,t)\int_0^t h(t-s)\Big(\nabla u(x,t)-\nabla u(x,s)\Big)ds\,dx\right| \leq \eta\|\nabla u\|_2^2+\frac{1}{4\eta}(\int_0^t h(s)\,ds)(h\circ\nabla u)(t),$$

$$\begin{aligned}&\mu_1\int_\Omega g_1(u_t(x,t))\int_0^t h(t-s)(u(x,t)-u(x,s))ds\,dx\\ &\qquad\leq \eta\mu_1\|g_1(u_t(x,t))\|_2^2 + \tfrac{\mu_1 c_*^2}{4\eta}(\int_0^t h(s)\,ds)(h\circ\nabla u)(t),\end{aligned}$$

$$\begin{aligned}&\mu_2\int_\Omega g_2(z(x,1,t))\int_0^t h(t-s)(u(x,t)-u(x,s))ds\,dx\\ &\qquad\leq \eta\mu_2\|g_2(z(x,1,t))\|_2^2 + \tfrac{\mu_2 c_*^2}{4\eta}(\int_0^t h(s)\,ds)(h\circ\nabla u)(t)\end{aligned}$$

and

$$\begin{aligned}
&\left|\alpha(t)\int_\Omega\Big(\int_0^t h(t-s)\nabla u(x,s)\,ds\Big)\Big(\int_0^t h(t-s)(\nabla u(x,t)-\nabla u(x,s))ds\Big)\,dx\right|\\
= &\left|\alpha(t)\int_\Omega\Big[\int_0^t h(t-s)\Big(\nabla u(x,t)-\nabla u(x,s)\Big)\,ds-\int_0^t h(t-s)\nabla u(x,t)\,ds\Big]\right.\\
&\qquad\left.\times\Big[\int_0^t h(t-s)\Big(\nabla u(x,t)-\nabla u(x,s)\Big)ds\Big]dx\right|\\
\leq\ & \eta\alpha^2(t)\int_\Omega\left|\int_0^t h(t-s)\nabla u(x,s)\,ds\right|^2dx+\tfrac{1}{4\eta}\int_\Omega\left|\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right|^2dx\\
\leq\ & \eta\alpha^2(t)\int_\Omega\Big(\int_0^t h(t-s)(|\nabla u(x,t)-\nabla u(x,s)|+|\nabla u(x,t)|)\,ds\Big)^2dx\\
&+\tfrac{1}{4\eta}\int_\Omega\Big(\int_0^t h(t-s)|\nabla u(x,t)-\nabla u(x,s)|\,ds\Big)^2dx\\
\leq\ & (2\eta\alpha^2(t)+\tfrac{1}{4\eta})\int_\Omega\Big(\int_0^t h(s)|\nabla u(x,t)-\nabla u(x,s)|ds\Big)^2dx\\
&+2\eta\alpha^2(t)\Big(\int_0^t h(s)\,ds\Big)^2\int_\Omega|\nabla u(x,t)|\,dx\\
\leq\ & 2\eta(1-l)^2\|\nabla u(x,t)\|_2^2+\Big(2\eta\alpha^2(t)+\tfrac{1}{4\eta}\Big)\Big(\int_0^t h(s)\,ds\Big)(h\circ\nabla u)(t).
\end{aligned}$$

Combining all estimates above, we get

$$\begin{aligned}\chi'(t)&\leq\eta(1+2(1-l)^2)\|\nabla u\|_2^2-\Big(\Big(\int_0^t h(s)\,ds\Big)-\eta\Big)\|u_t\|_2^2\\
&+\Big(2\eta\alpha^2(t)+\tfrac{1}{4\eta}+\tfrac{c_*^2}{4\eta}(\mu_1+\mu_2)+\tfrac{1}{4\eta}\Big)\Big(\int_0^t h(s)\,ds\Big)(h\circ\nabla u)(t)-\tfrac{h_0c_*^2}{4\eta}(h'\circ\nabla u)(t)\\
&+\eta\mu_1\|g_1(u_t)\|_2^2+\eta\mu_2\|g_2(z(x,1,t))\|_2^2.\end{aligned}\tag{4.13}$$

□

Proof of proposition 4.3. Since h is positive, then for any $t_0>0$ we have $\int_0^t h(s)\,ds\geq\int_0^{t_0}h(s)\,ds=\tilde{h}_0$ for all $t\geq t_0$.

Thus, making use of this and combining (2.23), (4.8), (4.10) and (4.13) we have

$$\begin{aligned}
L'(t)=\tfrac{d}{dt}L(t) \;&=\; ME'(t)+\varepsilon_1\alpha(t)\Psi'(t)+\varepsilon_1\alpha'(t)\Psi(t)+\varepsilon_2\alpha(t)I'(t)\\
&\quad+\varepsilon_2\alpha'(t)I(t)+\alpha(t)\chi'(t)+\alpha'(t)\chi(t)\\
&\le M\Big[-\tfrac12\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2+\tfrac12\alpha(t)(h'\circ\nabla u)(t)\Big]\\
&\quad+\varepsilon_1\alpha(t)\Big[\|u_t\|_2^2-(l-\gamma-\gamma c_*^2((\mu_1+\mu_2))\|\nabla u\|_2^2\\
&\quad+\tfrac{\mu_1}{4\gamma}\textstyle\int_\Omega|g_1(u_t(x,t))|^2\,dx+\tfrac{\mu_2}{4\gamma}\int_\Omega|g_2(z(x,1,t))|^2\,dx+\tfrac{(1-l)\alpha(t)}{4\gamma}(h\circ\nabla u)(t)\Big]\\
&\quad+\varepsilon_2\alpha(t)\Big[-2I(t)-\tfrac{e^{-2\tau}}{\tau}\textstyle\int_\Omega G_2(z(x,1,t))dx+\tfrac1\tau\int_\Omega G_2(u_t(x,t))\,dx\Big]\\
&\quad+\alpha(t)\Big[\eta(1+2(1-l)^2)\|\nabla u\|_2^2-\Big(\Big(\textstyle\int_0^t h(s)\,ds\Big)-\eta\Big)\|u_t\|_2^2\\
&\quad+\Big(2\eta\alpha^2(t)+\tfrac{1}{4\eta}+\tfrac{c_*^2}{4\eta}(\mu_1+\mu_2)+\tfrac{1}{4\eta}\Big)\Big(\textstyle\int_0^t h(s)\,ds\Big)(h\circ\nabla u)(t)\\
&\quad-\tfrac{h_0c_*^2}{4\eta}(h'\circ\nabla u)(t)+\eta\mu_1\|g_1(u_t)\|_2^2+\eta\mu_2\|g_2(z(x,1,t))\|_2^2\Big]\\
&\quad+\varepsilon_1\alpha'(t)\Psi(t)+\varepsilon_2\alpha'(t)I(t)+\alpha'(t)\chi(t)\\
&\le -\alpha(t)\left(\tilde h_0-\eta-\varepsilon_1\right)\|u_t\|_2^2+\alpha(t)\left(\tfrac M2-\tfrac{h_0c_*^2}{4\eta}\right)(h'\circ\nabla u)(t)\\
&\quad-\tfrac M2\alpha'(t)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2\\
&\quad-\alpha(t)\Big[\varepsilon_1(l-\gamma-\gamma c_*^2(\mu_1+\mu_2))-\eta(1+2(1-l)^2)\Big]\|\nabla_x u\|_2^2-2\varepsilon_2\alpha(t)I(t)\\
&\quad-\alpha(t)\left(MC+\varepsilon_2\tfrac{e^{-2\tau}}{\tau}\alpha_1-\left(\eta\mu_2+\tfrac{\varepsilon_1\mu_2}{4\gamma}\right)c_3\right)\textstyle\int_\Omega g_2(z(x,1,t))z(x,1,t)\,dx\\
&\quad-\alpha(t)\left(MC-\tfrac{\varepsilon_2}{\tau}\alpha_2\right)\textstyle\int_\Omega g_1(u_t)u_t\,dx\\
&\quad+\alpha(t)\left(\tfrac{\varepsilon_1(1-l)}{4\gamma}+\left(2\eta\alpha^2(t)+\tfrac{1}{4\eta}+\tfrac{c_*^2}{4\eta}(\mu_1+\mu_2)+\tfrac{1}{4\eta}\right)\left(\textstyle\int_0^t h(s)\,ds\right)\right)(h\circ\nabla u)(t)\\
&\quad+\alpha(t)\left(\eta\mu_1+\tfrac{\varepsilon_1\mu_1}{4\gamma}\right)\|g_1(u_t)\|_2^2\\
&\quad+\varepsilon_1\alpha'(t)\Psi(t)+\varepsilon_2\alpha'(t)I(t)+\alpha'(t)\chi(t).
\end{aligned}\tag{4.14}$$

By using (4.2), (4.4), Young's and Poincaré's inequalities, we have

$$\begin{aligned}
&+\varepsilon_1\alpha'(t)\Psi(t)+\varepsilon_2\alpha'(t)I(t)+\alpha'(t)\chi(t)\\
\le\;& \varepsilon_1\alpha'(t)\textstyle\int_\Omega|u_t||u|\,dx+\varepsilon_2\alpha'(t)\int_\Omega\int_0^1 e^{-2\tau\rho}G_2(z(x,\rho,t))\,d\rho\,dx\\
&+\alpha'(t)\textstyle\int_\Omega u_t(x,t)\int_0^t h(t-s)(u(t)-u(s))\,ds\,dx\\
\le\;& \tfrac{(1+\varepsilon_1)\alpha'(t)}{2}\textstyle\int_\Omega u_t^2\,dx+\tfrac{\varepsilon_1\alpha'(t)}{2}c_*^2\int_\Omega|\nabla u|^2\,dx\\
&+\tfrac{\alpha'(t)}{2}\left(\textstyle\int_0^t h(s)\,ds\right)c_*^2(h\circ\nabla u)(t)+\varepsilon_2\alpha'(t)\int_\Omega\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx.
\end{aligned}$$

Hence (4.14) takes the form

$$\begin{aligned}
L'(t)\;\le\;& -\alpha(t)\left(\tilde h_0-\eta-\varepsilon_1+\tfrac{(1+\varepsilon_1)\alpha'(t)}{2\alpha(t)}\right)\|u_t\|_2^2+\alpha(t)\left(\tfrac M2-\tfrac{h_0c_*^2}{4\eta}\right)(h'\circ\nabla u)(t)\\
&-\tfrac{M\alpha(t)}{2}\left(\tfrac{\alpha'(t)}{\alpha(t)}\right)(\textstyle\int_0^t h(s)ds)\|\nabla u\|_2^2\\
&-\alpha(t)\Big[\varepsilon_1\left(\tfrac{c_*^2\alpha'(t)}{2\alpha(t)}\right)+\varepsilon_1\left(l-\gamma-\gamma c_*^2(\mu_1+\mu_2)\right)-\eta\left(1+2(1-l)^2\right)\Big]\|\nabla_x u\|_2^2-2\varepsilon_2\alpha(t)I(t)\\
&-\alpha(t)\left(MC+\varepsilon_2\tfrac{e^{-2\tau}}{\tau}\alpha_1-\left(\eta\mu_2+\tfrac{\varepsilon_1\mu_2}{4\gamma}\right)c_3\right)\textstyle\int_\Omega g_2(z(x,1,t))z(x,1,t)\,dx\\
&-\alpha(t)\left(MC-\tfrac{\varepsilon_2}{\tau}\alpha_2\right)\textstyle\int_\Omega g_1(u_t)u_t\,dx\\
&+\alpha(t)\left(\tfrac{\varepsilon_1(1-l)}{4\gamma}+\left(\tfrac{c_*^2\alpha'(t)}{2\alpha(t)}+2\eta\alpha^2(t)+\tfrac{1}{4\eta}+\tfrac{c_*^2}{4\eta}(\mu_1+\mu_2)+\tfrac{1}{4\eta}\right)\left(\textstyle\int_0^t h(s)\,ds\right)\right)(h\circ\nabla u)(t)\\
&+\alpha(t)\left(\eta\mu_1+\tfrac{\varepsilon_1\mu_1}{4\gamma}\right)\|g_1(u_t)\|_2^2\\
&+\alpha(t)\left(\tfrac{\varepsilon_2\alpha'(t)}{\alpha(t)}\right)\textstyle\int_\Omega\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx.
\end{aligned}\tag{4.15}$$

At this point, we choose, first, $\varepsilon_1 > 0$ so small that

$$\tilde{h}_0 - \varepsilon_1 > 0.$$

Next, we choose $\gamma > 0$ so small such that

$$l - \gamma - \gamma c_*^2(\mu_1 + \mu_2) > 0.$$

and $\eta > 0$ sufficiently small such that

$$\varepsilon_1 \left(l - \gamma - \gamma c_*^2(\mu_1 + \mu_2)\right) - \eta\left(1 + 2(1-l)^2\right) > 0.$$

and

$$\tilde{h}_0 - \eta - \varepsilon_1 > 0.$$

Then, we pick $M > 0$ sufficiently large so that

$$\begin{gathered} \frac{M}{2} - \frac{h_0 c_*^2}{4\eta} > 0 \\ MC + \varepsilon_2 \frac{e^{-2\tau}}{\tau}\alpha_1 - \left(\eta\mu_2 + \frac{\varepsilon_1\mu_2}{4\gamma}\right) c_3 > 0 \\ MC - \frac{\varepsilon_2}{\tau}\alpha_2 > 0 \end{gathered}$$

We then use $\lim\limits_{t\to+\infty} \frac{\alpha'(t)}{\alpha(t)} = 0$ (which can be deduced from $(\mathbf{H1})$)to choose $t_1 \geq t_0$ so That, (4.14) takes the form

$$\frac{d}{dt}L(t) \leq -C_3\alpha(t)E(t) + C_4\alpha(t)(h \circ \nabla u)(t) + C_5\alpha(t)\|g_1(u_t)\|_2^2 \tag{4.16}$$

where C_3, C_4 and C_5 are three positive constants. This completes the proof of Proposition 4.3. □

Now, we estimate the last term in the right hand side of (4.16). We denote by

$$\Omega^+ = \{x \in \Omega : |u'| \geq \varepsilon'\}, \quad \Omega^- = \{x \in \Omega : |u'| \leq \varepsilon'\}.$$

From (2.1) and (2.2), it follows that

$$\int_{\Omega^+} |g_1(u')|^2\, dx \leq \mu_1 \int_{\Omega^+} u' g_1(u')\, dx \leq -\mu_1 E'(t). \tag{4.17}$$

Case 1: H is linear on $[0, \varepsilon']$. In this case one can easily check that there exists $\mu_1' > 0$, such that $|g_1(s)| \leq \mu_1'|s|$ for all $|s| \leq \varepsilon'$, and thus

$$\int_{\Omega^-} |g_1(u')|^2\, dx \leq \mu_1' \int_{\Omega^-} u' g_1(u')\, dx \leq -\mu_1' E'(t). \tag{4.18}$$

Substitution of (4.17) and (4.18) into (4.16) gives

$$(L(t) + \mu E(t))' \leq -c_1\alpha(t)H_2(E(t)) + C_4\alpha(t)h \circ \nabla u \tag{4.19}$$

where $\mu = C_5(\mu_1 + \mu_1')$ and here and in the sequel we take C_i to be a generic positive constant.
Case 2: $H'(0) = 0$ and $H'' > 0$ on $]0, \varepsilon']$.

Since H is convex and increasing, H^{-1} is concave and increasing. By (2.1), the reversed Jensen's inequality for concave function, and (2.23), it follows that

$$\begin{aligned} \int_{\Omega^-} |g_1(u')|^2\, dx &\leq \int_{\Omega^-} H^{-1}(u' g_1(u'))\, dx \\ &\leq |\Omega| H^{-1}\left(\frac{1}{|\Omega|}\int_{\Omega^-} u' g_1(u')\, dx\right) \leq CH^{-1}(-C'E'(t)). \end{aligned} \tag{4.20}$$

A combination of (4.16), (4.17) and (4.20) yields

$$(L(t) + C_5\mu_1 E(t))' \leq -C_3\alpha(t)E(t) + C_4\alpha(t)(h \circ \nabla u)(t) + \tilde{C}_5\alpha(t)H^{-1}(-C'E'(t)), \quad t \geq t_0. \tag{4.21}$$

Let us denote by H^* the conjugate function of the convex function H, i.e.,

$$H^*(s) = \sup_{t\in\mathbb{R}_+}(st - H(t)).$$

Then H^* is the Legendre transform of H, which is given by

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s \geq 0 \tag{4.22}$$

and satisfies the following inequality

$$st \leq H^*(s) + H(t), \quad \forall s,t \geq 0. \tag{4.23}$$

The relation (4.22), the fact that $H'(0) = 0$ and $(H')^{-1}, H$ are increasing functions yield

$$H^*(s) \leq s(H')^{-1}(s), \quad \forall s \geq 0. \tag{4.24}$$

Making use of $E'(t) \leq 0$, $H''(t) \geq 0$, (4.21) and (4.24) we derive for $\varepsilon_0 > 0$ small enough

$$\begin{array}{rc}
& [H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} + \tilde{C}_5 C' E(t)]' \\
= & \varepsilon_0 E'(t)H''(\varepsilon_0 E(t))(L(t) + C_5\mu_1 E(t)) + H'(\varepsilon_0 E(t))(L'(t) + C_5\mu_1 E'(t)) + \tilde{C}_5 C' E'(t) \\
\leq & -C_3\alpha(t)H'(\varepsilon_0 E(t))E(t) + C_4\alpha(t)H'(\varepsilon_0 E(t))(h\circ\nabla u)(t) \\
& +\tilde{C}_5\alpha(t)H'(\varepsilon_0 E(t))H^{-1}(-C'E'(t)) + \tilde{C}_5 C' E'(t) \\
\leq & -C_3\alpha(t)H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5\alpha(t)H^*(H'(\varepsilon_0 E(t))) + C_4\alpha(t)H'(\varepsilon_0 E(0))(h\circ\nabla u)(t) \\
\leq & -C_3\alpha(t)H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5\alpha(t)H'(\varepsilon_0 E(t))\varepsilon_0 E(t) + C_4\alpha(t)H'(\varepsilon_0 E(0))(h\circ\nabla u)(t) \\
\leq & -\tilde{C}_3\alpha(t)H'(\varepsilon_0 E(t))E(t) + C_4\alpha(t)H'(\varepsilon_0 E(0))(h\circ\nabla u)(t) \\
= & -\tilde{C}_3\alpha(t)H_2(E(t)) + C_4\alpha(t)H'(\varepsilon_0 E(0))(h\circ\nabla u)(t).
\end{array} \tag{4.25}$$

We note that, in the second inequality, we have used (4.23) and $0 \leq H'(\varepsilon_0 E(t)) \leq H'(\varepsilon_0 E(0))$.

Let

$$\tilde{L}(t) = \begin{cases} L(t) + \mu E(t) & \text{if } H \text{ is linear on } [0,\varepsilon'] \\ H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} + \tilde{C}_5 C' E(t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon'], \end{cases} \tag{4.26}$$

then from (4.19) and (4.25), it holds that

$$\tilde{L}'(t) \leq -c_4\alpha(t)H_2(E(t)) + c_5\alpha(t)\alpha(t)(h\circ\nabla u)(t), \quad \forall t \geq t_0. \tag{4.27}$$

On the other hand, by choosing $M > 0$ larger if needed, we can observe from Lemma 4.2 that $L(t)$ is equivalent to $E(t)$. So, $\tilde{L}(t)$ is also equivalent to $E(t)$. Moreover, because the fact that $\zeta'(t) \leq 0$, there exists $\overline{\varepsilon} > 0$, such that

$$\zeta(t)\tilde{L}(t) + 2c_5 E(t) \leq \overline{\varepsilon}E(t), \quad \forall t \geq t_0. \tag{4.28}$$

Finally, let

$$\mathcal{L}(t) = \varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t)), \quad \text{for } 0 < \varepsilon < \frac{1}{\overline{\varepsilon}},$$

then we observe, from (4.27), (H1), (2.23) and (4.28), that

$$\begin{array}{rcl}
\mathcal{L}'(t) & = & \varepsilon(\zeta'(t)\tilde{L}(t) + \zeta(t)\tilde{L}'(t) + 2c_5 E'(t)) \\
& \leq & -c_4\varepsilon\alpha(t)\zeta(t)H_2(E(t)) + c_5\varepsilon\alpha(t)\zeta(t)(h\circ\nabla u)(t) + 2c_5\varepsilon E'(t) \\
& \leq & -c_4\varepsilon\alpha(t)\zeta(t)H_2(E(t)) - c_5\varepsilon\alpha(t)(h'\circ\nabla u)(t) + 2c_5\varepsilon E'(t) \\
& \leq & -c_4\varepsilon\alpha(t)\zeta(t)H_2(E(t)) \leq -c_4\varepsilon\alpha(t)\zeta(t)H_2\left(\frac{1}{\overline{\varepsilon}}\left(\zeta(t)\tilde{L}(t) + 2c_5 E(t)\right)\right) \\
& \leq & -c_4\varepsilon\alpha(t)\zeta(t)H_2(\varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t))) = -c_4\varepsilon\alpha(t)\zeta(t)H_2(\mathcal{L}(t)).
\end{array} \tag{4.29}$$

We have used the fact H_2 is increasing in the last two inequalities. Noting that $H_1' = -1/H_2$ (see (2.12)), we infer from (4.29)

$$\mathcal{L}'(t)H_1'(\mathcal{L}(t)) \geq c_4\varepsilon\alpha(t)\zeta(t), \quad \forall t \geq t_0.$$

A simple Integration over (t_0, t) then yields

$$H_1(\mathcal{L}(t)) \geq H_1(\mathcal{L}(t_0)) + c_4\varepsilon \int_0^t \alpha(t)\zeta(s)\,ds - c_4\varepsilon \int_0^{t_0} \alpha(t)\zeta(s)\,ds.$$

Choose $\varepsilon > 0$ sufficiently small so that $H_1(\mathcal{L}(t_0)) - c_4\varepsilon \int_0^{t_0} \alpha(t)\zeta(s)\,ds > 0$, then, thanks to the fact H_1^{-1} is decreasing, we infer

$$\begin{array}{rcl} \mathcal{L}(t) & \leq & H_1^{-1}\left(H_1(\mathcal{L}(t_0)) - c_4\varepsilon \int_0^{t_0} \alpha(t)\zeta(s)\,ds + c_4\varepsilon \int_0^t \alpha(t)\zeta(s)\,ds\right) \\ & \leq & H_1^{-1}\left(c_4\varepsilon \int_0^t \alpha(t)\zeta(s)\,ds\right). \end{array}$$

Consequently, the equivalence of $\mathcal{L}, \tilde{L}, L$ and E, yield

$$E(t) \leq C_0 H_1^{-1}\left(\omega \int_0^t \alpha(t)\zeta(s)\,ds\right).$$

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