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Evaluation of some definite integrals of Ramanujan, using hypergeometric approach

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Abstract In this paper, we evaluate two definite integrals (with suitable convergence conditions) of Srinivasa Ramanujan, using Laplace transform, hypergeometric summation theorem for ${}_4F_3(-1)$, properties of Pochhammer's symbol and Gamma function.

1 Introduction and Preliminaries

A natural generalization of Gauss function ${}_2F_1$ is the general hypergeometric series ${}_pF_q$ [3, p.42, Eq.(1)] with p numerator parameters $a_1, ..., a_p$, and q denominator parameters $b_1, ..., b_q$ is defined by

$${}_pF_q\left(\begin{array}{ll} a_1, ..., a_p\,; & \\ b_1, ..., b_q\,; & \end{array} z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{z^n}{n!}\,, \tag{1.1}$$

where $a_j \in \mathbb{C}$ $(j = 1, ..., p)$ and $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, ..., q)$ $\left(\mathbb{Z}_0^- := \mathbb{Z} \cup \{0\} = \{0, -1, -2, ...\}\right)$. and $(\,p,\ q \in \mathbb{N} := \{1, 2, ...\})$. Also the Pochhammer symbol $(\lambda)_\upsilon$ $(\lambda \in \mathbb{C}, \upsilon \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is defined, in general, by

$$(\lambda)_\upsilon := \frac{\Gamma(\lambda + \upsilon)}{\Gamma(\lambda)} = \begin{cases} 1, & (\upsilon = 0\,;\ \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda + 1)...(\lambda + n - 1), & (\upsilon = n \in \mathbb{N}\,;\ \lambda \in \mathbb{C}). \end{cases} \tag{1.2}$$

The ${}_pF_q$ series in the eq.(1.1) is convergent for $|z| < \infty$ if $p \leq q$, and for $|z| < 1$ if$p = q + 1$. Furthermore, if we set

$$\omega = \left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right), \tag{1.3}$$

it is known that the ${}_pF_q$ series, with $p = q + 1$, is
i. absolutely convergent for $|z| = 1$, if $Re(\omega) > 0$,
ii. conditionally convergent for $|z| = 1, z \neq 1$, if $-1 < Re(\omega) \leq 0$.
Also binomial expansion is given by

$$(1 - z)^{-a} = {}_1F_0\,(a\,;\ -\ ;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (z)^n \quad ; \quad |z| < 1. \tag{1.4}$$

In [1, p.28, Eq.(4.4.3)], the classical summation theorem for hypergeometric series ${}_4F_3(-1)$ is given by

$${}_4F_3\left(\begin{array}{l} a,\ \ 1 + \frac{a}{2},\ \ b,\ \ c \qquad\quad ; \\ \frac{a}{2},\ \ 1 + a - b,\ \ 1 + a - c\,; \end{array} \ -1\right) = \frac{\Gamma(1 + a - b)\Gamma(1 + a - c)}{\Gamma(1 + a)\Gamma(1 + a - b - c)}\,, \tag{1.5}$$

subject to the following convergence conditions

$$Re(a - 2b - 2c) > -2\ \ ;\ \ \frac{a}{2},\ 1 + a - b,\ 1 + a - c \neq 0, -1, -2, -3, ... \tag{1.6}$$

Recurrence relation:

$$\Gamma(z+1) = z\Gamma(z). \tag{1.7}$$

Relation between circular function and Gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \ ; \ z \neq 0, \pm 1, \pm 2, \pm 3, ... \tag{1.8}$$

2 Ramanujan's Integrals

Each of the integral representations [2, p.11, Eq.(1.5.1(27)), Eq.(1.5.1(28))] holds true under certain conditions:

$$\mathbf{I}_1 = \int_0^\infty \cos(2zt) sech(\pi t) dt = \frac{1}{2} sech(z) \ , \qquad |\ (Im(z)\ | < \frac{\pi}{2}. \tag{2.1}$$

$$\mathbf{I}_2 = \int_0^\infty \cosh(2zt) sech(\pi t) dt = \frac{1}{2} sec(z) \ , \qquad |\ (Re(z)\ | < \frac{\pi}{2}, \tag{2.2}$$

where $z \in \mathbb{C}$. Formula (2.1) is known as Ramanujan's integral.

Proof of integral $\mathbf{I}_1$: In order to prove the integral (2.1), we begin with L.H.S.

$$\mathbf{I}_1 = \int_0^\infty \left(\frac{e^{2izt} + e^{-2izt}}{e^{\pi t} + e^{-\pi t}} \right) dt. \tag{2.3}$$

Applying binomial expansion (1.4), we obtain

$$\mathbf{I}_1 = \int_0^\infty \left(e^{-\pi t + 2izt} + e^{-\pi t - 2izt} \right) {}_1F_0 \left(1; -; -e^{-2\pi\ t} \right) dt \tag{2.4}$$

$$= \int_0^\infty \left(e^{-\pi t + 2izt} + e^{-\pi t - 2izt} \right) \sum_{r=0}^\infty \frac{(1)_r}{r!} (-e^{-2\pi t})^r dt. \tag{2.5}$$

Change the order of integration and the summation, we get

$$\mathbf{I}_1 = \sum_{r=0}^\infty \frac{(1)_r (-1)^r}{r!} \int_0^\infty \left(e^{-(\pi - 2iz + 2\pi r)t} + e^{-(\pi + 2iz + 2\pi r)t} \right) dt. \tag{2.6}$$

Making use of the following Laplace integral formula:

$$L[1;p] = \int_0^\infty e^{-pt} dt = \frac{1}{p}, \tag{2.7}$$

provided

$$Re(p) > 0 \ , \tag{2.8}$$

in the above eq.(2.6), we get

$$\mathbf{I}_1 = \sum_{r=0}^\infty \frac{(1)_r (-1)^r}{r!} \left(\frac{1}{\pi - 2iz + 2\pi r} + \frac{1}{\pi + i2z + 2\pi r} \right). \tag{2.9}$$

Using some algebraic properties of Pochhammer's symbols in the eq. (2.9), we get

$$\mathbf{I}_1 = \left(\frac{2\pi}{\pi^2 + 4z^2} \right) \sum_{r=0}^\infty \frac{(1)_r (\frac{3}{2})_r (\frac{\pi - 2iz}{2\pi})_r (\frac{\pi + 2iz}{2\pi})_r (-1)^r}{(\frac{1}{2})_r (\frac{3\pi - 2iz}{2\pi})_r (\frac{3\pi + 2iz}{2\pi})_r r!} \tag{2.10}$$

$$= \left(\frac{2\pi}{\pi^2 + 4z^2} \right) {}_4F_3 \left(\begin{matrix} 1, \frac{3}{2}, \frac{\pi - 2iz}{2\pi}, \frac{\pi + 2iz}{2\pi}; \\ \frac{1}{2}, \frac{3\pi - 2iz}{2\pi}, \frac{3\pi + 2iz}{2\pi}; \end{matrix} \ -1 \right). \tag{2.11}$$

We have checked convergence conditions of ${}_4F_3(-1)$ in the above eq. (2.11), with the help of eq. (1.6). So using hypergeometric summation theorem (1.5), which yield

$$\mathbf{I}_1 = \left(\frac{2\pi}{\pi^2 + 4z^2}\right)\Gamma\left(\frac{3}{2} - \frac{iz}{\pi}\right)\Gamma\left(\frac{3}{2} + \frac{iz}{\pi}\right). \tag{2.12}$$

Finally, applying properties (1.7) and (1.8) of Gamma function in the above eq. (2.12), we get R.H.S. of the eq. (2.1).

Convergence conditions of the two Laplace integrals of eq. (2.6), using eq. (2.8):
Suppose $z = x + iy$ where x and y are real numbers, then

$$Re(\pi + 2iz + 2\pi r) > 0 \;\; and \;\; Re(\pi - 2iz + 2\pi r) > 0. \tag{2.13}$$

OR

$$(\pi + 2y + 2\pi r) > 0 \quad and \quad (\pi - 2y + 2\pi r) > 0, \tag{2.14}$$

where $r = 0, 1, 2, 3, ...$
Therefore

$$(\pi + 2y) > 0 \quad and \quad (\pi - 2y) > 0.$$

OR

$$\frac{-\pi}{2} < y \quad and \quad y < \frac{\pi}{2}.$$

OR

$$\frac{-\pi}{2} < y < \frac{\pi}{2},$$

$$|y| < \frac{\pi}{2}.$$

OR

$$|Im(z)| < \frac{\pi}{2}. \tag{2.15}$$

Similarly, proof of I_2 is much akin that of I_1, which we have already presented in a detailed manner.

3 Conclusion

Thus certain Ramanujan's integrals and other definite integrals, which may be different from those of presented here, can also be evaluated in a similar way. Therefore, the results presented in this paper can be expressed in terms of hypergeometric functions (see eq.2.11).

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