Palestine Journal of Mathematics

Vol. 7(2)(2018) , 598–607

GORENSTEIN FI-FLAT DIMENSION WITH RESPECT TO A SEMIDUALIZING MODULE

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Communicated by Ayman Badawi

MSC 2010 Classifications: 18G20, 18G25.

Keywords and phrases: GFI_F-closed ring; G_C-FI-flat module; G_C-FI-flat dimension; Semidualizing module.

Abstract This paper presents some properties of G_C-FI-flat modules, where C is a semidualizing module and we investigate the relation between the G_C-yoke with the C-yoke of a module as well as the relation between the G_C-FI-flat resolution and the FI-flat resolution of a module over GFI_F-closed rings. We also obtain a criterion for computing the G_C-FI-flat dimension of modules.

1 Introduction

Projective, injective and flat modules play an important role in basic homological algebra. Homological properties of the Gorenstein projective, injective and flat modules have been studied by many authors, some references are [4, 5, 11]. The study of semidualizing modules over commutative Noetherian rings was initiated independently (with different names) by Foxby [7], Golod [10], and Vasconcelos [22]. Over a commutative Noetherian ring, Holm and Jørgensen in [12] introduced the C-Gorenstein projective, C-Gorenstein injective and C-Gorenstein flat modules using semidualizing modules and their associated projective, injective and flat classes which are also called G_C-projective, G_C-injective and G_C-FI-flat module respectively. White introduced in [23] the G_C-projective modules and gave a functorial description of the G_C-projective dimension of modules with respect to a semidualizing module C over a commutative ring; and in particular, many classical results about the Gorenstein projectivity of modules were generalized in [23]. Further, Selvaraj et. al introduced Gorenstein FI-injective and Gorenstein FI-flat modules in [18] and studied the covers of Gorenstein FI-flat modules in [19] and Tate homology in [20]. FI-cotorsion module was studied in [2] and Strongly FI-Cotorsion modules were introduced by Biju et. al in [3]. Also, the dimension of Gorenstein FI-flat modules has been analysed with respect to relative homology in [22]. In this paper we give a functorial description of the G_C-FI-flat dimension of modules with respect to a semidualizing module.

This paper is organized as follows. In Section 2, we recall some notions and definitions which will be needed in the later section. In this section We also introduce C-FI-flat C-FI-injective and G_C-FI-flat modules and also establish the relation between the G_C-yoke with the C-yoke of a module as well as the relation between the G_C-FI-flat resolution and the FI-flat resolution of a module over a GFI_F-closed ring. In Section 3, we discuss some properties of G_C-FI-flat dimension of modules. In particular, as an application of the results obtained in Section 2, we obtain a criterion for computing such a dimension as in the following result.

Proposition 1.1. *Let R be a GFI_F-closed ring and let M be a left R-module and $n \geq 0$. We prove that the G_C-FI-flat dimension of M is at most n if and only if for every non-negative integer t such that $0 \leq t \leq n$, there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ in R-Mod such that X_t is G_C-FI-flat and $X_i \in \overline{Add_R C}$ for $i \neq t$.*

2 G_C-FI-flat modules

Unless stated otherwise, throughout this paper all rings are associative with identity and all modules are unitary modules. Let R be a ring, we denote by R-Mod (Mod-R) the category of left (right) R-modules respectively. FI-injective and FI-flat modules were introduced by Mao et al. [15]. A right R-module M is called FI-injective if $Ext^1_R(G, M) = 0$ for any absolutely pure right R-module G. A left R-module N is said to be FI-flat if $Tor^R_1(G, N) = 0$ for any absolutely pure right R-module G.

At the beginning of this section, we recall some notions from [13, 23].

Definition 2.1. [23] A degree wise finite projective (resp., free) resolution of an R-module M is a projective (resp., free) resolution P of M such that each P_i is a finitely generated projective (resp., free). Note that M admits a degree wise finite projective resolution if and only if it admits a degree wise finite free resolution. However, it is possible for a module to admit a bounded degree wise finite projective resolution but not to admit a bounded degree wise finite free resolution. For example, if $R = k_1 \oplus k_2$, where k_1 and k_2 are fields, then $M = k_1 \oplus 0$ is a projective R-module, but it does not admit a bounded free resolution.

Definition 2.2. [13] Let R and S be rings. An (S, R)-bimodule C is called semidualizing if the following conditions are satisfied:

(1) ${}_SC$ admits a degree wise finite S-projective resolution;

(2) C_R admits a degree wise finite R^{op}-projective resolution;

(3) The homothety map ${}_SS_S \to Hom_{R^{op}}(C, C)$ is an isomorphism;

(4) The homothety map ${}_RR_R \to Hom_S(C, C)$ is an isomorphism;

(5) $Ext^i_S(C, C) = 0$ for any $i \geq 1$;

(6) $Ext^i_{R^{op}}(C, C) = 0$ for any $i \geq 1$.

Definition 2.3. [13] Let C be a semidualizing module for a ring R. An R-module is C-projective if it has the form $C \otimes_R P$ for some projective module P. An R-module is called C-injective if it has the form $Hom_R(C, I)$ for some injective module I. Set

$$\mathcal{P}_C(R) = \{C \otimes_R P \,|\, P \, is \; R - projective\},$$

and

$$\mathcal{I}_C(R) = \{Hom_R(C, I) \,|\, I \, is \; R - injective\}.$$

Definition 2.4. [13] An R-module is called C-flat if it has the form $C \otimes_R F$ for some flat module F. Set $\mathcal{F}_C(R) = \{C \otimes_R F \,|\, F$ is R-flat $\}$.

Setting $C = R$ in the above definitions, we see that $\mathcal{P}_C(R)$, $\mathcal{I}_C(R)$ and $\mathcal{F}_C(R)$ are the classes of ordinary projective, injective and flat R-modules, which we usually denote $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ respectively.

Now we introduce C-FI-flat and C-FI-injective modules as follows.

Definition 2.5. An R-module is called C-FI-flat if it has the form $C \otimes_R F$ for some FI-flat module F. Set $\mathcal{FI}f_C(R) = \{C \otimes_R F \,|\, F$ is FI-flat $\}$.

Definition 2.6. An R-module is called C-FI-injective if it has the form $Hom_R(C, E)$ for some FI-injective module E. Set $\mathcal{FI}i_C(R) = \{Hom_R(C, E) \,|\, E$ is FI-injective$\}$.

When $C = R$, we omit the subscript and recover the classes of FI-flat and FI-injective R-modules. Any semidualizing module defines two important classes of modules, namely the Auslander and Bass classes, with a certain nice duality property.

Definition 2.7. [23] The Auslander class $\mathcal{A}_C(R)$ with respect to C consists of all modules M satisfying:

(A1) $Tor_i^R(C, M) = 0$ for any $i \geq 1$;

(A2) $Ext_R^i(C, C \otimes_R M) = 0$ for any $i \geq 1$; and

(A3) The natural evaluation homomorphism $\mu_M : M \to Hom_R(C, C \otimes_R M)$ is an isomorphism.

The Bass class $\mathcal{B}_C(R)$ with respect to C consists of all modules N satisfying:

(B1) $Ext_R^i(C, N) = 0$ for any $i \geq 1$;

(B2) $Tor_i^R(C, Hom_R(C, N)) = 0$ for any $i \geq 1$; and

(B3) The natural evaluation homomorphism $\nu_N : C \otimes_R Hom_R(C, N) \to N$ is an isomorphism.

Let $M \in R\text{-}Mod$. Write M^I (resp., $M^{(I)}$) is the direct product (resp., sum) of copies of a module M indexed by a set I. We denote $Add_R M$ (resp., $Prod_R M$) the subclass of $R\text{-}Mod$ consisting of all modules isomorphic to direct summands of direct sums (resp., direct products) of copies of M. We start with the following

Proposition 2.8. $\mathcal{FI}f_C(R) = Add_R C$.

Proof. Let $F \in R\text{-}Mod$ be FI-flat. Then F is isomorphic to a direct summand of $K^{(J)}$ for some cardinal J, where K is FI-flat generator. So $C \otimes_R F$ is isomorphic to a direct summand of $C \otimes_R K^{(J)}$ ($\cong C^{(J)}$), and hence $C \otimes_R F \in Add_R C$. Thus we have $\mathcal{FI}f_C(R) \subseteq Add_R C$. Conversely, for any $M \in Add_R C$, there exists $N \in R\text{-}Mod$ such that $M \oplus N \cong C^{(J)}$ for some cardinal J. Note that $\mathcal{B}_C(R)$ is closed under direct sums and direct summands by [13, Proposition 4.2]. Since $C \cong C \otimes_R R \in \mathcal{B}_C(R)$ by [13, Lemma 5.1], both $C^{(J)}$ and M are in $\mathcal{B}_C(R)$. Since $Hom_R(C, M) \oplus Hom_R(C, N) \cong Hom_R(C, C^{(J)}) \cong R^{(J)}$, $Hom_R(C, M) \in R\text{-}Mod$ is FI-flat. Thus $M \in \mathcal{FI}f_C(R)$ by [13, Lemma 5.1]. Therefore $Add_R C \subseteq \mathcal{FI}f_C(R)$. □

Now we recall the following definitions.

Definition 2.9. [6] A left R-module M is said to be Gorenstein flat, if there exists an exact sequence of flat left R-modules,

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong Im(F_0 \to F^0)$ and such that $B \otimes_R -$ leaves the sequence exact whenever B is an injective right R-module. $\mathcal{GF}(R)$ denotes class of all Gorenstein flat R-modules.

Definition 2.10. [18] A left R-module M is said to be Gorenstein FI-flat if there is a $\mathscr{A} \otimes -$ exact exact sequence $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ of FI-flat left R-modules with $M = ker(F^0 \longrightarrow F^1)$ where $\mathscr{A}$ denotes class of all FI-injective right R-modules.

Example 2.11. Since every flat R-module is FI-flat and injective R-module is FI-injective, we can see that Gorenstein flat modules are Gorenstein FI-flat R-modules. However, the converse need not be true. But it is true by [15, Proposition 2.3].

Definition 2.12. [1] A ring R is said to be left GF-closed if $\mathcal{GF}(R)$ is closed under extensions.

Definition 2.13. [18] A ring R is said to be left GFI_F-closed if $\mathcal{GFI}_{\mathcal{F}}(R)$ is closed under extensions, i.e., for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of left R-modules if M_1 and M_3 are in $\mathcal{GFI}_{\mathcal{F}}(R)$, then M_2 is in $\mathcal{GFI}_{\mathcal{F}}(R)$, where $\mathcal{GFI}_{\mathcal{F}}(R)$ is the class of all Gorenstein FI-flat R-modules.

Example 2.14. Since Gorenstein flat modules are Gorenstein FI-flat modules, every GF-closed rings are GFI_F-closed rings.

Definition 2.15. A complete $\mathcal{FIF}_C$-resolution is a $\mathcal{FI}i_C(R) \otimes_R -$ exact exact sequence:

$$\mathcal{X} : \cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots \tag{2.1}$$

in $R\text{-}Mod$ with all F_i and F^i are FI-flat. A module $M \in R\text{-}Mod$ is called G_C-FI-flat if there exists a complete $\mathcal{FIF}_C$-resolution as in (1) with $M = Coker(F_1 \to F_0)$. Set $\mathcal{GFI}f_C(R)$ is the class of G_C-FI-flat modules in $R\text{-}Mod$.

It is trivial that in case ${}_RC_R = {}_RR_R$, the G_C-FI-flat modules are just the usual Gorenstein FI-flat modules.

Using the definition, we immediately get the following results.

Proposition 2.16. *If $(F_i)_{i\in I}$ is a family of G_C-FI-flat modules, then $\bigoplus F_i$ is G_C-FI-flat.*

Proposition 2.17. *A module M is G_C-FI-flat if and only if $Tor^R_{\geq 1}(Hom_R(C,E),M) = 0$ and M admits a $\mathcal{FI}f_C(R)$-resolution Y with $Hom_R(C,E)\otimes_R Y$ exact for any FI-injective E.*

Recall the following definition.

Definition 2.18. [1] Let R be a ring and let $\mathfrak{X}$ be a class of left R-modules.

(1) $\mathfrak{X}$ is closed under extensions: If for every short exact sequence of left R-modules $0\to A\to B\to C\to 0$, the conditions A and C are in $\mathfrak{X}$ implies B is in $\mathfrak{X}$.

(2) $\mathfrak{X}$ is closed under kernels of epimorphisms: If for every short exact sequence of left R-modules $0\to A\to B\to C\to 0$, the conditions B and C are in $\mathfrak{X}$ implies A is in $\mathfrak{X}$.

(3) $\mathfrak{X}$ is projectively resolving: If it contains all projective modules and it is closed under both extensions and kernels of epimorphisms. i.e., for every short exact sequence of R-modules $0\to A\to B\to C\to 0$ with $C\in\mathfrak{X}$, the conditions $A\in\mathfrak{X}$ and $B\in\mathfrak{X}$ are equivalent.

The following result is due to [17].

Proposition 2.19. *Let C be a semidualizing R-module. Then the class $\mathcal{GFI}f_C(R)$ is closed under kernels of epimorphisms and extensions.*

Proposition 2.20. *If F is FI-flat R-module, then F and $C\otimes_R F$ are G_C-FI-flat. Thus, every R-module admits a G_C-FI-flat resolution.*

Proof. Follows from [12, Example 2.8(a), Propositions 2.1, 2.13(1) and 2.15] and since the class of G_C-FI-flat modules contains the class of FI-flat modules, every R-module admits a G_C-FI-flat resolution. □

Theorem 2.21. *Let R be a GFI_F-closed ring and C is semidualizing module, then the class $\mathcal{GFI}f_C(R)$ of G_C-FI-flat R-modules is projectively resolving and closed under direct summands.*

Proof. Using the dual of Theorem 2.8 in [23] and together with the [17, Lemma 5.2], we see that $\mathcal{GFI}f_C(R)$ is projectively resolving. Now, comparing Proposition 2.5 with Proposition 1.4 in [11], we get $\mathcal{GFI}f_C(R)$ is closed under direct summands. □

Proposition 2.22. *Let R be a GFI_F-closed ring. Then every cokernel in a complete $\mathcal{FIF}_C$-resolution is G_C-FI-flat.*

Proof. Follows from Proposition 2.17, Theorem 2.21 and [17, Lemma 5.4]. □

Lemma 2.23. *Let R be a GFI_F-closed ring and let $M\in R$-Mod be G_C-FI-flat. Then there exists $\mathcal{FI}i_C(R)\otimes -$ exact sequences:*

$$0\to M\to G\to N\to 0$$

and

$$0\to K\to F\to M\to 0$$

in R-Mod with N, K are G_C-FI-flat, $G\in Add_RC$, and F is FI-flat.

Proof. It follows from the definition of G_C-FI-flat modules and Proposition 2.22 . □

The following result plays a crucial role in this section.

Lemma 2.24. *Let R be a GFI_F-closed ring and suppose that*

$$0 \to A \to G_1 \xrightarrow{f} G_0 \to M \to 0 \tag{2.2}$$

is an exact sequence in R-Mod with G_0, G_1 are G_C-FI-flat. Then we have the following exact sequences:

$$0 \to A \to C_1 \to G \to M \to 0, \tag{2.3}$$

and

$$0 \to A \to H \to F \to M \to 0 \tag{2.4}$$

with $C_1 \in Add_R C$, F is FI-flat, and G, H are G_C-FI-flat.

Proof. Since G_1 is G_C-FI-flat, there exists a short exact sequence $0 \to G_1 \to C_1 \to G' \to 0$ with $C_1 \in Add_R C$ and G' is G_C-FI-flat by Lemma 2.23. Then we have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & Im(f) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & C_1 & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Im(f) & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G' & = & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Since G_0 and G' are G_C-FI-flat, G is also G_C-FI-flat by Theorem 2.21. Connecting the middle rows in the above two diagrams, we get the first desired exact sequence (2.3).

Since G_0 is G_C-FI-flat, there exists an exact sequence $0 \to G'' \to F \to G_0 \to 0$ with F is FI-flat and G'' is G_C-FI-flat by Lemma 2.23. Then we have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G'' & = & G'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Im(f) & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G'' & = & G'' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & Im(f) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since G_1 and G'' are G_C-FI-flat, H is also G_C-FI-flat by Theorem 2.21. Connecting the middle rows in the above two diagrams, we get the second desired exact sequence (2.4). □

Gao et al. [8] investigated yoke and Gorenstein yoke modules, i.e., yoke of Gorenstein flat resolutions of modules over a right coherent ring. In a similar manner we introduce C-yoke and G_C-yoke module as follows.

Definition 2.25. Let n be a positive integer. An R-module A is called an C-yoke module (of M) if there exists an exact sequence

$$0 \to A \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

in R-Mod with all F_i are C-FI-flat.

Definition 2.26. Let n be a positive integer, a module A is called an G_C-yoke module (of M) if there exists an exact sequence

$$0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

in R-Mod with all G_i are G_C-FI-flat.

The following result establishes the relation between the G_C-yoke with the C-yoke of a module as well as the relation between the G_C-FI-flat resolution and the FI-flat resolution of a module.

Lemma 2.27. *Let R be a GFI_F-closed ring and let $n \geq 1$ and*

$$0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

be an exact sequence in R-Mod with all G_i are G_C-FI-flat. Then we have the following:

(i) There exist exact sequences:

$$0 \to A \to C_{n-1} \to \cdots \to C_1 \to C_0 \to N \to 0$$

and

$$0 \to M \to N \to G \to 0$$

in R-Mod with all $C_i \in Add_R C$ and G is G_C-FI-flat.

(ii) There exist exact sequences

$$0 \to B \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

and

$$0 \to H \to B \to A \to 0$$

in R-Mod with all F_i are FI-flat and H is G_C-FI-flat.

Proof. We proceed by induction on n.

(i) When $n = 1$, we have an exact sequence $0 \to A \to G_0 \to M \to 0$ in R-Mod. Since we have a $\mathcal{FI}i_C(R) \otimes_R -$ exact exact sequence $0 \to G_0 \to C_0 \to G \to 0$ in R-Mod with $C_0 \in Add_R C$ and G is G_C-FI-flat by Lemma 2.23, we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & C_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

The middle row and the last column in the above diagram are the desired two exact sequences.

Now assume that $n \geq 2$ and we have an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod with all G_i are G_C-FI-flat. Put $K = Coker(G_{n-1} \to G_{n-2})$. By Lemma 2.24, we have an exact sequence

$$0 \to A \to C_{n-1} \to G'_{n-2} \to K \to 0$$

in R-Mod with $C_{n-1} \in Add_R C$ and G'_{n-2} is G_C-FI-flat. Put $A' = Im(C_{n-1} \to G'_{n-2})$. Then we get an exact sequence $0 \to A' \to G'_{n-2} \to G_{n-3} \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod. Hence, by the induction hypothesis, we obtain the assertion.

(ii) When $n = 1$, we have an exact sequence $0 \to A \to G_0 \to M \to 0$ in R-Mod. Since we have a $\mathcal{FI}i_C(R) \otimes_R -$ exact exact sequence $0 \to H \to F_0 \to G_0 \to 0$ in R-Mod with F_0 is

FI-flat and H is G_C-FI-flat by Lemma 2.23, we get the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

The middle row and the first column in the above diagram are the desired two exact sequences.

Now assume that $n \geq 2$ and we have an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod with all G_i are G_C-FI-flat. Put $K = Ker(G_1 \to G_0)$. By Lemma 2.24, we get an exact sequence

$$0 \to K \to G_1' \to F_0 \to M \to 0$$

in R-Mod with F_0 is FI-flat and G_1' is G_C-FI-flat. Put $M' = Im(G_1' \to P_0)$. Then we have an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_2 \to G_1' \to G_0 \to M \to 0$ in R-Mod. So, by the induction hypothesis, we obtain the assertion. □

Here is a version of Schannuel's Lemma for $\mathcal{FIF}_C$-resolutions.

Proposition 2.28. *Let M be a left R-module, and consider two exact sequences of left R-modules,*

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

and

$$0 \to H_n \to H_{n-1} \to \cdots \to H_0 \to M \to 0,$$

where $G_0, \cdots, G_{n-1}$ and $H_0, \cdots, H_{n-1}$ are G_C-FI-flat. If R is GFI_F-closed, then G_n is G_C-FI-flat if and only if H_n is G_C-FI-flat.

Proof. It follows from Proposition 2.16 and Proposition 2.21 . □

3 G_C-FI-flat dimensions of modules

The class of G_C-FI-flat modules can be used to define the G_C-FI-flat dimension.

Definition 3.1. For a module $M \in R$-Mod, the G_C-FI-flat dimension of M, denoted by $G_C - FI_f d_R(M)$, is defined as $inf\{n|$ there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod with all G_i are G_C-FI-flat $\}$.
We have $G_C - FI_f d_R(M) \geq 0$, and $G_C - FI_f d_R(M) = \infty$ if no such integer exists.

We start with the following standard result.

Lemma 3.2. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in R-Mod.*

(i) $G_C - FI_f d_R(N) \leq max\,\{G_C - FI_f d_R(M), G_C - FI_f d_R(L) + 1\}$, and the equality holds if $G_C - FI_f d_R(M) \neq G_C - FI_f d_R(L)$.

(ii) $G_C - FI_f d_R(L) \leq max\,\{G_C - FI_f d_R(M), G_C - FI_f d_R(N) - 1\}$, and the equality holds if $G_C - FI_f d_R(M) \neq G_C - FI_f d_R(N)$.

(iii) $G_C - FI_f d_R(M) \leq max\,\{G_C - FI_f d_R(L), G_C - FI_f d_R(N)\}$*, and the equality holds if* $G_C - FI_f d_R(N) \neq G_C - FI_f d_R(L) + 1$.

Proof. It is immediate. □

The proof of the following Theorem is similar to [11, Theorem 3.15].

Theorem 3.3. *Assume that R is GFI_F-closed and C-is a semidualizing module. If any two of the modules M, M' or M'' in a short exact sequence $0 \to M' \to M \to M''$ have finite G_C-FI-flat dimension, then so has the third.*

Proposition 3.4. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in R-Mod. If $L \neq 0$ and N is G_C-FI-flat, then $G_C - FI_f d_R(L) = G_C - FI_f d_R(M)$.*

Proof. It follows by Lemma 3.2(3). □

We give a criterion for computing the G_C-FI-flat dimension of modules as follows. It generalizes [11, Theorem 3.14]. We denote $\overline{Add_R C} = Add_R C \cup Add_R R$.

Proposition 3.5. *Let R be a GFI_F-closed ring. The following statements are equivalent for any $M \in R$-Mod and $n \geq 0$:*

(i) $G_C - FI_f d_R(M) \leq n$;

(ii) For every non-negative integer t such that $0 \leq t \leq n$, there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ in R-Mod such that X_t is G_C-FI-flat and $X_i \in \overline{Add_R C}$ for $i \neq t$.

Proof. $(ii) \Rightarrow (i)$. It is trivial.
$(i) \Rightarrow (ii)$. We proceed by induction on n. Suppose $G_C - FI_f d_R(M) \leq 1$. Then there exists an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ in R-Mod with G_0 and G_1 are G_C-FI-flat. By Lemma 2.24 with $A = 0$, we get the exact sequences $0 \to C_1 \to G_0' \to M \to 0$ and $0 \to G_1' \to F_0 \to M \to 0$ in R-Mod with $C_1 \in Add_R C$, F_0 is FI-flat, and G_0', G_1' are G_C-FI-flat.

Now suppose $G_C - FI_f d_R(M) = n \geq 2$. Then there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod with G_i are G_C-FI-flat for any $0 \leq i \leq n$. Set $A = Coker(G_3 \to G_2)$. By applying Lemma 2.24 to the exact sequence $0 \to A \to G_1 \to G_0 \to M \to 0$, we get an exact sequence $0 \to G_n \to \cdots \to G_2 \to G_1' \to F_0 \to M \to 0$ in R-Mod with G_1' are G_C-FI-flat and F_0 is FI-flat. Set $N = Coker(G_2 \to G_1')$. Then we have $G_C - FI_f d_R(N) \leq n - 1$. By the induction hypothesis, there exists an exact sequence

$$0 \to X_n \to \cdots \to X_t \to \cdots \to X_1 \to F_0 \to M \to 0$$

in R-Mod such that F_0 is FI-flat and X_t is G_C-FI-flat and $X_i \in \overline{Add_R C}$ for $i \neq t$ and $1 \leq t \leq n$.

Now we need only to prove (ii) for $t = 0$. Set $B = Coker(G_2 \to G_1)$. By the induction hypothesis, we get an exact sequence $0 \to X_n \to \cdots \to X_3 \to X_2 \to G_1' \to B \to 0$ in R-Mod with G_1' G_C-FI-flat and $X_i \in \overline{Add_R C}$ for any $2 \leq i \leq n$. Set $D = Coker(X_3 \to X_2)$. Then by applying Lemma 2.24 to the exact sequence $0 \to D \to G_1' \to G_0 \to M \to 0$, we get the exact sequence $0 \to D \to C_1 \to G_0' \to M \to 0$ in R-Mod with $C_1 \in Add_R C$ and G_0' is G_C-FI-flat. Thus we obtain the desired exact sequence

$$0 \to X_n \to \cdots \to X_2 \to X_1 \to G_0' \to M \to 0$$

in R-Mod with all $X_i \in \overline{Add_R C}$ and G_0' is G_C-FI-flat. □

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Received: April 304, 2017.

Accepted: December 27, 2017.