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Recent decomposition results on $QTAG$-modules

Ayazul Hasan, Jules Clement Mba and Rafiquddin

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Abstract A $QTAG$-module M is said to have the exchange property if whenever M' is a module containing M and $M' = M \oplus N = \Sigma_{i\in I} K_i$, then there are submodules $L_i \subseteq K_i$ $(i \in I)$ such that $M' = M \oplus \Sigma_{i\in I} L_i$. The purpose of this paper is essentially to study exchange property for $QTAG$-module. We show that every stiff module has the exchange property. We have further studied stiff modules and modules with loose socles to constructing the modules that are neither transitive nor fully transitive.

1 Introduction and preliminary terminology

Let R be any ring. Consider the following two conditions on a module M_R:

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

A module M_R satisfying (I) and (II) is called a TAG-module, and a module satisfying only condition (I) is called a $QTAG$-module. The study of $QTAG$-modules was initiated by Singh [8]. This is a very fascinating structure that has been the subject of research of many authors. Different notions and structures of $QTAG$-modules have been studied, and a theory was developed, introducing several notions, interesting properties, and different characterizations of submodules. Many interesting results have been obtained, but there is still a lot to explore.

Let all rings discussed here be associative with unity $(1 \neq 0)$ and modules are unital $QTAG$-modules. A module in which the lattice of its submodule is totally ordered is called a serial module; in addition, if it has finite composition length it is called a uniserial module. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module, and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M,\ e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) : y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_n(M)$ denotes the submodule of M generated by the elements of height at least n and $H^n(M)$ is the submodule of M generated by the elements of exponents at most n. The module M is h-divisible if $M = M^1 = \bigcap_{n=0}^{\infty} H_n(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded, if there exists an integer n such that $H_M(x) \leq n$ for every uniform element $x \in M$.

A submodule N of M is h-pure in M if $N \cap H_n(M) = H_n(N)$, for every integer $n \geq 0$. A submodule $B \subseteq M$ is a basic submodule of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A submodule $N \subseteq M$ is said to be high, if it is a complement of M^1 i.e., $M = N \oplus M^1$. The sum of all simple submodules of M is called the socle of M and is denoted by $Soc(M)$. The cardinality of a minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-Ulm Kaplansky invariant of M and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$.

The submodules $H_n(M), n \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules [5] are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{n=0}^{\infty} (N + H_n(M))$. Therefore, the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$ and h-dense in M if $\overline{N} = M$.

It is interesting to note that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules [6]. Notations and terminology are followed by [1, 2]. As usual, End(M) denotes the endomorphism ring of a module M.

2 Some general results

We begin by defining the following.

Definition 2.1. Let μ be a cardinal. We say that a $QTAG$-module M has the μ-exchange property if, for any $QTAG$-module M' containing M as a submodule, and for any submodules N and K_i $(i \in I)$ where the cardinal of I does not exceed μ, the condition $M' = M \oplus N = \Sigma_{i\in I} K_i$ implies that there exist submodules $L_i \subseteq K_i$ $(i \in I)$ such that $M' = M \oplus \Sigma_{i\in I} L_i$.

Remark 2.2. If M has the μ-exchange property for every cardinal μ, then we say that M has the exchange property.

From the above discussion, the following consequences are immediate:

$(2a)$ If a $QTAG$-module M has the μ-exchange property, and if M' is any $QTAG$-module such that $M' = M \oplus N \oplus A = A \oplus \Sigma_{i\in I} K_i$ where the cardinal of I does not exceed μ, then there are submodules $L_i \subseteq K_i$ $(i \in I)$ such that $M' = M \oplus A \oplus \Sigma_{i\in I} L_i$.

$(2b)$ If M is a $QTAG$-module and $M = M_1 \oplus M_2$, then M has the μ-exchange property if and only if M_1 and M_2 have the μ-exchange property.

$(2c)$ If a $QTAG$-module M has the 2-exchange property, then M has the μ-exchange property for every finite μ.

$(2d)$ If a $QTAG$-module M is represented in two ways as a direct sum of countably generated many submodules each having the $\aleph_0$-exchange property, then these two direct decompositions of M have isomorphic refinements.

$(2e)$ Every closed module has the exchange property.

To develop the study, we need to prove some results and we start with the following lemma.

Lemma 2.3. *Let M be a $QTAG$-module with a decomposition $M = M_1 \oplus M_2$, and let ϕ be the projection of M onto M_1. If N is an h-pure submodule of M such that the restriction of ϕ to $Soc(N)$ is a height-preserving isomorphism of $Soc(N)$ onto $Soc(M_1)$, then $M = N \oplus M_2$.*

Proof. Clearly $N \cap M_2 = 0$. If $x \in Soc(M_1)$, then there is a unique element $y \in Soc(N)$ such that $\phi(y) = x$, and there is an element $z \in Soc(M_2)$ such that $y = x + z$. In particular, $x = y - z$, so that $Soc(M_1) \subseteq N + M_2$. Thus $Soc(M) \subseteq N + M_2$, and in order to show that $M = N \oplus M_2$ it suffices to show that $N + M_2$ is h-pure in M. Choose any $a \in Soc(M)$. Then $a = x + b$ with $x \in Soc(M_1)$ and $b \in Soc(M_2)$, and $H_M(a) = \min\{H_{M_1}(x), H_{M_2}(b)\}$. Moreover, $a = y - z + b$ and $H_M(x) = H_{M_1}(x) = H_M(y) = H_N(y) \leq H_{M_2}(z)$. If $H_{M_2}(b) < H_{M_1}(x)$, then $H_{M_2}(b - z) = H_{M_2}(b) < H_N(y)$, and we have, in this case, that $H_{N+M_2}(a) = H_{M_2}(b) = H_M(a)$. If $H_{M_2}(b) \geq H_{M_1}(x)$, then $H_{M_2}(b - z) \geq H_{M_1}(x) = H_N(y)$, and we infer that $H_{N+M_2}(a) = H_N(y) = H_{M_1}(x) = H_M(a)$. Consequently, the height of an element of exponent one is the same in $N + M_2$ as in M, and we conclude that $N + M_2$ is an h-pure submodule of M. □

As an immediate consequence, we yield the following.

Lemma 2.4. *If M is a QTAG-module such that $M = M_1 \oplus M_2$, and N is an h-pure submodule of M for which $Soc(N) = Soc(M_1)$, then $M = N \oplus M_2$.*

Lemma 2.5. *If M is a $QTAG$-module such that $M = M_1 \oplus M_2 \oplus M_3 = M_4 \oplus M_5$, and $Soc(M_1) \subseteq Soc(M_4) \subseteq Soc(M_1) \oplus Soc(M_2)$, then there is a submodule N of M_4 such that N is isomorphic to a submodule of M_2, $Soc(N) = Soc(M_4) \cap Soc(M_2)$, and $M = M_1 \oplus N \oplus M_5$.*

Proof. Let ϕ be the projection of M onto $M_1 \oplus M_2$, and let ψ be the projection of M onto M_4. Set $U = \psi(M_1)$. Then $Soc(U) = Soc(M_1)$, and U is h-pure in M. Set $V = \phi(U)$ and $W = \phi(M_4)$. Then $Soc(W) = Soc(M_4)$, and W is h-pure in M. Furthermore, $Soc(V) = Soc(M_1)$, V is h-pure in M, and $V \subseteq W \subseteq M_1 \oplus M_2$. Hence by Lemma 2.4, we infer that $M_1 \oplus M_2 = V \oplus M_2$. Moreover, if $T = W \cap M_2$, then $W = V \oplus T$. Since the restriction of ϕ to M_4 is an isomorphism of M_4 onto W, and $\phi(U) = V$, it follows that $M_4 = U \oplus N$, where N is the submodule of M_4 onto T under ϕ. The restriction of ϕ to $Soc(M_4)$ is the identity mapping, and therefore $Soc(N) = Soc(T) = Soc(M_4) \cap Soc(M_2)$. This last formula also implies that the restriction to N of the projection of M onto M_2 is an isomorphism of N into M_2. Finally, as $M = U \oplus N \oplus M_5$, it again follows by Lemma 2.4 that $M = M_1 \oplus N \oplus M_5$. □

We are now in a position to state and prove the main result of this section.

Theorem 2.6. *If M is a $QTAG$-module without elements of infinite height, and M is represented in two ways as a direct sum of submodules each having the 2-exchange property, then these two direct decompositions of M possess isomorphic refinements.*

Proof. By combining those direct summands of bounded order in the two decompositions of M, we may assume that the decompositions are of the form

$$M = M_1 \oplus \Sigma_{i \in I} M_i = M_2 \oplus \Sigma_{j \in J} M_j \tag{2.1}$$

where M_1 and M_2 are direct sums of uniserial modules, and each M_i and each M_j is an unbounded module having the 2-exchange property. Notice that this implies that no M_i or no M_j is a direct sum of uniserial modules. For each $i \in I$, choose a finite subset $J_i \subseteq J$ and two submodules A_i and B_i such that B_i is bounded, $M_i = A_i \oplus B_i$, and

$$Soc(A_i) \subseteq \Sigma_{j \in J_i} M_j \tag{2.2}$$

Write $B = M_1 \oplus \Sigma_{i \in I} B_i$. For each $j \in J$, choose a finite subset $I_j \subseteq I$ and two submodules K_j and L_j such that L_j is bounded, $M_j = K_j \oplus L_j$, and

$$Soc(K_j) \subseteq \Sigma_{i \in I_j} A_i \tag{2.3}$$

Write $L = M_2 \oplus \Sigma_{j \in J} L_j$. Then

$$M = B \oplus \Sigma_{i \in I} A_i = L \oplus \Sigma_{j \in J} K_j \tag{2.4}$$

and B and L are direct sums of uniserial modules. Notice that in order to show that the decompositions (2.1) have isomorphic refinements, it suffices to show that the decompositions (2.4) have isomorphic refinements.

We will now construct two transfinite sequences of subsets $I_\alpha \subseteq I$ and $J_\alpha \subseteq J$ such that the following conditions hold for each ordinal α:

(i) I_α and J_α are each nonempty and countable;

(ii) $I_\alpha \cap I_\beta = J_\alpha \cap J_\beta = \phi$ for all $\beta < \alpha$;

(iii) $\Sigma_{j \in J^{\alpha+1}} Soc(K_j) \subseteq \Sigma_{i \in I^{\alpha+1}} Soc(A_i) \subseteq L \oplus \Sigma_{j \in J^{\alpha+1}} Soc(K_j)$

where $I^\alpha = \cup_{\beta<\alpha} I$ and $J^\alpha = \cup_{\beta<\alpha} J$.

First, let us suppose that the subsets I_α and J_α have been obtained for all α less than an ordinal γ. It then follows from (iii) that

$$\Sigma_{j \in J^\gamma} Soc(K_j) \subseteq \Sigma_{i \in I^\gamma} Soc(A_i) \subseteq L \oplus \Sigma_{j \in J^\gamma} Soc(K_j) \tag{2.5}$$

Define $A^\gamma = \Sigma_{i \in I^\gamma} A_i, \quad U^\gamma = \Sigma_{i \notin I^\gamma} A_i, \quad K^\gamma = \Sigma_{j \in J^\gamma} K_j, \quad V^\gamma = \Sigma_{j \notin J^\gamma} K_j.$

Then

$$M = B \oplus A^\gamma \oplus U^\gamma = L \oplus K^\gamma \oplus V^\gamma, \tag{2.6}$$

and (2.5) can be rewritten as $Soc(K^\gamma) \subseteq Soc(A^\gamma) \subseteq L \oplus Soc(K^\gamma)$. Therefore it follows from Lemma 2.5 that there exists a submodule W_γ such that W_γ is isomorphic to a submodule of L, $Soc(W_\gamma) = Soc(A^\gamma) \cap Soc(L)$, and

$$M = B \oplus K^\gamma \oplus W_\gamma \oplus U^\gamma. \tag{2.7}$$

Since each A_i and each K_j is not a direct sum of uniserial modules, it follows from (2.6) and (2.7) that $I^\gamma = I$ if and only if $J^\gamma = J$.

Suppose that $J^\gamma \neq J$ and choose an index $j_0 \in J - J^\gamma$. With $S_0 = \{j_0\}$ and $T_0 = I_{j_0}$, define the sets S_n and T_n, for each positive integer n, by $S_n = \cup_{i \in T_{n-1}} J_i$ and $T_n = \cup_{j \in S_n} I_j$. If $I_\gamma = \cup_{n<\infty} T_n - I^\gamma$ and $J_\gamma = \cup_{n<\infty} S_n - J^\gamma$, then I_γ and J_γ are countable, and it follows from (2.2), (2.3), and (2.5) that $(i) - (iii)$ hold for $\alpha = \gamma$.

Since each I_α and each J_α is nonempty, there must exist an ordinal δ such that $I^\delta = I$ and $J^\delta = J$.

Now for each $\alpha < \delta$, $K^{\alpha+1} = K^\alpha \oplus \Sigma_{j \in J_\alpha} K_j$ and $U^\alpha = U^{\alpha+1} \oplus \Sigma_{i \in I_\alpha} A_i$.

Therefore taking in (2.7) successive values of γ, say $\gamma = \alpha$ and $\gamma = \alpha + 1$, we obtain

$$M = B \oplus K^\alpha \oplus W_\alpha \oplus U^{\alpha+1} \oplus \Sigma_{i \in I_\alpha} A_i \tag{2.8}$$

and

$$M = B \oplus K^\alpha \oplus W_{\alpha+1} \oplus U^{\alpha+1} \oplus \Sigma_{j \in J_\alpha} K_j \tag{2.9}$$

Moreover, $Soc(W_\alpha) \subseteq Soc(W_{\alpha+1})$, and we infer from Lemma 2.5 that there exists a submodule N_α of $W_{\alpha+1}$ such that

$$M = B \oplus K^\alpha \oplus W_\alpha \oplus U^{\alpha+1} \oplus N_\alpha \oplus \Sigma_{j \in J_\alpha} K_j \tag{2.10}$$

One consequence of (2.9) and (2.10) is that $Soc(W_\alpha) \oplus Soc(N_\alpha) = Soc(W_{\alpha+1})$ for all $\alpha < \delta$. Since $W_0 = 0$, a transfinite induction yields

$$Soc(W_\alpha) = \Sigma_{\beta<\alpha} Soc(N_\beta) \tag{2.11}$$

Set $N^\alpha = \Sigma_{\beta<\alpha} N_\beta$ for each $\alpha \leq \delta$. If N^α is an h-pure submodule of M, then it follows from (2.10), (2.11), and Lemma 2.4 that $N^{\alpha+1}$ is a direct summand and hence an h-pure submodule of M. And Again a transfinite induction yields that N^α is h-pure in M for all $\alpha \leq \delta$. If $\gamma = \delta$ is substituted in (2.7), we obtain that $M = B \oplus L^\delta \oplus W_\delta$, and as N^δ is h-pure and $Soc(N^\delta) = Soc(W_\delta)$, it follows by Lemma 2.4 that

$$M = B \oplus K^\delta \oplus N^\delta = B \oplus K^\delta \oplus \Sigma_{\alpha<\delta} N_\delta \tag{2.12}$$

Comparing (2.4) and (2.12) we infer that

$$L \cong B \oplus \Sigma_{\alpha<\delta} N_\delta \tag{2.13}$$

and comparing (2.9) and (2.10) we infer that

$$\Sigma_{i \in I_\alpha} A_i \cong N_\alpha \oplus \Sigma_{j \in J_\alpha} K_j \tag{2.14}$$

The two decompositions of (2.14) have isomorphic refinements by (2d) for all $\alpha < \delta$, and this fact, together with (2.13), imply that the decompositions (2.4) possess isomorphic refinements, completing the proof of the theorem. □

3 Stiff modules and loose socles

We start here with a few more definitions.

Definition 3.1. Let M be a $QTAG$-module. Then M is called essentially indecomposable if whenever $M = M_1 \oplus M_2$, either M_1 or M_2 is bounded.

Definition 3.2. A $QTAG$-module M is said to be stiff if for every endomorphism ψ of M there is a decomposition $M = M_1 \oplus M_2$ and an integer t such that M_1 is bounded and $\psi(x) = tx$ for all $x \in Soc(M_2)$.

From the above definitions, the following lemmas are immediate:

Lemma 3.3. *If M and M' are both essentially indecomposable $QTAG$-modules, if the basic submodules of M and M' are isomorphic, and if M has an unbounded direct summand which is isomorphic to a direct summand of M', then $M \cong M'$.*

Lemma 3.4. *Every stiff $QTAG$-module is essentially indecomposable.*

With the help of the above discussion we are able to infer the following result.

Lemma 3.5. *Every stiff $QTAG$-module has the exchange property.*

Proof. Let M be a stiff $QTAG$-module, and suppose that P is a module such that

$$P = M \oplus Q = A_1 \oplus A_2.$$

Suppose that φ_k is an isomorphism of A_k into M for $k = 1, 2$. Let ϕ_k denote the projection of P onto A_k, let ϕ'_k denote the restriction of ϕ_k to M, and let $\psi_k = \varphi_k \phi'_k$ $(k = 1, 2)$. Then each ψ_k is an endomorphism of M, and hence there exist a decomposition $M = M_1 \oplus M_2$ and integers t_1 and t_2 such that M_1 is bounded and, for each k, $\psi_k(x) = t_k x$ all $x \in Soc(M_2)$. Now we may assume that $\varphi_1\phi_1(x) = \varphi_1\phi'_1(x) = x$, all $x \in Soc(M_2)$. It follows that the restriction of ϕ_1 to M_2 is one-one, and that ϕ_1 restricted to $Soc(M_2)$ preserves heights. Set $N = \phi_1(M_2)$ and $K = \varphi_1(N)$. Then as elements of exponent one have the same height in N or K as in P, N and K are h-pure submodules of P. Furthermore, $Soc(M_2) = Soc(K)$, and consequently by Lemma 2.4, we get $M = M_1 \oplus K$. Now $\varphi_1(N) = K \subseteq \varphi_1(A_1) \subseteq M$, and therefore $\varphi_1(A_1)/\varphi_1(N)$ is bounded. Hence A_1/N is bounded, and it follows that N is a direct summand of A_1, say $A_1 = N \oplus L$. If ϕ is the projection of P onto N determined by the decomposition $P = N \oplus L \oplus A_2$, then the restrictions of ϕ and ϕ_1 to $Soc(M_2)$ are equal. Thus by Lemma 2.3, we get

$$P = M_1 \oplus M_2 \oplus Q = M_2 \oplus L \oplus A_2.$$

and an application of $(2a)$ and $(2e)$ completes the argument. □

Analysis. Let c denote the cardinal of the continuum. Let C be a closed module and B is a basic submodule of C such that $d(H_\omega(B_i)) = i$, for all i. Then there exist an h-pure submodule T of C which contains B, and an element $c \in Soc(C)$ which is not contained in T such that the following condition is satisfied:

$(*)$ If U and V are h-pure submodules of C both containing T and such that $c \notin V$, and if ψ is a homomorphism of U into V, then there exist a decomposition $U = M_1 \oplus M_2$ and an integer t such that M_1 is bounded, and $\psi(x) = tx$ for all $x \in Soc(M_2)$.

Moreover, there exist 2^c distinct h-pure submodules U_i $(i \in I)$ such that $U_i \supseteq T$, $c \notin U_i$ and $U_i = U_j$ if and only if $Soc(U_i) = Soc(U_j)$, $(i, j \in I)$. In view of $(*)$, each U_i is certainly stiff. Furthermore, there is an isomorphism of U_i into U_j only if $Soc(U_i) \subseteq Soc(U_j)$, and hence distinct modules of the family U_i $(i \in I)$ are nonisomorphic. Now applying Theorem 2.6, in conjunction with Lemmas 3.3-3.5, to the direct sums of the modules U_i $(i \in I)$, we obtain the following result.

Theorem 3.6. *If μ is a cardinal such that $c \leq \mu \leq 2^c$, then there are 2^μ nonisomorphic $QTAG$-modules without elements of infinite height and of cardinal μ.*

Motivated by stiff modules, we introduce the following:

Definition 3.7. Let M be a $QTAG$-module. An h-dense subsocle S of M is said to be loose if whenever $\psi \in \mathrm{End}(M)$ and $\psi(S) \subseteq S$ there exists an integer $t < \omega$ such that $\psi|Soc(H_t(M))$ is multiplication by an integer.

It is obvious that a module M is stiff if it has a loose socle.

Now we are able to prove the following.

Theorem 3.8. *Let M be a $QTAG$-module such that $M^1 = 0$, M has a countably generated basic submodule and $g(\overline{M}/M) < 2^{\aleph_0}$; where $\overline{M}$ is the closure of M. Then M contains a loose subsocle of $\overline{M}$ and, consequently, an h-pure, h-dense stiff submodule.*

Proof. Since M has a countably generated basic submodule, End$(\overline{M})$ has cardinality of the continuum. Let $\mathcal{F}$ be the family of all $\psi \in$ End$(\overline{M})$ such that, for every $t < \omega$, $\psi|Soc(H_t(\overline{M}))$ is not multiplication by an integer. We need only find an h-dense subsocle S of M such that $\psi(S) \nsubseteq S$ for all $\psi \in \mathcal{F}$. Such an S will be a loose subsocle of $\overline{M}$, and if K is maximal among the submodules of M supported by S, then K is an h-pure, h-dense submodule of M. Since $\overline{M}$ is also the closure of any such K, K will be stiff.

In order to construct S, we first fix a well-ordering $\{\psi_\alpha\}_{\alpha<\beta}$ of $\mathcal{F}$ where β does not exceed the first ordinal having cardinality of the continuum. Let $U = Soc(B)$ where B is a basic submodule of M and z be a fixed element of $Soc(M)$ not contained in U. We wish to find two families $\{x_\alpha\}_{\alpha<\beta}$ and $\{y_\alpha\}_{\alpha<\beta}$ of elements of $Soc(M)$ such that (i) $\psi_\alpha(x_\alpha) = y_\alpha + z$ for all $\alpha < \beta$ and (ii) the submodule S generated by U and all the x_α's and y_α's has a direct decomposition $S = U \oplus \bigoplus_{\alpha<\beta}[\langle x_\alpha R\rangle \oplus \langle y_\alpha R\rangle]$ and does not contain z. We proceed by induction. Suppose $\gamma < \beta$ and that for each $\alpha < \gamma$ we have an x_α and y_α satisfying (i) and such that the submodule V generated by U and all the x_α's and y_α's with $\alpha < \gamma$ has the direct decomposition $V = U \oplus \bigoplus_{\alpha<\gamma}[\langle x_\alpha R\rangle \oplus \langle y_\alpha R\rangle]$ and $z \notin V$. We wish to find an $x_\gamma \in Soc(M)$ such that $\langle V,\ (x_\gamma \psi_\gamma(x_\gamma))R\rangle$ does not contain z and has the direct decomposition $V \oplus \langle x_\gamma R\rangle \oplus \langle \psi_\gamma(x_\gamma)R\rangle$. Assume that no such x_γ exists and write $Soc(M) = V \oplus \langle zR\rangle \oplus T$. Then for each $a \in T$ there exists $b \in V \oplus \langle zR\rangle$ and a positive integer k such that $\psi_\gamma(a) = b + ka$. It is easily seen that the integer k is independent of the choice of a. Thus the endomorphism $\phi = k - \psi_\gamma$ maps T into the submodule $V \oplus \langle zR\rangle$ which has cardinality less than that of the continuum. Since $g(Soc(\overline{M})/Soc(M)) < 2^{\aleph_0}$, we then conclude that $g(\phi(\overline{M})) < 2^{\aleph_0}$. Therefore, there exists $t < \omega$ such that $Soc(H_t(\overline{M})) \subseteq$ Kerϕ, which contradicts the fact that $\psi_\gamma \in \mathcal{F}$. The desired x_γ exists and we set $y_\gamma = \psi_\gamma(x_\gamma) - z$.

We conclude then that there exists an $S = U \oplus \bigoplus_{\alpha<\beta}[\langle x_\alpha R\rangle \oplus \langle y_\alpha R\rangle]Soc(M)$ such that $z \notin S$ and, for each α, $\psi_\alpha(x_\alpha) = y_\alpha + z$. S is an h-dense subsocle of M (and, consequently, an h-dense subsocle of $\overline{M}$) since U is an h-dense subsocle of M. Since $\psi_\alpha(x_\alpha) = y_\alpha + z \notin S$ for each α, we have that $\psi(S) \nsubseteq S$ for all $\psi \in \mathcal{F}$. □

Along similar lines we have the following theorem.

Theorem 3.9. *Let M a $QTAG$-module. If the closure $\overline{M}$ of M is an unbounded closed module with a countably generated basic submodule and if N is a countably generated submodule of $\overline{M}$, then $\overline{M}$ contains a loose subsocle S such that $S \cap N = 0$.*

4 Applications

The Ulm-sequence of x is defined as $U(x) = (H(x),\ H(x_1),\ H(x_2), \dots)$. This is analogous to the U-sequences in groups [1]. These sequences are partially ordered because $U(x) \leq U(y)$ if $H(x_i) \leq H(y_i)$ for every i. Ulm invariants and Ulm sequences play an important role in the study of $QTAG$-modules. Using these concepts transitive and fully transitive modules were defined in [7]. A $QTAG$-module M is fully transitive if for $x, y \in M$, $U(x) \leq U(y)$, there is an endomorphism ψ of M such that $\psi(x) = \psi(y)$ and it is transitive if for any two elements $x, y \in M$, with $U(x) \leq U(y)$, there is an automorphism ϕ of M such that $\phi(x) = \phi(y)$. It is well known that countably generated h-reduced $QTAG$-modules and $QTAG$-modules without element of infinite height (i.e. $H_\omega(M) = 0$) are both transitive and fully transitive. The question of whether all QTAG-modules are transitive or fully transitive was unanswered. But some results were given of $QTAG$-modules that are neither transitive nor fully transitive earlier in [3]. Here we continue the similar study of modules that are neither transitive nor fully transitive with the aid of stiff modules and modules with loose socles.

We need the following lemma.

Lemma 4.1. *Let M a $QTAG$-module such that either (i) M/M^1 is stiff or (ii) M has a high submodule having a loose socle. If $\psi \in End(M)$ and if $x \in M^1$, then there is an integer t such that $\psi(x) - tx \in H_1(M^1)$.*

Proof. Let $\overline{\psi} \in \text{End}(M/M^1)$ be defined by $\overline{\psi}(y + M^1) = \psi(y) + M^1$.

(i) Suppose that M/M^1 is stiff. Then choose t and k such that $\overline{\psi}|Soc(H_k(M/M^1))$ is multiplication by t. Let $z \in H_k(M) - M^1$ be such that $d\left(\dfrac{zR}{xR}\right) = 1$. Then $z + M^1 \in Soc(H_k(M/M^1))$ and therefore $(\psi(z) - tz) = u \in M^1$. Thus $(\psi(x) - tx) = v \in H_1(M^1)$ where $d\left(\dfrac{vR}{uR}\right) = 1$.

(ii) Suppose that M has a high submodule N such that N has a loose socle. Let $\sigma : M \to M/M^1$ be the canonical map. It is easily seen that $\overline{\psi}(\sigma(Soc(N))) \subseteq \sigma(Soc(N))$. Since $\sigma|N$ is an isomorphism of N onto an h-pure, h-dense submodule of M/M^1, the closure of M/M^1 is also the closure of $\sigma(N) \cong N$. Therefore $\sigma(Soc(N))$ is a loose subsocle of M/M^1. Thus there is $k < \omega$ such that $\overline{\psi}|Soc(H_k(M/M^1))$ is multiplication by an integer. The proof is now completed as in the first case. □

Remark 4.2. If N and K are high submodules of M, it is easily verified that $\sigma(Soc(N)) = \sigma(Soc(K))$. Consequently, every high submodule of M has a loose socle if one does.

Theorem 4.3. *Let M be an h-reduced $QTAG$-module such that either (i) M/M^1 is stiff or (ii) M has a high submodule with a loose socle. Then*

(a) *if M^1 is the direct sum of two or more uniserial modules, then M is neither transitive nor fully transitive; and*

(b) *if M^1 is not uniserial, then M is not fully transitive.*

Proof. (a) Suppose $M^1 = \bigoplus_{i \in I} \langle x_i R \rangle$. Then $H_1(M^1) = 0$ and each x_i has $(\omega, \infty, \infty, \dots)$ as its Ulm sequence. However, if $i \neq j$, there is no endomorphism of M mapping x_i to x_j. Indeed, Lemma 4.1 implies that each $\langle x_i R \rangle$ is a fully invariant submodule of M.

(b) Assume that M^1 is not uniserial. Then there exist elements x and y in M^1 such that $\langle xR, yR \rangle = \langle xR \rangle \oplus \langle yR \rangle$ is an h-pure submodule of M^1 and $U_M(x) \leq U_M(y)$. We shall show that $\psi(x) \neq y$ for all $\psi \in \text{End}(M)$. If $\psi \in \text{End}(M)$, we have, by Lemma 4.1, that $\psi(x) - tx \in H_1(M^1)$ and $y \notin x + H_1(M^1)$ because of the h-purity of $\langle xR \rangle \oplus \langle yR \rangle$ in M^1. □

Theorem 4.4. *Let M a $QTAG$-module. If the closure $\overline{M}$ of M is an unbounded closed module with a countably generated basic submodule and suppose N is an h-pure submodule such that $\overline{M}/N$ is an h-divisible module of cardinality less than $2^{\aleph_0}$. Then if L is a non-uniserial h-reduced $QTAG$-module with a countably generated basic submodule, there exists a $QTAG$-module P such that (i) $P/P^1 \cong N$ (ii) $P^1 = L$ (iii) P is not fully transitive.*

Proof. It is clear that the proof of Theorem 3.8 can be slightly modified so as to yield a loose subsocle S of $\overline{M}$ contained in N and such that $Soc(N)/S$ is countably generated. Then, if E is the injective envelope of L and if K is an h-pure submodule of N supported by S, $N/K \cong E/L$. Let P be a subdirect sum of N and E with kernels K and L. Then it follows that $P/P^1 \cong N$, $P^1 = L$ and K is a high submodule of P. Since K has a loose socle, the desired conclusion follows from Theorem 4.3. □

We close the study with

5 Concluding discussion

We construct a counter example to a conjecture due to Mehdi et al. [4]. Recall that two $QTAG$-modules M, M' are quasi-isomorphic (denoted by $M \doteq M'$) if there exist submodules N and N' of M and M', respectively, such that $N \cong N'$ and M/N and M'/N' are bounded. They have raised the following question: Let M and M' be two quasi-isomorphic $QTAG$-modules such that $M/M^1 \cong M'/M'^1$. What are the conditions under which $M \cong M'$? Although an affirmative answer can be given when M/M^1 is a direct sum of uniserial modules, the answer is in the negative for the general case. On the basis of techniques already used in this paper, we construct a module M with the following properties: $M/M^1 \cong \overline{M}$, $M^1 \cong \overline{M}$ and M contains a high submodule having a loose socle, where $\overline{M}$ is the closure of M. An argument similar to that M is isomorphic to no proper submodule of itself. We then have $M \doteq H_1(M)$ and $M/M^1 \cong \overline{M} \cong H_1(\overline{M}) \cong H_1(M/M^1) = H_1(M)/(H_1(M))^1$, but $M \ncong H_1(M)$.

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Author information

Ayazul Hasan, Department of Mathematics, Faculty of Science, Jazan University, Jazan- P.O. Box 2097, Kingdom of Saudi Arabia.
E-mail: ayaz.maths@gmail.com

Jules Clement Mba, Department of Pure and Applied Mathematics, University of Johannesburg, P.O. Box 524 Auckland Park, 2006 Johannesburg, South Africa.
E-mail: jmba@uj.ac.za

Rafiquddin, Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India.
E-mail: rafiqamt786@rediffmail.com

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