Palestine Journal of Mathematics

Vol. 7(2)(2018) , 663–666

A NOTE ON THE MINIMUM DOMINATING ENERGY OF A GRAPH

Hosamani S. M, Patil P. V and Malghan S. H

Communicated by Ayman Badawi

MSC 2010 Classifications: 05C50.

Keywords and phrases: dominating set, Energy, Minimum dominating energy.

The first-named author acknowledges the support by the Science and Engineering Research Board, New Delhi India under the Major Research Project No. SERB/F/4168/2012-13 Dated 03.10.2013.

Abstract The minimum dominating energy of a graph has been reported recently in [15]. In this paper some new bounds for the minimum dominating energy $E_D(G)$ of a graph G are presented.

1 Introduction

The concept of energy of a graph was introduced by I. Gutman [6]. Let $G = (V, E)$ be a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. For any integer x, $\lceil x \rceil$ is the largest integer greater than or equal to x. For undefined terminologies we refer the reader to [5].

For details on mathematical aspects of the theory of graph energy see the reviews [8], papers [9, 10, 11, 12]. The basic properties including various upper bounds for energy of a graph have been established in [10, 11], and it is found remarkable chemical applications in the molecular orbital theory of conjugated molecules [7, 16].

2 The Minimum Dominating Energy

The minimum dominating matrix[15] has been defined as follows.

Definition 1. Let G be a simple graph of order n and size m. A subset D of V is called a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set. Let D be any minimum dominating set of G. The minimum dominating matrix of G is the $n \times n$ matrix $A_D(G) = (a_{i,j})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in D; \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_D(G)$ is denoted by

$$f_n(G, \lambda) := det(\lambda I - A_c(G))$$

The minimum dominating eigenvalues of a graph G are the eigenvalues of $A_D(G)$. Since $A_D(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum dominating energy of G is then defined as

$$E_D(G) = \sum_{i=1}^{n} |\lambda_i|.$$

3 Main Results

For the sake of completeness, we mention below result which is important throughout the paper.

Lemma 3.1. *[15] If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_D(G)$, then*

$$\sum_{i=1}^{n} |\lambda_i|^2 = 2m + |D|. \tag{3.1}$$

We need following result, which will be helpful to prove our result.

Theorem 1. [14] Suppose a_i and b_i, $1 \leq i \leq n$ are positive real numbers, then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \Big(\sum_{i=1}^{n} a_ib_i\Big)^2 \tag{3.2}$$

where $M_1 = max_{_{1\leq i\leq n}}(a_i)$; $M_2 = max_{_{1\leq i\leq n}}(b_i)$; $m_1 = min_{_{1\leq i\leq n}}(a_i)$ and $m_2 = min_{_{1\leq i\leq n}}(b_i)$

Theorem 2. Let G be a graph of order n and size m with $|D| = k$. Suppose zero is not an eigenvalue of $A_D(G)$. Then

$$E_D(G) \geq \frac{2\sqrt{\lambda_1\lambda_n}\sqrt{(2m+k)n}}{\lambda_1 + \lambda_n}. \tag{3.3}$$

where λ_1 and λ_n are minimum and maximum of the absolute value of $\lambda_i' s$.

Proof. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_D(G)$. We assume that $a_i = |\lambda_i|$ and $b_i = 1$, which by Theorem 1 implies

$$\begin{aligned}
\sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} 1^2 &\leq \frac{1}{4}\left(\sqrt{\frac{\lambda_n}{\lambda_1}} + \sqrt{\frac{\lambda_1}{\lambda_n}}\right)^2 \Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 \\
(2m+k)n &\leq \frac{1}{4}\Big(\frac{(\lambda_1+\lambda_n)^2}{\lambda_1\lambda_n}\Big)(E_D(G))^2 \\
E_D(G) &\geq \frac{2\sqrt{\lambda_1\lambda_n}\sqrt{(2m+k)n}}{\lambda_1 + \lambda_n}.
\end{aligned}$$

□

Theorem 3. [13] Let a_i and b_i, $1 \leq i \leq n$ are nonnegative real numbers, then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \Big(\sum_{i=1}^{n} a_ib_i\Big)^2 \leq \frac{n^2}{4}\big(M_1M_2 - m_1m_2\big)^2 \tag{3.4}$$

where M_i and m_i are defined similarly to Theorem 1.

Theorem 4. Let G be a graph of order n and size m with $|D| = k$, then

$$E_D(G) \geq \sqrt{(2m+k)n - \frac{n^2}{4}(\lambda_n - \lambda_1)^2} \tag{3.5}$$

where λ_1 and λ_n are minimum and maximum of the absolute value of $\lambda_i' s$.

Proof. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_D(G)$. We assume that $a_i = 1$ and $b_i = |\lambda_i|$, which by Theorem 3 implies

$$\begin{aligned} \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\lambda_i|^2 - \big(\sum_{i=1}^{n} |\lambda_i|\big)^2 &\leq \frac{n^2}{4}\big(\lambda_n - \lambda_1\big)^2 \\ (2m+k)n - (E_D(G))^2 &\leq \frac{n^2}{4}\big(\lambda_n - \lambda_1\big)^2 \\ E_D(G) &\geq \sqrt{(2m+k)n - \frac{n^2}{4}(\lambda_n - \lambda_1)^2}. \end{aligned}$$

□

Theorem 5. [1] Suppose a_i and b_i, $1 \leq i \leq n$ are positive real numbers, then

$$|n \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i| \leq \alpha(n)(A-a)(B-b) \tag{3.6}$$

where a, b, A and B are real constants, that for each i, $1 \leq i \leq n$, $a \leq a_i \leq A$ and $b \leq b_i \leq B$. Further, $\alpha(n) = n\lfloor \frac{n}{2} \rfloor \big(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\big)$.

Theorem 6. Let G be a graph of order n and size m with $|D| = k$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $A_D(G)$. Then

$$E_D(G) \geq \sqrt{2mn + nk - \alpha(n)(|\lambda_1| - |\lambda_n|)^2} \tag{3.7}$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor \big(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\big)$.

Proof. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_D(G)$. We assume that $a_i = |\lambda_i| = b_i$, $a = |\lambda_n| = b$ and $A = |\lambda_1| = b$, which by Theorem 5, implies

$$|n \sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} |\lambda_i|\right)^2| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2 \tag{3.8}$$

Since, $E_D(G) = \sum_{i=1}^{n} |\lambda_i|$, $\sum_{i=1}^{n} |\lambda_i|^2 = 2m + k$, the above inequality becomes,

$$(2m+k)n - E_D(G)^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2,$$

wherefrom (7) follows. □

Theorem 7. [4] Let a_i and b_i, $1 \leq i \leq n$ are nonnegative real numbers, then

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i^2 \leq (r+R)\big(\sum_{i=1}^{n} a_i b_i\big) \tag{3.9}$$

where r and R are real constants, so that for each i, $1 \leq i \leq n$, holds, $ra_i \leq b_i \leq Ra_i$.

Theorem 8. Let G be a graph of order n and size m with $|D| = k$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $A_D(G)$. Then

$$E_D(G) \geq \frac{|\lambda_1||\lambda_n|n + 2m + k}{|\lambda_1| + |\lambda_n|} \tag{3.10}$$

where λ_1 and λ_n are minimum and maximum of the absolute value of $\lambda_i's$.

Proof. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $A_c(G)$. We assume that $b_i = |\lambda_i|$, $a_i = 1$ $r = |\lambda_n|$ and $R = |\lambda_1|$, which by Theorem 7 implies

$$\sum_{i=n}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} 1 \leq (|\lambda_1| + |\lambda_n|) \sum_{i=1}^{n} |\lambda_i|. \tag{3.11}$$

Since, $E_D(G) = \sum_{i=1}^{n} |\lambda_i|$, $\sum_{i=1}^{n} |\lambda_i|^2 = 2m + k$, from the above , inequality (10) directly follows from Theorem 7. □

References

[1] M. Biernacki, H. Pidek, C. Ryll-Nardzewsk, Sur une inégalité entre des intégrales définies, Univ. Marie CurieŰSktoodowska **A4**, 1-4 (1950).

[2] V. Consonni, R. Todeschini, New spectral index for molecule description, MATCH Commun. Math. Comput. Chem. **60**, 3-14(2008).

[3] D. M. Cvetković, P. Rowlinson, S. Simić, Eigen spaces of graphs, Cambridge University Press (1997).

[4] J. B. Diaz, F. T. Metcalf, Stronger forms of a class of inequalities of G. PólyaŰG.Szegö and L. V. Kantorovich, *Bull. Amer. Math. Soc.* **69**, 415-418(1963).

[5] F. Harary, Graph Theory, *Addison-Wesely*, Reading, (1969).

[6] I. Gutman, The energy of a graph, *Ber. Math-Statist. Sekt.* Forschungszentrum Graz **103**, 1-22(1978).

[7] I. Gutman, O. E. Polansky, Mathematical Concepts in Organic Chemistry. *Springer-Verlag*, Berlin, (1986).

[8] I. Gutman, X. Li, J. Zhang, Graph Enrgy (Ed-s: M. Dehmer, F. Emmert), Streib., Analysis of Complex Networks, From Biology to Linguistics, *Wiley -VCH, Weinheim* 145-174(2009).

[9] I. Gutman, B. Zhou, Laplacian energy of a graph, *Lin. Algebra Appl.* **414**, 29-37(2006).

[10] J.H. Koolen, V. Moulton, Maximal energy graphs, *Adv. Appl. Math.* **26**, 47-52(2001).

[11] B. J. McClelland, Properties of the latent roots of a matrix: The estimation of π-electron energies, *J. Chem. Phys.* **54**, 640-643(1971).

[12] I. Ž. Milovanović, E. I. Milovanović, A. Zakić, A short note on graph energy, *MATCH Commun. Math. Comput. Chem.* **72**, 179-182(2014).

[13] N. Ozeki, On the estimation of inequalities by maximum and minimum values, *J. College Arts Sci. Chiba Univ.* **5**, 199-203(1968), in Japanese.

[14] G. Polya, G. Szego, Problems and Theorems in analysis, Series, Integral Calculus, Theory of Functions, *Springer*, Berlin, (1972).

[15] M.R. Rajesh Kanna, B.N. Dharmendra, G. Sridhara, The minimum dominating energy of a graph, *International Journal of Pure and Applied Mathematics*, **85(4)**, 707-718(2013).

[16] N. Trinajstić, Chemical graph theory, *CRC Press* (1992).

Author information

Hosamani S. M, Department of Mathematics, Rani Channamma University, Belagavi 591156, India.
E-mail: `sunilkumar.rcu@gmail.com`

Patil P. V, Department of Mathematics, Jain College of Engineering, Belagavi 591156, India.
E-mail: `prashant66.sdm@gmail.com`

Malghan S. H, Department of Mathematics, SECAB's ARS Inamadar Arts, Science and Commerece College for Women, Vijayapura-586709, INDIA.
E-mail: `malghan984@gmial.com`

Received: October 20, 2016.

Accepted: April 9, 2017.