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Certain identities for a continued fraction of order six

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Abstract On page 365 of his lost notebook [26], Ramanujan wrote five identities which shows the relation between the celebrated Rogers-Ramanujan continued fraction $R(q)$ and the other five continued fractions $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$ and $R(q^5)$. In this paper, we present some new identities providing the relations between a new continued fraction $H(q)$ of order six and the other continued fraction $H(q^n)$ for $n = 2, 3, 4, 5, 7, 9, 11, 13$ and 17 which are analogues to the results of Rogers- Ramanujan's continued fraction. In the process, we establish some new $P-Q$ type modular equations for the ratios of Ramanujan's theta functions.

1 Introduction

For $|\ q\ |< 1$,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{1.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \tag{1.2}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{3n(n-1)/2}, \tag{1.3}$$

are special cases of Ramanujan's general theta function

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ \ |ab| < 1. \tag{1.4}$$

Now, we define modular equation in brief. Let

$$K := K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{1.5}$$

where $0 < k < 1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\, n!} z^n, \quad 0 \le |z| < 1,$$

with $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$ for n a positive integer and a, b, c are complex numbers such that $c \neq 0, -1, -2, \ldots$. The number k is called the modulus of K and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.6}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1.6). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L} = \frac{\varphi^2(q)}{\varphi^2(q^n)}. \tag{1.7}$$

In Section 2, we collect preliminary results which are useful to prove main results. In Section 3, we establish continued fraction of order six. Several $P-Q$ modular equations were established in Section 4. In Section 5, we establish modular relations connecting the continued fractions $H(q)$ and $H(q^n)$ for $n = 2, 3, 4, 5, 7, 9, 11, 13$ and 17.

2 Preliminary Results

In this section, we collect the several results which are very useful in proving our main results.

Lemma 2.1. *[5, Ch.16, Entry 22(iv), p.37] Let from Entry 22(iv) of Chapter 16, the definition*

$$\chi(q) = (-q; q^2)_\infty, \tag{2.1}$$

where $(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$. *Then we have [5, Ch.17, Entry 12(v) and (vi), p.124]*

$$\chi(q) = 2^{1/6} \left(x(1-x)e^y\right)^{-\frac{1}{24}} \tag{2.2}$$

and

$$\chi(-q) = 2^{1/6} (1-x)^{\frac{1}{12}} \left(xe^y\right)^{-\frac{1}{24}}, \tag{2.3}$$

where $q := q(x) := e^{-y}$ *and*

$$y = \pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}.$$

Lemma 2.2. *[20, Theorem 3.2] If* $M = \dfrac{\chi(-q)\chi(-q^3)}{\chi(q)\chi(q^3)}$ *and* $N = \dfrac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$, *then*

$$N^2 - \frac{1}{N^2} = 2\left(M - \frac{1}{M}\right). \tag{2.4}$$

Lemma 2.3. *[5, Ch.16, Entry 30(ii) and (iii), p.46] For* $\mid ab \mid < 1$, *we have*

$$f(a, b) + f(-a, -b) = 2f(a^3b, ab^3), \tag{2.5}$$

$$f(a, b) - f(-a, -b) = 2af\left(\frac{b}{a}, \frac{a}{b}a^4b^4\right). \tag{2.6}$$

Lemma 2.4. *[5, Ch.16, Entry 19, p.35] For* $\mid ab \mid < 1$, *we have*

$$f(a, b) = (-a; ab)_\infty(-b; ab)_\infty(ab; ab)_\infty. \tag{2.7}$$

Lemma 2.5. *[5, Ch.18, Entry 24 (iii), Eq.(24.21), p.215] If* β *is of degree* 2 *over* α, *then*

$$\sqrt{\beta} = \frac{1 - \sqrt{1-\alpha}}{1 + \sqrt{1-\alpha}}. \tag{2.8}$$

Lemma 2.6. *[5, Ch.19, Entry 5(ii), p.230] If* β *is of degree* 3 *over* α, *then*

$$(\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} = 1. \tag{2.9}$$

Lemma 2.7. *[5, Ch.18, Entry 24 (iii), Eq.(24.22), p.215] If β is of degree 4 over α, then*

$$\sqrt[4]{\beta}=\frac{1-\sqrt[4]{1-\alpha}}{1+\sqrt[4]{1-\alpha}}. \tag{2.10}$$

Lemma 2.8. *[5, Ch.19, Entry 13(i), p.280] If β is of degree 5 over α, then*

$$(\alpha\beta)^{1/2}+((1-\alpha)(1-\beta))^{1/2}+2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}=1. \tag{2.11}$$

Lemma 2.9. *[5, Ch.19, Entry 19(i), p.314] If β is of degree 7 over α, then*

$$(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}=1. \tag{2.12}$$

Lemma 2.10. *[5, Ch.20, Entry 7(i), p.363] If β is of degree 11 over α, then*

$$(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}+2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}=1. \tag{2.13}$$

Lemma 2.11. *[6, Ch.36, Entry 62, 63 and 64, pp.387-388] Let*

$$L=1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}, \tag{2.14}$$

$$M=64\left(\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}-\sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right), \tag{2.15}$$

and

$$N=32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}. \tag{2.16}$$

1. *If β is of degree 9 over α, then*

$$L^6-N(14L^3+LM)-3N^2=0. \tag{2.17}$$

2. *If β is of degree 13 over α, then*

$$\sqrt{L}\left(L^3+8N\right)-\sqrt{N}(11L^2+M)=0. \tag{2.18}$$

3. *If β is of degree 17 over α, then*

$$L^3-N^{1/3}\left(10L^2+M\right)+13N^{2/3}L+12N=0. \tag{2.19}$$

3 Main Theorem

In this section, we obtain a new continued fraction of order six and its related identities. Ramanujan has recorded several continued fractions in his notebooks. One of the fascinating continued fraction identity recorded by Ramanujan as Entry 11 in his second notebook is as follows [25], [5, Entry 11, p. 21]:

$$\begin{aligned}&\frac{(-a;q)_\infty(b;q)_\infty-(a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty+(a;q)_\infty(-b;q)_\infty}\\&=\frac{a-b}{1-q\,+}\;\frac{(a-bq)(aq-b)}{1-q^3\;+}\;\frac{q(a-bq^2)(aq^2-b)}{1-q^5\;+}\cdots,\end{aligned} \tag{3.1}$$

where $q,\ a$ and b are complex numbers with $|\, q\,|< 1$.
The identity (3.1) was first proved by K. G. Ramanathan [24] and the other proofs were given by the following authors C. Adiga, B. C. Berndt, S. Bhargava and G. N. Watson [2] and L. Jacobsen [10]. Recently, many mathematicians have studied several interesting properties of Rogers-Ramanujan continued fraction [7, 15, 22, 23, 27], Ramanujan's cubic continued fraction

[3, 4, 8, 11, 16] and other continued fractions [1, 14, 20, 12].
Replacing b by $-b$ in the above equation (3.1), we find that

$$\frac{(-a;q)_\infty(-b;q)_\infty-(a;q)_\infty(b;q)_\infty}{(-a;q)_\infty(-b;q)_\infty+(a;q)_\infty(b;q)_\infty} = \frac{a+b}{1-q}\,+\,\frac{(a+bq)(aq+b)}{1-q^3}\,+\,\frac{q(a+bq^2)(aq^2+b)}{1-q^5}\,+\cdots. \tag{3.2}$$

Again replacing q by ab in the above equation (3.2), we find that

$$\frac{(-a;ab)_\infty(-b;ab)_\infty-(a;ab)_\infty(b;ab)_\infty}{(-a;ab)_\infty(-b;ab)_\infty+(a;ab)_\infty(b;ab)_\infty} = \frac{a+b}{1-ab}\,+\,\frac{ab(1+b^2)(1+a^2)}{1-(ab)^3}\,+\,\frac{a^2b^2(1+ab^3)(1+a^3b)}{1-(ab)^5}\,+\cdots, \mid ab\mid<1. \tag{3.3}$$

Using the Jacobi's triple product identity (2.7), the above equation (3.3) can be written as

$$\frac{f(a,b)-f(-a,-b)}{f(a,b)+f(-a,-b)} = \frac{a+b}{1-ab}\,+\,\frac{ab(1+b^2)(1+a^2)}{1-(ab)^3}\,+\,\frac{a^2b^2(1+ab^3)(1+a^3b)}{1-(ab)^5}\,+\cdots. \tag{3.4}$$

Using (2.5) and (2.6), the above equation (3.4) reduces to the following identities:

$$\frac{af\left(\frac{b}{a},\frac{a}{b}a^4b^4\right)}{f\left(a^3b,ab^3\right)} = \frac{a+b}{1-ab}\,+\,\frac{ab(1+b^2)(1+a^2)}{1-(ab)^3}\,+\,\frac{a^2b^2(1+ab^3)(1+a^3b)}{1-(ab)^5}\,+\cdots. \tag{3.5}$$

$$\frac{2f(a,b)}{f(a^3b,ab^3)} = 1+\frac{a+b}{1-ab}\,+\,\frac{ab(1+b^2)(1+a^2)}{1-(ab)^3}\,+\,\frac{a^2b^2(1+ab^3)(1+a^3b)}{1-(ab)^5}\,+\cdots. \tag{3.6}$$

$$\frac{2f(-a,-b)}{f(a^3b,ab^3)} = -1+\frac{a+b}{1-ab}\,+\,\frac{ab(1+b^2)(1+a^2)}{1-(ab)^3}\,+\,\frac{a^2b^2(1+ab^3)(1+a^3b)}{1-(ab)^5}\,+\cdots. \tag{3.7}$$

When $a=b=q$ in the continued fraction (3.5), we obtain

$$\frac{q\psi(q^8)}{\varphi(q^4)} = \frac{q}{1-q^2}\,+\,\frac{(q+q^3)^2}{1-q^6}\,+\cdots, \tag{3.8}$$

which is known as Ramanujan-Selberg continued fraction. For more details one can see [17, 19].
When $a=q$ and $b=q^2$ in the continued fraction (3.5), we obtain

$$\frac{qf(q,q^{11})}{f(q^5,q^7)} = \frac{q(1+q)}{1-q^3}\,+\,\frac{q^3(1+q^4)(1+q^2)}{1-q^9}\,+\cdots, \tag{3.9}$$

which is a continued fraction of order twelve due to Mahadeva Naika et al. [14].
When $a=q$ and $b=q^3$ in the continued fraction (3.5), we obtain

$$\frac{q^{1/2}f(q,q^7)}{f(q^3,q^5)} = \frac{q^{1/2}(1+q)}{1-q^2}\,+\,\frac{q^2(1+q)(1+q^3)}{1-q^6}\,+\cdots, \tag{3.10}$$

which is known as Ramanujan-Göllnitz-Gordon continued fraction. For more details one can see [9, 13].
When $a=b=q$ in the continued fraction (3.4), we obtain

$$\frac{\varphi(q)-\varphi(-q)}{\varphi(q)+\varphi(-q)} = \frac{2q}{1-q^2}\,+\,\frac{q^2(1+q^2)(1+q^2)}{1-q^6}\,+\,\frac{q^4(1+q^4)(1+q^4)}{1-q^{10}}\,+\cdots, \tag{3.11}$$

which is due to M. S. Mahadeva Naika et al.[18].
When $a=b=q$ in (3.6), we find that

$$\frac{\varphi(q^4)}{2\varphi(q)} = \frac{1}{1}\,+\,\frac{2q}{1-q^2}\,+\,\frac{q^2(1+q^2)(1+q^2)}{1-q^6}\,+\,\frac{q^4(1+q^4)(1+q^4)}{1-q^{10}}\,+\cdots, \tag{3.12}$$

which is due to C. Adiga et al.[1]. For more details, one can refer [19].
When $a=q$ and $b=q^5$ in (3.5), we obtain following theorem :

Theorem 3.1.

$$H(q) := \frac{qf(q^4,q^{20})}{f(q^8,q^{16})} = \frac{q(1+q^4)}{1-q^6} \underset{+}{} \frac{q^6(1+q^{10})(1+q^2)}{1-q^{18}} \underset{+}{} \ldots, \; |q|<1, \tag{3.13}$$

which is a continued fraction of order six.
Recently K. R. Vasuki et al.[28] have also studied a continued fraction of order six which is quite different from (3.13).

Theorem 3.2. *We have*

$$\frac{qf(q^4,q^{20})}{f(q^8,q^{16})} = \frac{1-\frac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}}{1+\frac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}}. \tag{3.14}$$

Proof. Putting $a=q$ and $b=q^5$ in the left hand side of (3.4), we find that

$$\frac{qf(q^4,q^{20})}{f(q^8,q^{16})} = \frac{f(q,q^5)-f(-q,-q^5)}{f(q,q^5)+f(-q,-q^5)}. \tag{3.15}$$

Using the Jacobi's triple product identity, we deduce that

$$\frac{f(-q,-q^5)}{f(q,q^5)} = \frac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}. \tag{3.16}$$

Using (3.16) in (3.15), we obtain (3.14). □

Remark 3.3. For $q=e^{-\pi\sqrt{n}}$, we have

$$H(e^{-\pi\sqrt{n}}) = \frac{1-\dfrac{g_nG_{9n}}{G_ng_{9n}}}{1+\dfrac{g_nG_{9n}}{G_ng_{9n}}}, \tag{3.17}$$

where

$$G_n := 2^{-\frac{1}{4}}q^{-\frac{1}{24}}\chi(q) \qquad \textit{and} \qquad g_n := 2^{-\frac{1}{4}}q^{-\frac{1}{24}}\chi(-q). \tag{3.18}$$

4 $P-Q$ modular relations between the ratios of theta functions

In this section, we establish several $P-Q$ type modular relation identities using the Ramanujan's modular equations recorded in his notebooks. We use the following notations

$$\mathbb{P}_{k,n} := \left(P^kQ_n^k + \frac{1}{P^kQ_n^k}\right) \tag{4.1}$$

and

$$\mathbb{Q}_{k,n} := \left(\frac{P^k}{Q_n^k} + \frac{Q_n^k}{P^k}\right). \tag{4.2}$$

Lemma 4.1. *If* $P = \dfrac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$, *then*

$$(1-\alpha)^{\frac{1}{4}} = \frac{P^4-1+r}{4P}, \tag{4.3}$$

where $r^2 = P^8+14P^4+1$.

Proof. Using the equations (2.4), (2.1), (2.2) and (2.3), we obtain (4.3). □

Lemma 4.2. *If* $Q_n = \dfrac{\chi(-q^n)\chi(q^{3n})}{\chi(q^n)\chi(-q^{3n})}$, *then*

$$(1-\beta)^{\frac{1}{4}} = \frac{Q_n^4-1+s}{4Q_n}, \tag{4.4}$$

where $s^2 = Q_n^8+14Q_n^4+1$ *and* β *is of degree* n *over* α.

Proof. Using the equations (2.4), (2.1) , (2.2) and (2.3), we obtain (4.4). □

Theorem 4.3. *If* $P = \dfrac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$ *and* $Q_2 = \dfrac{\chi(-q^2)\chi(q^6)}{\chi(q^2)\chi(-q^6)}$*, then*

$$P^3Q_2^4 + P^4Q_2^2 + PQ_2^4 - 2P^2Q_2^2 - P^3 + Q_2^2 - P = 0. \tag{4.5}$$

Proof. Using the equations (4.3) and (4.4) in the equation (2.8), we find that

$$\begin{aligned}
&(P^8Q_2^8 + P^8Q_2^4s + P^4Q_2^8r + 6P^8Q_2^4 + 6P^4Q_2^8 + P^4Q_2^4rs + 8P^2Q_2^8 - P^8s\\
&+ 6P^4Q_2^4r + 6P^4Q_2^4s - Q_2^8r + P^8 + 36P^4Q_2^4 + Q_2^8 - 32P^5Q_2^2 + 8P^2Q_2^4s\\
&- P^4rs + 48P^2Q_2^4 - Q_2^4rs + P^4r - 6P^4s - 6Q_2^4r + Q_2^4s + 6P^4 - 32PQ_2^2r\\
&+ 6Q_2^4 - 8P^2s + 32PQ_2^2 + 8P^2 + rs - r - s + 1)(P^8Q_2^8 + P^8Q_2^4s + P^4Q_2^8r\\
&+ 6P^8Q_2^4 + 6P^4Q_2^8 + P^4Q_2^4rs + 8P^2Q_2^8 - P^8s + 6P^4Q_2^4r + 6P^4Q_2^4s - Q_2^8r\\
&+ P^8 + 36P^4Q_2^4 + Q_2^8 + 32P^5Q_2^2 + 8P^2Q_2^4s - P^4rs + 48P^2Q_2^4 - Q_2^4rs\\
&+ P^4r - 6P^4s - 6Q_2^4r + Q_2^4s + 6P^4 + 32PQ_2^2r + 6Q_2^4 - 8P^2s\\
&- 32PQ_2^2 + 8P^2 + rs - r - s + 1) = 0.
\end{aligned} \tag{4.6}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas the other factor is not zero in this neighborhood. By the Identity Theorem the first factor vanishes identically.

$$\begin{aligned}
&P^8Q_2^8 + P^8Q_2^4s + P^4Q_2^8r + 6P^8Q_2^4 + 6P^4Q_2^8 + P^4Q_2^4rs + 8P^2Q_2^8 - P^8s\\
&+ 6P^4Q_2^4r + 6P^4Q_2^4s - Q_2^8r + P^8 + 36P^4Q_2^4 + Q_2^8 - 32P^5Q_2^2 + 8P^2Q_2^4s\\
&- P^4rs + 48P^2Q_2^4 - Q_2^4rs + P^4r - 6P^4s - 6Q_2^4r + Q_2^4s + 6P^4 - 32PQ_2^2r\\
&+ 6Q_2^4 - 8P^2s + 32PQ_2^2 + 8P^2 + rs - r - s + 1 = 0.
\end{aligned} \tag{4.7}$$

Collecting the terms containing r on one side of the equation (4.7) and then squaring both sides, we find that

$$\begin{aligned}
&P^8Q_2^{16} + 4P^6Q_2^{16} + P^8Q_2^{12}s + 12P^8Q_2^{12} + 6P^4Q_2^{16} + 4P^6Q_2^{12}s + 48P^6Q_2^{12}\\
&+ 4P^2Q_2^{16} + 5P^8Q_2^8s + 16P^7Q_2^{10} + 6P^4Q_2^{12}s + 6P^8Q_2^8 + 72P^4Q_2^{12} + Q_2^{16}\\
&+ 20P^6Q_2^8s - 16P^5Q_2^{10} + 4P^2Q_2^{12}s + 16P^7Q_2^6s + 24P^6Q_2^8 + 48P^2Q_2^{12}\\
&- 5P^8Q_2^4s + 96P^7Q_2^6 + 30P^4Q_2^8s - 16P^3Q_2^{10} + Q_2^{12}s + 12P^8Q_2^4 - 16P^5Q_2^6s\\
&+ 36P^4Q_2^8 + 12Q_2^{12} - 20P^6Q_2^4s - 96P^5Q_2^6 + 20P^2Q_2^8s + 16PQ_2^{10} - 6P^4s\\
&+ 48P^6Q_2^4 - 16P^3Q_2^6s + 24P^2Q_2^8 - P^8s + 16P^7Q_2^2 - 30P^4Q_2^4s - 96P^3Q_2^6\\
&+ 5Q_2^8s + P^8 + 16P^5Q_2^2s - 440P^4Q_2^4 + 16PQ_2^6s + 6Q_2^8 - 4P^6s - 16P^5Q_2^2\\
&- 20P^2Q_2^4s + 96PQ_2^6 + 4P^6 + 16P^3Q_2^2s + 48P^2Q_2^4 - 16P^3Q_2^2 - 16P^7Q_2^2s\\
&- 5Q_2^4s + 6P^4 - 16PQ_2^2s + 12Q_2^4 - 4P^2s + 16PQ_2^2 + 4P^2 - s + 1 = 0.
\end{aligned} \tag{4.8}$$

Collecting the terms containing s on one side of the equation (4.8) and squaring both sides, we find that

$$\begin{aligned}
&(P^3Q_2^4 + P^4Q_2^2 + PQ_2^4 - 2P^2Q_2^2 - P^3 + Q_2^2 - P)(P^{12}Q_2^{10} - P^9Q_2^{12} + 3P^{10}Q_2^{10}\\
&+ 7P^{11}Q_2^8 - 3P^7Q_2^{12} - P^8Q_2^{10} + 14P^9Q_2^8 - 3P^5Q_2^{12} + 22P^{10}Q_2^6 - 6P^6Q_2^{10}\\
&+ P^{11}Q_2^4 - 9P^7Q_2^8 - P^3Q_2^{12} - P^4Q_2^{10} + 26P^9Q_2^4 - 9P^5Q_2^8 + P^{10}Q_2^2 - 44P^6Q_2^6\\
&+ 3P^2Q_2^{10} - 39P^7Q_2^4 + 14P^3Q_2^8 + 16P^8Q_2^2 + Q_2^{10} + P^9 - 39P^5Q_2^4 + 7PQ_2^8 + P^3\\
&- 34P^6Q_2^2 + 22P^2Q_2^6 + 3P^7 + 26P^3Q_2^4 + 16P^4Q_2^2 + 3P^5 + PQ_2^4 + P^2Q_2^2) = 0.
\end{aligned} \tag{4.9}$$

By examining the behavior of the above factors near $q=0$, we can find a neighborhood about the origin, where the first factor is zero; whereas the other factor is not zero in this neighborhood. By the Identity Theorem the first factor vanishes identically. This completes the proof. □

Theorem 4.4. *If* $P=\dfrac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$ *and* $Q_3=\dfrac{\chi(-q^3)\chi(q^9)}{\chi(q^3)\chi(-q^9)}$, *then*

$$\mathbb{P}_{2,3}+2\mathbb{P}_{1,3}-4=\mathbb{Q}_{2,3}\left(\mathbb{P}_{1,3}-1\right). \tag{4.10}$$

Proof. The equation (2.9) can be written as

$$\alpha\beta=\left(1-((1-\alpha)(1-\beta))^{1/4}\right)^4. \tag{4.11}$$

Using (4.3) and (4.4) in (4.11), we deduce that

$$\begin{aligned}
&-8P^{13}Q_3^{13}-8P^{13}Q_3^9s-8P^9Q_3^{13}r-72P^{13}Q_3^9-72P^9Q_3^{13}+16P^{16}Q_3^4\\
&+48P^{10}Q_3^{10}-8P^9Q_3^9rs+16P^4Q_3^{16}-16P^{13}Q_3^5s-72P^9Q_3^9r-72P^9Q_3^9s\\
&-16P^5Q_3^{13}r+72P^{13}Q_3^5-648P^9Q_3^9+72P^5Q_3^{13}+16P^{12}Q_3^4r+48P^{10}Q_3^6s\\
&+48P^6Q_3^{10}r+16P^4Q_3^{12}s+192P^{12}Q_3^4+288P^{10}Q_3^6-16P^9Q_3^5rs+288P^6Q_3^{10}\\
&-16P^5Q_3^9rs+192P^4Q_3^{12}-8P^{13}Q_3s+72P^9Q_3^5r-144P^9Q_3^5s-144P^5Q_3^9r\\
&+72P^5Q_3^9s-8PQ_3^{13}r+8P^{13}Q_3+648P^9Q_3^5-128P^7Q_3^7+48P^6Q_3^6rs\\
&+648P^5Q_3^9+8PQ_3^{13}-48P^{10}Q_3^2s+80P^8Q_3^4r+288P^6Q_3^6r+288P^6Q_3^6s\\
&+80P^4Q_3^8s-48P^2Q_3^{10}r+48P^{10}Q_3^2-8P^9Q_3rs+96P^8Q_3^4+1728P^6Q_3^6\\
&-32P^5Q_3^5rs+96P^4Q_3^8+48P^2Q_3^{10}-8PQ_3^9rs+8P^9Q_3r-72P^9Q_3s+72PQ_3^9\\
&-128P^7Q_3^3s+144P^5Q_3^5r+144P^5Q_3^5s-128P^3Q_3^7r-72PQ_3^9r+8PQ_3^9s\\
&+72P^9Q_3+128P^7Q_3^3-48P^6Q_3^2rs-648P^5Q_3^5+128P^3Q_3^7-48P^2Q_3^6rs\\
&+48P^6Q_3^2r-288P^6Q_3^2s-80P^4Q_3^4r-80P^4Q_3^4s-288P^2Q_3^6r+48P^2Q_3^6s\\
&+288P^6Q_3^2-16P^5Q_3rs+384P^4Q_3^4-128P^3Q_3^3rs+288P^2Q_3^6-16PQ_3^5rs\\
&+16P^5Q_3r+72P^5Q_3s+128P^3Q_3^3r+128P^3Q_3^3s+72PQ_3^5r+16PQ_3^5s\\
&-72P^5Q_3-128P^3Q_3^3+48P^2Q_3^2rs-72PQ_3^5-16P^4s-48P^2Q_3^2r-8PQ_3\\
&-16Q_3^4r+16P^4+48P^2Q_3^2-8PQ_3rs+16Q_3^4+8PQ_3r+8PQ_3s-48P^2Q_3^2s.
\end{aligned} \tag{4.12}$$

Collecting the terms containing r on one side of the equation (4.12) and squaring both sides, we find that

$$\begin{aligned}
&(-P^5Q_3^5+P^6Q_3^2-2P^4Q_3^4+P^2Q_3^6-P^5Q_3+4P^3Q_3^3-PQ_3^5+P^4-2P^2Q_3^2\\
&+Q_3^4-PQ_3)(P^7Q_3^{23}+P^8Q_3^{20}-3P^6Q_3^{22}+P^4Q_3^{24}+P^7Q_3^{19}s+P^9Q_3^{17}\\
&+15P^7Q_3^{19}+13P^5Q_3^{21}+3P^3Q_3^{23}+PQ_3^{25}+P^8Q_3^{16}s-3P^6Q_3^{18}s+P^4Q_3^{20}s\\
&+12P^8Q_3^{16}-29P^6Q_3^{18}+33P^4Q_3^{20}+7P^2Q_3^{22}+P^9Q_3^{13}s+8P^7Q_3^{15}s\\
&+13P^5Q_3^{17}s+3P^3Q_3^{19}s+PQ_3^{21}s+12P^9Q_3^{13}+42P^7Q_3^{15}+261P^5Q_3^{17}\\
&+85P^3Q_3^{19}+21PQ_3^{21}+5P^8Q_3^{12}s-8P^6Q_3^{14}s+26P^4Q_3^{16}s+7P^2Q_3^{18}s\\
&+6P^8Q_3^{12}+226P^6Q_3^{14}+354P^4Q_3^{16}+129P^2Q_3^{18}+5P^9Q_3^9s+10P^7Q_3^{11}s\\
&+170P^5Q_3^{13}s+64P^3Q_3^{15}s+14PQ_3^{17}s+6P^9Q_3^9+190P^7Q_3^{11}+1242P^5Q_3^{13}\\
&+670P^3Q_3^{15}+105PQ_3^{17}-5P^8Q_3^8s+210P^6Q_3^{10}s+196P^4Q_3^{12}s+80P^2Q_3^{14}s\\
&+76P^8Q_3^8+1350P^6Q_3^{10}+1258P^4Q_3^{12}+486P^2Q_3^{14}+Q_3^{16}-5P^9Q_3^5s
\end{aligned}$$

$$
\begin{aligned}
&+144P^7Q_3^7s+364P^5Q_3^9s+294P^3Q_3^{11}s+31PQ_3^{13}s+12P^9Q_3^5+549P^7Q_3^7\\
&+562P^5Q_3^9+1002P^3Q_3^{11}+22PQ_3^{13}+63P^8Q_3^4s+192P^6Q_3^6s+342P^4Q_3^8s\\
&+94P^2Q_3^{10}s+Q_3^{12}s-63P^8Q_3^4-591P^6Q_3^6+477P^4Q_3^8-158P^2Q_3^{10}+12Q_3^{12}\\
&-P^9Q_3s-99P^7Q_3^3s-90P^5Q_3^5s-24P^3Q_3^7s-27PQ_3^9s+P^9Q_3+99P^7Q_3^3\\
&+153P^5Q_3^5-753P^3Q_3^7-9PQ_3^9+57P^6Q_3^2s-117P^4Q_3^4s-72P^2Q_3^6s\\
&-57P^6Q_3^2+117P^4Q_3^4+387P^2Q_3^6+6Q_3^8-9P^5Q_3s+111P^3Q_3^3s\\
&+9P^5Q_3-111P^3Q_3^3-99PQ_3^5-45P^2Q_3^2s-5Q_3^4s+45P^2Q_3^2+12Q_3^4\\
&+36PQ_3^5s+5Q_3^8s+9PQ_3s-9PQ_3-s+1)=0.
\end{aligned}
\tag{4.13}
$$

By examining the behavior of the above factors near $q=0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factor is not zero in this neighborhood. By the Identity Theorem the first factor vanishes identically. This completes the proof. □

Theorem 4.5. *If* $P=\dfrac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$ *and* $Q_4=\dfrac{\chi(-q^4)\chi(q^{12})}{\chi(q^4)\chi(-q^{12})}$, *then*

$$
\begin{aligned}
&P^7Q_4^8-4P^6Q_4^8+7P^5Q_4^8+P^8Q_4^4-8P^4Q_4^8+7P^3Q_4^8+4P^6Q_4^4-4P^2Q_4^8-P\\
&+22P^4Q_4^4-P^7-4P^6+4P^2Q_4^4-7P^5-8P^4+Q_4^4-7P^3-4P^2+PQ_4^8=0.
\end{aligned}
\tag{4.14}
$$

Proof. Proof of the equation (4.14) is similar to the proof of the equation (4.5) except that in the place of the equation (2.9); the equation (2.10) is used, hence we omit the details. □

Theorem 4.6. *If* $P=\dfrac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$ *and* $Q_5=\dfrac{\chi(-q^5)\chi(q^{15})}{\chi(q^5)\chi(-q^{15})}$, *then*

$$
\mathbb{P}_{2,5}=\mathbb{Q}_{3,5}-5\mathbb{Q}_{2,5}+15\mathbb{Q}_{1,5}-20. \tag{4.15}
$$

Proof. Proof of the equation (4.15) is similar to the proof of the equation (4.5) except that in the place of the equation (2.9); the equation (2.11) is used, hence we omit the details. □

Theorem 4.7. *If* $P=\dfrac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$ *and* $Q_7=\dfrac{\chi(-q^7)\chi(q^{21})}{\chi(q^7)\chi(-q^{21})}$, *then*

$$
\mathbb{P}_{3,7}-14\mathbb{P}_{2,7}+49\mathbb{P}_{1,7}-70=\mathbb{Q}_{4,7}+7\mathbb{Q}_{2,7}(2-\mathbb{P}_{1,7}). \tag{4.16}
$$

Proof. Proof of the equation (4.16) is similar to the proof of the equation (4.5) except that in the place of the equation (2.9); the equation (2.12) is used, hence we omit the details. □

Theorem 4.8. *If* $P=\dfrac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$ *and* $Q_9=\dfrac{\chi(-q^9)\chi(q^{27})}{\chi(q^9)\chi(-q^{27})}$, *then*

$$
\mathbb{P}_{2,9}(1-\mathbb{Q}_{1,9})+\mathbb{Q}_{2,9}+2\mathbb{Q}_{1,9}=4. \tag{4.17}
$$

Proof. Proof of the equation (4.17) is similar to the proof of the equation (4.5) except that in the place of the equation (2.9); the equation (2.17) is used, hence we omit the details. □

Theorem 4.9. *If* $P=\dfrac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$ *and* $Q_{11}=\dfrac{\chi(-q^{11})\chi(q^{33})}{\chi(q^{11})\chi(-q^{33})}$, *then*

$$
\begin{aligned}
&\mathbb{P}_{5,11}+22\mathbb{P}_{4,11}+55\mathbb{P}_{3,11}-220\mathbb{P}_{2,11}+616\mathbb{P}_{1,11}-704\\
&=\mathbb{Q}_{6,11}-22\mathbb{Q}_{4,11}+11\mathbb{Q}_{2,11}(25-\mathbb{P}_{3,11}+10\mathbb{P}_{2,11}-15\mathbb{P}_{1,11}).
\end{aligned}
\tag{4.18}
$$

Proof. Proof of the equation (4.18) is similar to the proof of the equation (4.5) except that in the place of the equation (2.9); the equation (2.13) is used, hence we omit the details. □

Theorem 4.10. *If* $P = \frac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$ *and* $Q_{13} = \frac{\chi(-q^{13})\chi(q^{39})}{\chi(q^{13})\chi(-q^{39})}$, *then*

$$\begin{aligned}&\mathbb{P}_{6,13} + 338\mathbb{P}_{4,13} + 1989\mathbb{P}_{2,13} + 3328 = \mathbb{Q}_{7,13} + 13\mathbb{Q}_{6,13} + 52\mathbb{Q}_{5,13}\\&- 13\mathbb{Q}_{4,13}(\mathbb{P}_{2,13} - 2) - 26\mathbb{Q}_{3,13}(5 + 11\mathbb{P}_{2,13}) - 13\mathbb{Q}_{2,13}(\mathbb{P}_{4,13}\\&+ 4\mathbb{P}_{2,13} - 1) + 13\mathbb{Q}_{1,13}(155 + 86\mathbb{P}_{2,13} - 10\mathbb{P}_{4,13}).\end{aligned} \tag{4.19}$$

Proof. Proof of the equation (4.19) is similar to the proof of the equation (4.5) except that in the place of the equation (2.9); the equation (2.18) is used, hence we omit the details. □

Theorem 4.11. *If* $P = \frac{\chi(-q)\chi(q^3)}{\chi(q)\chi(-q^3)}$ *and* $Q_{17} = \frac{\chi(-q^{17})\chi(q^{51})}{\chi(q^{17})\chi(-q^{51})}$, *then*

$$\begin{aligned}&\mathbb{P}_{8,17} + 17\mathbb{P}_{6,17}(54 + \mathbb{Q}_{2,17} + 24\mathbb{Q}_{1,17}) - \mathbb{P}_{4,17}(306 + 51\mathbb{Q}_{4,17} - 1105\mathbb{Q}_{3,17}\\&- 1020\mathbb{Q}_{2,17} - 2703\mathbb{Q}_{1,17}) + \mathbb{P}_{2,17}(21318 + 408\mathbb{Q}_{1,17} + 8330\mathbb{Q}_{2,17} + 1054\mathbb{Q}_{3,17}\\&+ 714\mathbb{Q}_{4,17} + 578\mathbb{Q}_{5,17} - 51\mathbb{Q}_{6,17}) + 31858 = 36227\mathbb{Q}_{1,17} - 10812\mathbb{Q}_{2,17}\\&+ 8959\mathbb{Q}_{3,17} - 578\mathbb{Q}_{4,17} + 1819\mathbb{Q}_{5,17} - 204\mathbb{Q}_{6,17} + 34\mathbb{Q}_{7,17} + \mathbb{Q}_{9,17}.\end{aligned} \tag{4.20}$$

Proof. Proof of the equation (4.20) is similar to the proof of the equation (4.5) except that in the place of the equation (2.9); the equation (2.19) is used, hence we omit the details. □

5 Modular equations between a continued fractions $H(q)$ and $H(q^n)$

In this section, we prove following modular relations connecting the continued fraction $H(q)$ with $H(q^n)$ for $n = 2, 3, 4, 5, 7, 9, 11, 13$ and 17 using the identities established in the previous section. We use the following notations

$$\mathbb{U}_n := \left(u^n v^n + \frac{1}{u^n v^n}\right) \tag{5.1}$$

and

$$\mathbb{V}_n := \left(\frac{u^n}{v^n} + \frac{v^n}{u^n}\right). \tag{5.2}$$

Theorem 5.1. *If* $u = H(q)$ *and* $v = H(q^2)$, *then*

$$u^4v^3 + u^2v^4 + u^4v - 2u^2v^2 - v^3 + u^2 - v = 0. \tag{5.3}$$

Proof. Using the equation (3.13) in (3.14) , we find that

$$P = \frac{1-u}{1+u} \tag{5.4}$$

and

$$Q_2 = \frac{1-v}{1+v}. \tag{5.5}$$

Using the equations (5.4) and (5.5) in the equation (4.5), we obtain (5.3). □

Theorem 5.2. *If* $u = H(q)$ *and* $v = H(q^3)$, *then*

$$\mathbb{U}_2 + 2\mathbb{U}_1 + \mathbb{V}_2(1 + \mathbb{U}_1) = 4. \tag{5.6}$$

Proof. Using the equation (3.13) in (3.14), we find that

$$P = \frac{1-u}{1+u} \tag{5.7}$$

and

$$Q_3 = \frac{1-v}{1+v}. \tag{5.8}$$

Using the equations (5.7) and (5.8) in the equation (4.10), we obtain (5.6). □

Theorem 5.3. *If $u = H(q)$ and $v = H(q^4)$, then*

$$u^8v^7 - 4u^8v^6 + 7u^8v^5 - 8u^8v^4 + u^4v^8 + 7u^8v^3 - 4u^8v^2 + 4u^4v^6 + u^8v + 22u^4v^4 - v^7 + 4u^4v^2 - 4v^6 - 7v^5 + u^4 - 8v^4 - 7v^3 - 4v^2 - v = 0. \tag{5.9}$$

Proof. Proof of the equation (5.9) is similar to the proof of the equation (5.6); except that in the place of the equation (4.5), the equation (4.14) is used. □

Theorem 5.4. *If $u = H(q)$ and $v = H(q^5)$, then*

$$\mathbb{U}_2 = \mathbb{V}_3 - 5\mathbb{V}_2 + 15\mathbb{V}_1 - 20. \tag{5.10}$$

Proof. Proof of the equation (5.10) is similar to the proof of the equation (5.6); except that in the place of the equation (4.5), the equation (4.15) is used. □

Theorem 5.5. *If $u = H(q)$ and $v = H(q^7)$, then*

$$\mathbb{U}_3 - 14\mathbb{U}_2 + 49\mathbb{U}_1 - 70 = \mathbb{V}_4 + 7\mathbb{V}_2(2 - \mathbb{U}_1). \tag{5.11}$$

Proof. Proof of the equation (5.11) is similar to the proof of the equation (5.6); except that in the place of the equation (4.5), the equation (4.16) is used. □

Theorem 5.6. *If $u = H(q)$ and $v = H(q^9)$, then*

$$\mathbb{U}_2 + \mathbb{U}_1(\mathbb{V}_2 - 2) + \mathbb{V}_2 = 4. \tag{5.12}$$

Proof. Proof of the equation (5.12) is similar to the proof of the equation (5.6); except that in the place of the equation (4.5), the equation (4.17) is used. □

Theorem 5.7. *If $u = H(q)$ and $v = H(q^{11})$, then*

$$\begin{aligned}&\mathbb{U}_5 + 22\mathbb{U}_4 + 55\mathbb{U}_3 - 220\mathbb{U}_2 + 616\mathbb{U}_1 - 704\\ &= \mathbb{V}_6 - 22\mathbb{V}_4 + 11\mathbb{V}_2(25 - 15\mathbb{U}_1 + 10\mathbb{U}_2 - \mathbb{U}_3).\end{aligned} \tag{5.13}$$

Proof. Proof of the equation (5.13) is similar to the proof of the equation (5.6); except that in the place equation (4.5), the equation (4.18) is used. □

Theorem 5.8. *If $u = H(q)$ and $v = H(q^{13})$, then*

$$\begin{aligned}&\mathbb{U}_6 + 338\mathbb{U}_4 + 1989\mathbb{U}_2 + 3328 = \mathbb{V}_7 + 13\mathbb{V}_6 + 52\mathbb{V}_5 + 13\mathbb{V}_4(2 - \mathbb{U}_2)\\ &- 22\mathbb{V}_3(11 + 5\mathbb{U}_2) + 13\mathbb{V}_2(1 - 4\mathbb{U}_2 - \mathbb{U}_4) + 13\mathbb{V}_1(15 + 13\mathbb{U}_4 + 86\mathbb{U}_2).\end{aligned} \tag{5.14}$$

Proof. Proof of the equation (5.14) is similar to the proof of the equation (5.6); except that in the place of the equation (4.5), the equation (4.19) is used. □

Theorem 5.9. *If $u = H(q)$ and $v = H(q^{17})$, then*

$$\begin{aligned}&\mathbb{V}_9 + 34\mathbb{V}_7 - 204\mathbb{V}_6 + 1819\mathbb{V}_5 - 578\mathbb{V}_4 + 8959\mathbb{V}_3 - 10812\mathbb{V}_2 + 36227\mathbb{V}_1 = \mathbb{U}_8\\ &- 17\mathbb{U}_6(54 - \mathbb{V}_2 - 24\mathbb{V}_1) + \mathbb{U}_4(51\mathbb{V}_4 - 1105\mathbb{V}_3 - 1020\mathbb{V}_2 - 2703\mathbb{V}_1 - 306)\\ &+ \mathbb{U}_2(21318 - 7480\mathbb{V}_1 + 8330\mathbb{V}_2 + 1054\mathbb{V}_3 + 714\mathbb{V}_4 + 578\mathbb{V}_5 - 51\mathbb{V}_6) + 31858.\end{aligned} \tag{5.15}$$

Proof. Proof of the equation (5.15) is similar to the proof of the equation (5.6); except that in the place of the equation (4.5), the equation (4.20) is used. □

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