A NOTE ON THE MINIMUM DOMINATING ENERGY OF A GRAPH

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Abstract The minimum dominating energy of a graph has been reported recently in [15]. In this paper some new bounds for the minimum dominating energy $E_D(G)$ of a graph $G$ are presented.

1 Introduction

The concept of energy of a graph was introduced by I. Gutman [6]. Let $G = (V, E)$ be a graph. The number of vertices of $G$ we denote by $n$ and the number of edges we denote by $m$, thus $|V(G)| = n$ and $|E(G)| = m$. For any integer $x$, $\lceil x \rceil$ is the largest integer greater than or equal to $x$. For undefined terminologies we refer the reader to [5].

For details on mathematical aspects of the theory of graph energy see the reviews [8], papers [9, 10, 11, 12]. The basic properties including various upper bounds for energy of a graph have been established in [10, 11], and it is found remarkable chemical applications in the molecular orbital theory of conjugated molecules [7, 16].

2 The Minimum Dominating Energy

The minimum dominating matrix[15] has been defined as follows.

**Definition 1.** Let $G$ be a simple graph of order $n$ and size $m$. A subset $D$ of $V$ is called a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in $D$. Any dominating set with minimum cardinality is called a minimum dominating set. Let $D$ be any minimum dominating set of $G$. The minimum dominating matrix of $G$ is the $n \times n$ matrix $A_D(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in D; \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_D(G)$ is denoted by

$$f_n(G, \lambda) := \det(\lambda I - A(G))$$

The minimum dominating eigenvalues of a graph $G$ are the eigenvalues of $A_D(G)$. Since $A_D(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum dominating energy of $G$ is then defined as

$$E_D(G) = \sum_{i=1}^{n} |\lambda_i|.$$
3 Main Results

For the sake of completeness, we mention below result which is important throughout the paper.

**Lemma 3.1.** [15] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A_D(G)$, then

\[
\sum_{i=1}^{n} |\lambda_i|^2 = 2m + |D|.
\] (3.1)

We need following result, which will be helpful to prove our result.

**Theorem 1.** [14] Suppose $a_i$ and $b_i$, $1 \leq i \leq n$ are positive real numbers, then

\[
\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4} \left( \sqrt{M_1 M_2} + \sqrt{m_1 m_2} \right)^2 \left( \sum_{i=1}^{n} a_i b_i \right)^2
\] (3.2)

where $M_1 = \max_{1 \leq i \leq n} (a_i)$; $M_2 = \max_{1 \leq i \leq n} (b_i)$; $m_1 = \min_{1 \leq i \leq n} (a_i)$ and $m_2 = \min_{1 \leq i \leq n} (b_i)$.

**Theorem 2.** Let $G$ be a graph of order $n$ and size $m$ with $j D j = k$. Suppose zero is not an eigenvalue of $A_D(G)$. Then

\[
E_D(G) \geq \frac{2 \sqrt{\lambda_1 \lambda_n} \sqrt{(2m + k)n}}{\lambda_1 + \lambda_n}.
\] (3.3)

where $\lambda_1$ and $\lambda_n$ are minimum and maximum of the absolute value of $\lambda_i$'s.

**Proof.** Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A_D(G)$. We assume that $a_i = |\lambda_i|$ and $b_i = 1$, which by Theorem 1 implies

\[
\sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} 1^2 \leq \frac{1}{4} \left( \sqrt{\frac{\lambda_n}{\lambda_1}} + \sqrt{\frac{\lambda_1}{\lambda_n}} \right)^2 \left( \sum_{i=1}^{n} |\lambda_i| \right)^2
\]

\[
E_D(G) \geq \frac{2 \sqrt{\lambda_1 \lambda_n} \sqrt{(2m + k)n}}{\lambda_1 + \lambda_n}.
\]

**Theorem 3.** [13] Let $a_i$ and $b_i$, $1 \leq i \leq n$ are nonnegative real numbers, then

\[
\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2
\] (3.4)

where $M_i$ and $m_i$ are defined similarly to Theorem 1.

**Theorem 4.** Let $G$ be a graph of order $n$ and size $m$ with $|D| = k$, then

\[
E_D(G) \geq \sqrt{(2m + k)n - \frac{n^2}{4} (\lambda_n - \lambda_1)^2}
\] (3.5)

where $\lambda_1$ and $\lambda_n$ are minimum and maximum of the absolute value of $\lambda_i$'s.
Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A_D(G)$. We assume that $a_i = 1$ and $b_i = |\lambda_i|$, which by Theorem 3 implies

$$
\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\lambda_i|^2 - \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 \leq \frac{n^2}{4} (\lambda_n - \lambda_1)^2
$$

$$
(2m + k)n - (E_D(G))^2 \leq \frac{n^2}{4} (\lambda_n - \lambda_1)^2
$$

$$
E_D(G) \geq \sqrt{(2m + k)n - \frac{n^2}{4} (\lambda_n - \lambda_1)^2}.
$$

\[ \square \]

Theorem 5. [1] Suppose $a_i$ and $b_i$, $1 \leq i \leq n$ are positive real numbers, then

$$
|n \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i| \leq \alpha(n)(A - a)(B - b)
$$

(3.6)

where $a, b, A$ and $B$ are real constants, that for each $i$, $1 \leq i \leq n$, $a \leq a_i \leq A$ and $b \leq b_i \leq B$. Further, $\alpha(n) = n \left( \frac{n}{2} \right) (1 - \frac{1}{n} \left( \frac{2}{n} \right))$.

Theorem 6. Let $G$ be a graph of order $n$ and size $m$ with $|D| = k$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $A_D(G)$. Then

$$
E_D(G) \geq \sqrt{2mn + nk - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}
$$

(3.7)

where $\alpha(n) = n \left( \frac{n}{2} \right) (1 - \frac{1}{n} \left( \frac{2}{n} \right))$.

Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A_D(G)$. We assume that $a_i = |\lambda_i| = b_i$, $a = |\lambda_n| = b$ and $A = |\lambda_1| = b$, which by Theorem 5, implies

$$
|n \sum_{i=1}^{n} |\lambda_i|^2 - \left( \sum_{i=1}^{n} |\lambda_i| \right)^2| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2
$$

(3.8)

Since, $E_D(G) = \sum_{i=1}^{n} |\lambda_i|$, $\sum_{i=1}^{n} |\lambda_i|^2 = 2m + k$, the above inequality becomes,

$$
(2m + k)n - E_D(G)^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2,
$$

wherefrom (7) follows.

\[ \square \]

Theorem 7. [4] Let $a_i$ and $b_i$, $1 \leq i \leq n$ are nonnegative real numbers, then

$$
\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i^2 \leq (r + R) \left( \sum_{i=1}^{n} a_i b_i \right)
$$

(3.9)

where $r$ and $R$ are real constants, so that for each $i$, $1 \leq i \leq n$, holds, $ra_i \leq b_i \leq Ra_i$.

Theorem 8. Let $G$ be a graph of order $n$ and size $m$ with $|D| = k$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $A_D(G)$. Then

$$
E_D(G) \geq \frac{|\lambda_1||\lambda_2|n + 2m + k}{|\lambda_1| + |\lambda_n|}
$$

(3.10)

where $\lambda_1$ and $\lambda_n$ are minimum and maximum of the absolute value of $\lambda_i$'s.
Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A_c(G)$. We assume that $b_i = |\lambda_i|$, $a_i = 1$ for $r = |\lambda_n|$ and $R = |\lambda_1|$, which by Theorem 7 implies

$$
\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \leq (|\lambda_1| + |\lambda_n|) \sum_{i=1}^{n} |\lambda_i|.
$$

(3.11)

Since, $E_D(G) = \sum_{i=1}^{n} |\lambda_i|$, $\sum_{i=1}^{n} |\lambda_i|^2 = 2m + k$, from the above inequality (10) directly follows from Theorem 7.

References


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