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ON DECOMPOSITION OF THE REAL LINE IN TERMS OF RATIO SETS

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Abstract An attempt has been made in this paper to decompose the real line $\mathbb{R}$ into two complementary sets whose ratio sets have empty interior. Another decomposition of the real line has been made into uncountable pairwise disjoint sets $\{X_\alpha\}_{\alpha<\omega_c}$, where ω_c is the smallest uncountable ordinal, i.e. $\bigcup\limits_{\alpha<\omega_c} X_\alpha = \mathbb{R}$ and $X_\alpha \cap X_\beta = \emptyset$ for all $\alpha < \omega_c, \beta < \omega_c$ such that ratio sets of X_α $(\alpha < \omega_c)$ have non-empty interior.

1 Introduction

First we recall a classical result on decomposition of real line $\mathbb{R}$ as follows:

Theorem 1.1. *([8] page 4) The real line can be decomposed into two complementary sets A and B such that A is of first category and B is of measure zero.*

Several authors ([5], [6], [9]) generalized the above result in different ways. Also, Miller [7] decomposed the real line in the sense of Difference set, where the Difference set of a linear set A, written as $D(A)$, is defined by $D(A) = \{a - b : a, b \in A\}$.
Motivated by the result of Miller [7], we are interested to decompose the real line in terms of ratio sets. The notion ratio set of linear set was introduced by N.C. Bose Majumder [1] in the following way:

Definition 1.2. The Ratio set of a linear set A of nonzero abscissa denoted by $R(A)$, is defined by $R(A) = \{\frac{a}{b} \text{ or } \frac{b}{a} : a, b \in A\}$.
Also ratio of two linear sets A and B is defined by $R(A, B) = \{\frac{a}{b} : a \in A, b \in B \setminus \{0\}\}$.

Bose Majumder [1] established that ratio set $R(A)$ of a linear set A with nonzero abscissa having positive Lebesgue measure contains an interval with left hand end point 1.

Definition 1.3. ([8]) A set A is said to have the property of Baire if it can be expressed as symmetric difference of an open set and a set of first category.

The category analogue of Bose Majumder's result was established by Ganguly and Basu [3] in the following way:
If A is a subset of nonzero reals with second category having the property of Baire, then the set $R(A, A)$ contains an interval of the form $[1, \xi)$ $(\xi > 1)$.
Bose Majumder's result was improved by Ganguly and Bandopadhyay [2] as follows:
If A is a linear set with positive abscissa having positive Lebesgue measure then there exists an interval I such that $R(A \cap I) = R(I)$.
Now we consider following three classes of subsets of $\mathbb{R}$:

(i) $\mathcal{M}^+$ denotes the collection of all Lebesgue measurable subset of $\mathbb{R}$ with positive measure.

(ii) $\mathcal{B}^+$ denotes the collection of all Baire second category subset of $\mathbb{R}$ with property of Baire.

(iii) $\mathcal{A} = \{A \subseteq \mathbb{R} : R(A)$ has non-empty interior $\}$.

Then by the result of Bose Majumder [1] and the result of Ganguly and Basu [3] the above classes can be summarized by the following formula:

$$\mathcal{M}^+ \cup \mathcal{B}^+ \subset \mathcal{A} \tag{1.1}$$

It is interesting to see that the Ratio set of Cantor ternary set C contains an interval ([4]). Thus

$$C \in \mathcal{A} \text{ but } C \notin \mathcal{M}^+ \bigcup \mathcal{B}^+$$

A linear set B is said to be 'big' in sense of measure if $B \in \mathcal{M}^+$ and B is said to be 'big' in sense of category if $B \in \mathcal{B}^+$. On the other hand a linear set X is said to be 'big' in sense of ratio if $X \in \mathcal{A}$. So, Cantor set C is 'small' both in sense of measure and category but 'big' in sense of ratio.

Our intention in this paper is to decompose the real line into two complementary sets A and B such that both the sets $R(A)$ and $R(B)$ have empty interior. Additionally we show that the real line can be decomposed into uncountable pairwise disjoint sets $\{X_\alpha\}_{\alpha<\omega_c}$, where ω_c is the smallest uncountable ordinal, i.e. $\bigcup\limits_{\alpha<\omega_c} X_\alpha = \mathbb{R}$ and $X_\alpha \cap X_\beta = \emptyset$ for all $\alpha < \omega_c, \beta < \omega_c$ such that ratio sets of X_α $(\alpha < \omega_c)$ have non-empty interior.

2 Results

Theorem 2.1. *There exist two complementary subsets A and B of $\mathbb{R}$ i.e. $\mathbb{R} = A \cup B$, $A \cap B = \emptyset$ such that $R(A)$ and $R(B)$ have empty interior.*

Proof. $P = \{x_i \notin \{0, 1\} : i \in \mathbb{N}$, the set of natural numbers$\}$ be the dense subset of $\mathbb{R}$ such that $x \in P$ implies $\frac{1}{x} \in P$. Clearly such dense set exists. Let $\{y_\alpha : \alpha < \omega_c\}$ be a well ordering of $\mathbb{R}^*$ $(= \mathbb{R} \setminus \{0\})$, where ω_c is the smallest ordinal which is equal to that of $\mathbb{R}$. Now A and B will be constructed by the following way:

Place y_0, the smallest real with respect to our well ordering in A_0. Put all numbers of the form $y_0.x_n$ in B_0 for $n \in \mathbb{N}$. Put all numbers of the form $y_0.x_n.x_m$ in A_0 such that $x_n.x_m \notin P \cup \{1\}$ for $n, m \in \mathbb{N}$. Put all the number of the form $y_0.x_n.x_m.x_p$ in B_0, $(n, m, p \in \mathbb{N})$ such that products of two and more $x_n, (n \in \mathbb{N})$ are not in $P \cup \{1\}$. Continue this process of alternatively putting elements in A_0 and B_0 using the ordinary induction, countably many times.
Clearly A_0 and B_0 are disjoint countable sets. So, there exists a smallest element say y_1 relative to our well ordering that has not been put in either A_0 or B_0. Put y_1 in A_1 with $A_0 \cap A_1 = \emptyset$ and $B_0 \cap A_1 = \emptyset$. Put all numbers of the form $y_1.x_n$ in B_1 for $n \in \mathbb{N}$ with $A_0 \cap B_1 = \emptyset, B_0 \cap B_1 = \emptyset$ and $A_1 \cap B_1 = \emptyset$. Put all numbers of the form $y_1.x_n.x_m$ in A_1 such that $x_n.x_m \notin P \cup \{1\}$ for $n, m \in \mathbb{N}$. Put all numbers of the form $y_1.x_n.x_m.x_p$ in B_1, $(n, m, p \in \mathbb{N})$ such that products of two and more $x_n, (n \in \mathbb{N})$ are not in $P \cup \{1\}$.
Continuing this procedure again by ordinary induction countably many times. Clearly A_0, B_0, A_1 and B_1 are pairwise mutually disjoint countable sets. So, there exist a smallest element y_{α_1} in our well ordering that has not been put in $\bigcup\limits_{\beta<\alpha_1} A_\beta$ or $\bigcup\limits_{\beta<\alpha_1} B_\beta$.

Put y_{α_1} in A_{α_1} with $A_\beta \cap A_{\alpha_1} = \emptyset$, $B_\beta \cap A_{\alpha_1} = \emptyset$, $\beta < \alpha_1$. Put all number of the form $y_{\alpha_1}.x_n$, $(n \in \mathbb{N})$ in B_{α_1} with $A_\beta \cap B_{\alpha_1} = \emptyset$, $B_\beta \cap B_{\alpha_1} = \emptyset$ and $A_{\alpha 1} \cap B_{\alpha 1} = \emptyset$, $\beta < \alpha_1$. Put all numbers of the form $y_{\alpha_1}.x_n.x_m$ in A_{α_1} such that $x_n.x_m \notin P \cup \{1\}$ for $n, m \in \mathbb{N}$. Put all numbers of the form $y_{\alpha_1}.x_n.x_m.x_p$ in B_{α_1},$(n, m, p \in \mathbb{N})$ such that products of two and more $x_n, (n \in \mathbb{N})$ are not in $P \cup \{1\}$.
By using transfinite induction we obtain two sets $A^* = \bigcup\limits_{\alpha<\omega_c} A_\alpha$, $B = \bigcup\limits_{\alpha<\omega_c} B_\alpha$ such that $\mathbb{R}^*$ $= A^* \cup B$. Clearly $A^* \cap B = \emptyset$. Consider $A = A^* \cup \{0\}$. Then $A \cup B = \mathbb{R}$ with $A \cap B = \emptyset$. If $a_1 (= 0)$ and $a_2 (\neq 0)$ are two elements of A, then $\frac{a_1}{a_2} = 0 \notin P$. Consider two distinct non zero

elements a_1 and a_2 in A. If they are both in the same "hierarchy" of our process, that is if

$$\begin{aligned} a_1 &= y_{\alpha_i}.x_{p_1}.x_{p_2}..........x_{p_{n_{(1)}}} \\ a_2 &= y_{\alpha_i}.x_{p_1}.x_{p_2}.....x_{p_{n_{(1)}}}.x_{q_1}.x_{q_2}....x_{q_{n_{(2)}}} \end{aligned}$$

where $n_{(1)}, n_{(2)}$ are both non-negative even integers. Then $\frac{a_2}{a_1} = x_{q_1}.x_{q_2}....x_{q_{n_{(2)}}} \notin P$, where $n_{(2)}$ is non-negative even integers. Since $\frac{a_2}{a_1} \notin P$, therefore $\frac{a_1}{a_2} \notin P$.
If a_1 and a_2 are not in same "hierarchy" in our construction then

$$a_1 = y_{\alpha_i}.x_{p_1}.x_{p_2}..........x_{p_{n_{(1)}}}$$

$$a_2 = y_{\alpha_j}.x_{q_1}.x_{q_2}.........x_{q_{n_{(2)}}}$$

where $n_{(1)}, n_{(2)}$ are both non-negative even integers. Now we shall show that $\frac{a_2}{a_1} \notin P$. If possible let $\frac{a_2}{a_1} = x_i \in P$, $i \in \mathbb{N}$. Then

$$a_2 = a_1.x_i = y_{\alpha_i}.x_{p_1}.x_{p_2}....x_{p_{n_{(1)}}}.x_i \tag{2.1}$$

If x_i, $(i \in \mathbb{N})$ is not reciprocal with each of $x_{p_1}, x_{p_2}, ..., x_{p_{n_{(1)}}}$ then from the relation (2.1), $a_2 \in B$, since n_1 is non-negative even integer, a contradiction.
If x_i, $(i \in \mathbb{N})$ is reciprocal with each of $x_{p_1}, x_{p_2}, ..., x_{p_{n_{(1)}}}$ then from the relation (2.1), $a_2 \in B$, since n_1 is non-negative even integer, a contradiction.
Hence $a_2 = y_{\alpha_j}x_{q_1}x_{q_2}....x_{q_{n_{(2)}}} \in A \cap B$, a contradiction. Similarly $\frac{a_1}{a_2} \notin P$. Therefore $R(A)$ does not contain any interval. A similar argument shows that $R(B)$ contains no interval. Hence the result. □

Remark: From the above theorem both $A, B \notin \mathcal{A}$. So, $\mathbb{R}$ is the union of two disjoint 'small' sets in sense of ratio.

Our next theorem will show that the real line can be expressed as uncountable union of pairwise disjoint 'big' sets in the sense of ratio.

Theorem 2.2. *There exist a pairwise disjoint collection of sets of reals* $\{X_\alpha : \alpha < \omega_c\}$ *such that* $R(X_\alpha) = \mathbb{R}$ *for every* $\alpha < \omega_c$ *and* $\mathbb{R} = \bigcup_{\alpha<\omega_c} X_\alpha$ *(Here* ω_c *denotes the smallest ordinal whose cardinal is equal to that of* $\mathbb{R}$*).*

Proof. We use diagonal method to construct a collection of sets $\{X_\alpha : \alpha < \omega_c\}$ with desire properties. Let $\{y_\alpha : \alpha < \omega_c\}$ be the well ordering of $\mathbb{R}$, where ω_c is the smallest ordinal whose cardinal is equal to that of $\mathbb{R}$.
For $y_1 \in \mathbb{R}$, there exist two reals a_{11}, b_{11} such that $b_{11} = a_{11}y_1$; For $y_2 \in \mathbb{R}$, there exist

$$a_{12}, b_{12} \in \mathbb{R} \setminus \{a_{11}, b_{11}\}$$

such that $b_{12} = a_{12}y_2$. For same y_1 there exist another pair of

$$a_{21}, b_{21} \in \mathbb{R} \setminus \{a_{11}, b_{11}, a_{12}, b_{12}\}$$

such that $b_{21} = a_{21}y_1$. Similarly for the same y_2 there exist

$$a_{22}, b_{22} \in \mathbb{R} \setminus \{a_{11}, b_{11}, a_{12}, b_{12}, a_{21}, b_{21}\}$$

such that $b_{22} = a_{22}y_2$.
Suppose $\alpha < \omega_c$ and all pairs (β_1, β_2), $\beta_1, \beta_2 < \alpha$ we have

$$a_{\beta_1\beta_2}, b_{\beta_1\beta_2} \in \mathbb{R} \setminus (\{a_{\gamma_1\gamma_2}, b_{\gamma_1\gamma_2} : \gamma_1, \gamma_2 < \beta_2\} \cup \{a_{\delta_1\beta_2}, b_{\delta_1\beta_2} : \delta_1 < \beta_1\}) \text{ such that } b_{\beta_1\beta_2} = a_{\beta_1\beta_2}y_{\beta_2}, \text{ for all } \beta_1 < \beta_2$$

and

$$a_{\beta_1\beta_2}, b_{\beta_1\beta_2} \in \mathbb{R} \setminus (\{a_{\gamma_1\gamma_2}, b_{\gamma_1\gamma_2} : \gamma_1, \gamma_2 < \beta_1\} \cup \{a_{\delta_1\beta_1}, b_{\delta_1\beta_1} : \delta_1 < \beta_1\} \cup \{a_{\beta_1\delta_1}, b_{\beta_1\delta_1} : \delta_1 < \beta_2\}) \text{ such that } b_{\beta_1\beta_2} = a_{\beta_1\beta_2}y_{\beta_2}, \text{ for all } \beta_1 \geq \beta_2.$$

Let us consider $A = \{a_{\beta_1\beta_2}, b_{\beta_1\beta_2} : \beta_1, \beta_2 < \alpha\}$. Since card$(A) \leq$ card $\alpha <$ card ω_c. Therefore, there exist

$$a_{\alpha\beta}, b_{\alpha\beta} \in \mathbb{R} \setminus (\{a_{\gamma\delta}, b_{\gamma\delta} : \gamma, \delta < \beta\} \cup \{a_{\zeta\beta}, b_{\zeta\beta} : \zeta < \alpha\}) \text{ such that } b_{\alpha\beta} = a_{\alpha\beta} y_\beta \text{ for each } \alpha < \beta < \omega_c$$

and

$$a_{\alpha\beta}, b_{\alpha\beta} \in \mathbb{R} \setminus (\{a_{\gamma\delta}, b_{\gamma\delta} : \gamma, \delta < \alpha\} \cup \{a_{\zeta\alpha}, b_{\zeta\alpha} : \zeta < \alpha\} \cup \{a_{\alpha\zeta}, b_{\alpha\zeta} : \zeta < \beta\})$$

$$\text{such that } b_{\alpha\beta} = a_{\alpha\beta} y_\beta \text{ for each } \beta \leq \alpha < \omega_c.$$

So, we get transfinite sequence of reals $\{a_{\alpha\beta}, b_{\alpha\beta}\}_{\alpha,\beta<\omega_c}$ such that $b_{\alpha\beta} = a_{\alpha\beta} y_\beta$ for $\alpha, \beta < \omega_c$. Now we consider $X_\alpha = \{a_{\alpha\beta}, b_{\alpha\beta} : \beta < \omega_c\}$, for each $\alpha < \omega_c$. Then clearly $\{X_\alpha\}_{\alpha<\omega_c}$ is an uncountable sequence of pairwise disjoint subsets of $\mathbb{R}$ with $R(X_\alpha) = \mathbb{R}$ for each $\alpha < \omega_c$. If $\bigcup_{\alpha<\omega_c} X_\alpha \neq \mathbb{R}$, put all the elements of $\mathbb{R} \setminus \bigcup_{\alpha<\omega_c} X_\alpha$ in X_1 we get desired result. □

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