The $k$-Distance degree index of a Graph

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Abstract In this paper, we introduce a new distance-based topological index of a graph $G$, called a $k$-distance degree index. It is defined as $N_k(G) = \sum_{k=1}^{diam(G)}(\sum_{v \in V(G)} d_k(v))k$, where $d_k(v) = |N_k(v)| = |\{u \in V(G) : d(v, u) = k\}|$ is the $k$-distance degree of a vertex $v$ in $G$, $d(v, u)$ is the distance between vertices $u$ and $v$ in $G$ and $diam(G)$ is the diameter of $G$. Exact formulas of the $N_k$-index for some well-known graphs are presented. Bounds for $N_k$-index and some other interesting results are established. It is shown that, $N_k$-index of any graph $G$ is an even integer number. In addition, an explicit formulae of a cartesian product of graphs are presented and we apply this result to compute the $N_k$-index of some graphs (of chemical and computer science interest) like hypercube $Q_d$, Hamming graphs $H(d, n)$, nanotube $R = P_n \square C_m$ and nanotori $S = C_n \square C_m$, etc.

1 Introduction

Throughout this paper, we consider only simple connected graphs, i.e., finite and connected graph without loops, multiple and directed edges. A graph $G = (V, E)$ is said to be connected if there is a path between every pair of its vertices. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph $G$, respectively. The distance $d(u, v)$ between any two vertices $u$ and $v$ of $G$ is equal to the length (number of edges in) a shortest path connecting them. For a vertex $v \in V$ and a positive integer $k$, the open $k$-neighborhood of $v$ in a graph $G$, denoted by $N_k(v)$ (or simply $N_k(v)$), is defined as $N_k(v) = \{u \in V(G) : d(u, v) = k\}$ and the closed $k$-neighborhood of $v$ is $N_k[v] = N_k(v) \cup \{v\}$. The $k$-degree of a vertex $v$ in $G$, denoted $d_k(v)$ (or simply $d_v$ if no misunderstanding), is defined as $d_k(v) = |N_k(v)|$. It is clearly that $d_1(v) = d(v)$ for every $v \in V$. A vertex of degree equals to zero in $G$ is called an isolated vertex and a vertex of degree one is called a pendant vertex. The graph with no vertices (and hence no edges) is the null graph. Any graph with just one vertex is referred to as trivial graph and denoted $K_1$. The complement $\overline{G}$ of a graph $G$ is a graph with vertex set $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in $G$. A totally disconnected graph $K_n$ is one in which no two vertices are adjacent (that is, one whose edge set is empty). If a graph $G$ consists of $p \geq 2$ disjoint copies of a graph $H$, then we write $G = pH$. For a vertex $v$ of $G$, the eccentricity $e(v) = \max\{d(v, u) : u \in V(G)\}$. The radius of $G$ is $rad(G) = \min\{e(v) : v \in V(G)\}$ and the diameter of $G$ is $diam(G) = \max\{e(v) : v \in V(G)\}$.

A topological index of a graph $G$ is a numerical parameter mathematically derived from the graph structure. It is a graph invariant thus it does not depend on the labeling or pictorial representation of the graph and it is the graph invariant number calculated from a graph representing a molecule. The topological indices of molecular graphs are widely used for establishing correlations between the structure of a molecular compound and its physic-chemical properties or biological activity. The topological indices which are definable by a distance function $d(\cdot, \cdot)$ are called a distance-based topological index. All distance-based topological indices can be derived
from the distance matrix or some closely related distance-based matrix, for more information on this matter see [2] and a survey paper [18] and the references therein.

There are many examples of such indices, especially those based on distances, which are applicable in chemistry and computer science. The Wiener index (1947), defined as

\[ W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v) \]

is the first and most studied of the distance based topological indices [17]. The hyper-Wiener index,

\[ WW(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V} (d(u, v) + d^2(u, v)) \]

was introduced in (1993) by M. Randic [13]. The Harrary index

\[ H(G) = \sum_{\{u,v\} \subseteq V} \frac{1}{d^2(u, v)} \]

was introduced in (1992) by Mihalic et al. [10]. In spite of this, the Harary index is nowadays defined as [8, 11]

\[ H(G) = \sum_{\{u,v\} \subseteq V} \frac{1}{d(u, v)} \]

The Schultz index

\[ S(G) = \sum_{\{u,v\} \subseteq V} (d(u) + d(v))d(u, v) \]

was introduced in (1989) by H. P. Schultz [14]. A. Dobrynin et al. in (1994) also proposed the Schultz index and called it the degree distance index and denoted DD(G) [1]. S. Klavzar and I Gutman, motivated by Schultz index, introduced in (1997) the second kind of Schultz index

\[ S^*(G) = \sum_{\{u,v\} \subseteq V} d(u)d(v)d(u, v) \]

called modified Schultz (or Gutman) index of G [9]. The eccentric connectivity index

\[ \xi = \sum_{v \in V} d(v)e(v) \]

was proposed by Sharma et al. [15]. For more details and examples of distance-based topological indices, we refer the reader to [2, 18, 12, 6] and the references therein.

For any terminology or notation not mention here, we refer to books [3, 5].

In this paper, we introduce a new distance-based topological index of a graph \(G = (V, E)\), called a \(k\)-distance degree index (shortly \(N_k\)-index). It is defined as

\[ N_k(G) = \sum_{k=1}^{\text{diam}(G)} \left( \sum_{v \in V(G)} d_k(v) \right) k. \]

We present the exact formulas of the \(N_k\)-index for some well-known graphs as the complete graph \(K_n\), the path \(P_n\), the cycle \(C_n\), the star \(K_{1,n-1}\), the complete bipartite \(K_{r,s}\) and the wheel \(W_n = K_1 + C_{n-1}\). Upper and lower bounds on \(N_k\)-index of \(G\) and other some interesting results are established. In addition, an explicit formula for the cartesian product of graphs are computed. Finally, the \(N_k\)-index formula of the cartesian product applied to some graphs like hypercube \(Q_n\), Hamming graphs \(H(r, s)\), nanotube \(R = P_r \square C_s\) and nanotori \(S = C_r \square C_s\), etc.

2 The \(N_k\)-index of graphs

**Definition 2.1.** For a connected graph \(G\) with \(n\) vertices, the \(N_k\)-index of \(G\), is defined as

\[ N_k(G) = \sum_{k=1}^{\text{diam}(G)} \left( \sum_{v \in V(G)} d_k(v) \right) k. \]
To illustrate the $N_k$-index of a graph, firstly, we consider the following remarks.

**Remark 2.2.** Let $G$ be a connected graph. Then for a vertex $v \in V(G)$

(i) Since, $d(v, u) = 0$, for $u \in V(G)$, if and only if $v = u$, it follows that $d_0(v) = |N_0(v)| = 1$.

(ii) If $k > e(v)$, then $d_k(v) = 0$.

Then, we discuss the following example.

**Example 2.3.** Let $G$ be a graph with four vertices $v_1, v_2, v_3, v_4$ as in Figure 1.

![Figure 1](image)

It is clear that $diam(G) = 2$. Hence,

$$N_k(G) = \sum_{k=1}^{diam(G)} \left( \sum_{v \in V(G)} d_k(v) \right) k$$

$$= \left( \sum_{v \in V(G)} d_1(v) \right) 1 + \left( \sum_{v \in V(G)} d_2(v) \right) 2$$

$$= (d_1(v_1) + d_1(v_2) + d_1(v_3) + d_1(v_4)).1 + (d_2(v_1) + d_2(v_2) + d_2(v_3) + d_2(v_4)).2$$

$$= (1 + 3 + 2 + 2) + 2(2 + 0 + 1 + 1) = 16.$$

Since, for any two vertices $u$ and $v$ in a graph $G$, either $u$ and $v$ are adjacent and then $u \in N_1(v/G)$ (also $v \in N_1(u/G)$) or $u$ and $v$ are not adjacent in $G$, then $u \notin N_1(v/G)$ and $v \notin N_1(u/G)$. If, without loss of the generality, $u \notin N_1(v/G)$, then $u \in N_k(v/G)$, for some $2 \leq k \leq diam(G)$. Using the definition of the complement $\overline{G}$ of $G$, if $u \notin N_1(v/G)$, then $u \in N_1(v/\overline{G})$.

Thus, $\sum_{k=2}^{diam(G)} N_k(v/G) = N_1(v/\overline{G})$. That means $\sum_{k=2}^{diam(G)} \sum_{v \in V(G)} d_k(v/G) = \sum_{v \in V(G)} d_1(v/\overline{G})$.

Then, by using the well-known result $d_1(v/\overline{G}) = n - 1 - d_1(v/G)$, the following result follows.

**Lemma 2.4.** Let $G$ be a connected graph with $n \geq 2$ vertices. Then

(i) $\sum_{k=1}^{diam(G)} \sum_{v \in V(G)} d_k(v) = n(n - 1)$.

(ii) $\sum_{k=0}^{diam(G)} \sum_{v \in V(G)} d_k(v) = n^2$.

Note that we can rewrite $N_k$-index of a graph $G$ as $N_k(G) = \sum_{v \in V(G)} \left( \sum_{k=1}^{diam(G)} d_k(v).k \right)$.

**Theorem 2.5.** For any a connected graph $G$ of order $n$, size $m$ and $diam(G) = 2$

$$N_k(G) = 2n(n - 1) - 2m.$$
Proof. Let $G$ be a connected graph of order $n$, size $m$ and diameter $diam(G) = 2$ and let $\overline{G}$ be the size of $\overline{G}$. Since for any two distinct vertices $v$ and $u$ in $G$, either $uv \in E(G)$ or $uv \in E(\overline{G})$, it follows that $d_2(v/G) = d_1(v/\overline{G})$, for every $v \in V(G)$. Hence,

$$N_k(G) = \sum_{k=1}^{2} \left( \sum_{v \in V(G)} d_k(v/G) \right) \cdot k$$

$$= \left( \sum_{v \in V(G)} d_1(v/G) \right) \cdot 1 + \left( \sum_{v \in V(G)} d_2(v/G) \right) \cdot 2$$

$$= \left( \sum_{v \in V(G)} d_1(v/G) \right) \cdot 1 + \left( \sum_{v \in V(G)} d_1(v/\overline{G}) \right) \cdot 2$$

$$= 2m + (2m) \cdot 2 = 2m + 4m$$

$$= 2m + 4\left(\frac{n(n-1)}{2} - m\right) = 2n(n - 1) - 2m.$$

□

We need the following definition to prove the next result.

**Definition 2.6.** [3] Power of a Graph: For a positive integer number $k$, $k^{th}$ power of a simple graph $G = (V, E)$ is the graph $G^k$ whose vertex set is $V(G)$, two distinct vertices being adjacent in $G^k$ if and only if their distance in $G$ is at most $k$.

**Theorem 2.7.** For a positive integer number $k$ and a connected nontrivial graph $G$, $N_k$-index is an even integer number.

**Proof.** Let $G$ be a connected nontrivial graph. Of order $n \geq 2$, size $m$ and diameter $diam(G)$. Since $V(G) = V(G^k)$ for every $1 \leq k \leq diam(G)$ and $G = G^1$, it follows that $d_k(v/G) = d_1(v/G^k)$, for every $v \in V(G)$. By the well-known results, for any graph $G$, $\sum_{v \in V(G)} d_1(v/G) = 2|E(G)|$, we obtain $\sum_{v \in V(G)} d_k(v/G) = \sum_{v \in V(G)} d_1(v/G^k) = 2|E(G^k)|$. Hence,

$$N_k(G) = \sum_{k=1}^{diam(G)} \left( \sum_{v \in V(G)} d_k(v/G) \right) \cdot k = \sum_{k=1}^{diam(G)} \left( 2|E(G^k)| \right) \cdot k = 2 \sum_{k=1}^{diam(G)} \left( |E(G^k)| \right) \cdot k.$$

Since $|E(G^k)|$ and $k$ are integer numbers for every $1 \leq k \leq diam(G)$, it follows that $\sum_{v \in V(G)} |E(G^k)| \cdot k$ is an integer number. Therefore, $N_k$-index is an even integer number. □

3 The $N_k$-index of some standard graphs

In this section, we compute the $N_k$-index of some well-known graphs such as complete graphs $K_n$, paths $P_n$, cycles $C_n$, wheel $W_{1,n}$, complete bipartite $K_{r,s}$ and multipartite graphs $K_{n_1,n_2,\ldots,n_t}$, $t \geq 3$.

**Proposition 3.1.** For $n \geq 2$,

$$N_k(K_n) = n(n - 1).$$

**Proof.** Consider a complete graph $K_n$ of order $n \geq 2$. Since $diam(K_n) = 1$, it follows that

$$N_k(K_n) = \sum_{k=1}^{diam(K_n)} \left( \sum_{v \in V(K_n)} d_k(v) \right) \cdot k = \sum_{k=1}^{1} \sum_{v \in V(K_n)} d(v) = n(n - 1).$$

□

**Proposition 3.2.** For $n \geq 2$,

$$N_k(P_n) = \frac{n^3 - n}{3}.$$
Proof. Consider a path graph $P_n$ of order $n \geq 2$. We prove the result of $N_k$-index of $P_n$ only for $n$ is even. The proof for $n$ is odd is analogous. Since $\text{diam}(P_n) = n - 1$, it follows that

$$N_k(P_n) = \sum_{k=1}^{n-1} \left( \sum_{v \in V(P_n)} d_k(v) \right) \cdot k$$

$$= ( \sum_{v \in V(P_n)} d_1(v) \cdot 1 + ( \sum_{v \in V(P_n)} d_2(v) \cdot 2 + \cdots + ( \sum_{v \in V(P_n)} d_i(v) \cdot i + \cdots + ( \sum_{v \in V(P_n)} d_{n-1}(v) \cdot (n-1)$$

$$= (1 + 2 + 2 + \cdots + 2 + 1 + (1 + 1 + 2 + 2 + \cdots + 2 + 1 + 1).2 + \cdots + \underbrace{1 + \cdot \cdot \cdot + 1}_{\text{i times}} + \underbrace{1 + 2 + 2 + \cdots + 2 + 1 + 1 + \cdot \cdot \cdot + 1}_{\text{t times}} + i + \cdots + \underbrace{(1 + 1 + \cdot \cdot \cdot + 1) \cdot \frac{n}{2}}_{\text{t times}} + \underbrace{(1 + 0 + \cdot \cdot \cdot + 0 + 1 + 1 + \cdot \cdot \cdot + 1)}_{\text{n times}} + \underbrace{(n - 1)}_{\text{(n - 1)}}$$

$$= 2\cdot(n - 1) \cdot 1 + 2\cdot(1 + 2 + 2 \cdot \cdot \cdot + 2 + (n - i) \cdot i + \cdots + 2\cdot\frac{n}{2}) + 2\cdot(1 + 0 + \cdot \cdot \cdot + 0 + 1 + 1 + \cdot \cdot \cdot + 1). (n - 2) + 2\cdot(1 + 0 + \cdot \cdot \cdot + 0 + 1 + 1 + \cdot \cdot \cdot + 1). (n - 1)$$

$$= 2\cdot(n - 1) \cdot 1 + 2\cdot(2(n - 2) + 2(n - i) \cdot i + \cdots + 2\cdot(2) \cdot (n - 2) + 2\cdot(1) \cdot (n - 1)$$

$$= \sum_{k=1}^{n-1} 2(n - k) \cdot k = 2n \sum_{k=1}^{n-1} k - 2 \sum_{k=1}^{n-1} k^2$$

$$= \frac{n^3 - n}{3}.$$

\[\square\]

Proposition 3.3. For $n \geq 3$,

$$N_k(C_n) = \begin{cases} \frac{n^3}{4}, & \text{if } n \text{ even;} \\ \frac{n^3}{4(n^2 - 1)}, & \text{if } n \text{ odd.} \end{cases}$$

Proof. Consider a cycle graph $C_n$ of order $n \geq 3$. Since $\text{diam}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$ then we consider the following cases

Case 1: If $n$ is even, then $\text{diam}(C_n) = \frac{n}{2}$ and $d_k(v) = 2$, $v \in V(C_n)$ and for every $2 \leq k \leq \frac{n}{2} - 1$ and $d_2(v) = 1$, for every $v \in V(C_n)$. Sequentially,

$$N_k(C_n) = \sum_{k=1}^{\frac{n}{2}} \left( \sum_{v \in V(C_n)} d_k(v) \right) \cdot k$$

$$= \frac{n}{2} \sum_{k=1}^{\frac{n}{2} - 1} \left( \sum_{v \in V(C_n)} 2 \right) \cdot k + \left( \sum_{v \in V(C_n)} 1 \right) \cdot \frac{n}{2}$$

$$= \frac{3}{2} \sum_{k=1}^{\frac{n}{2} - 1} (2n) \cdot k + \frac{n^2}{2}$$

$$= 2n \sum_{k=1}^{\frac{n}{2}} k + \frac{n^2}{2} = \frac{n^3}{4}.$$
The $N_k$-index of Graphs

Case 2: If $n$ is odd, then $diam(C_n) = \frac{n-1}{2}$ and $d_k(v) = 2$, $v \in V(C_n)$. Sequentially,

\[
N_k(C_n) = \sum_{k=1}^{\frac{n-1}{2}} \left( \sum_{v \in V(C_n)} d_k(v) \right).k
= \sum_{k=1}^{\frac{n-1}{2}} \left( \sum_{v \in V(C_n)} 2 \right).k
= \sum_{k=1}^{\frac{n-1}{2}} (2n).k
= 2n \sum_{k=1}^{\frac{n-1}{2}} k = \frac{n(n^2 - 1)}{4}.
\]

Thus, $N_k(C_n) = \begin{cases} \frac{n^3}{4}, & \text{if } n \text{ even;} \\ \frac{n(n^2-1)}{4}, & \text{if } n \text{ odd.} \end{cases}$

A graph $G$ is said to be a complete $t$-partite graph if there is a partition $V_1 \cup V_2 \cup \ldots \cup V_t = V(G)$ of the vertex set, such that $uv \in E(G)$, if and only if $u$ and $v$ are in different parts of the partition. If $|V_i| = n_i$, for every $1 \leq i \leq t$, then $G$ is denoted by $K_{n_1,n_2,\ldots,n_t}$.

Corollary 3.4. [16] For any complete $k$-partite graph $K_{n_1,n_2,\ldots,n_k}$, the number of its edge is

\[
m = \frac{1}{2} \left( \sum_{i=1}^{k} n_i \right) - \sum_{i=1}^{k} n_i^2.
\]

From Theorem 2.5 and Corollary 3.4, the following results are immediately follows.

Proposition 3.5. For $t \geq 2$, $n = n_1 + \ldots + n_t$ and $n_1 \geq n_2 \geq \ldots \geq n_t$ the $N_k$-index of a complete $t$-partite $K_{n_1,\ldots,n_t}$ graph is

\[
N_k(K_{n_1,\ldots,n_t}) = n(n-2) + \sum_{i=1}^{t} n_i^2.
\]

Proposition 3.6. For $2 \leq r \leq s$, the $N_k$-index of a complete bipartite graph $K_{r,s}$ is

\[
N_k(K_{r,s}) = 2(r+s)(r+s-1) - 2rs.
\]

Proposition 3.7. For $n \geq 2$, the $N_k$-index of a star graph is

\[
N_k(K_{1,n-1}) = 2(n-1)^2.
\]

Proposition 3.8. For $n \geq 4$ the $n_k$-index of a wheel $W_{1,n} = K_1 + C_n$ with $n + 1$ vertices is

\[
N_k(W_{1,n}) = 2n(n-1).
\]

4 Bounds for $N_k$-index of graphs

In this section, upper and lower bounds for $N_k$-index of a graph $G$ and some interesting result are established.

Theorem 4.1. Let $G$ be a connected graph with $n \geq 2$ vertices. Then

\[
n(n-1) \leq N_k(G) \leq n(n-1)^2.
\]

The lower bound attains on complete graphs $K_n$, for $n \geq 2$, whereas the upper bound attains on $K_2$. 

Proof. Let $G$ be a connected graph with $n \geq 2$ vertices. Then for $1 \leq k \leq \text{diam}(G)$,

$$\sum_{k=1}^{\text{diam}(G)} \left( \sum_{v \in V(G)} d_k(v) \right). 1 \leq \sum_{k=1}^{\text{diam}(G)} \left( \sum_{v \in V(G)} d_k(v) \right). k \leq \sum_{k=1}^{\text{diam}(G)} \left( \sum_{v \in V(G)} d_k(v) \right). \text{diam}(G).$$

Then by Theorem 2.4, $n(n-1) \leq N_k(G) \leq n(n-1)\text{diam}(G)$. Since for any connected graph $G$, $\text{diam}(G) \leq n-1$, it follows that $n(n-1) \leq N_k(G) \leq n(n-1)^2$. \hfill \qed

**Theorem 4.2.** Let $G$ be a connected graph with $n \geq 2$ vertices. Then $N_k(G) = n(n-1)$, if and only if $G = K_n$.

**Proof.** If $G = K_n$, for $n \geq 2$, then $N_k(G) = n(n-1)$. Conversely, Suppose, to the contrary, that $G \neq K_n$. Then $\text{diam}(G) \geq 2$ and $m = |E(G)| < \frac{n(n-1)}{2}$. Thus by Theorem 2.5,

$$N_k(G) \geq \sum_{k=1}^{2} \left( \sum_{v \in V(G)} d_k(v) \right). k = 2n(n-1) - 2m > n(n-1).$$

Corollary 4.3. Let $G$ be a graph with $n$ vertices and diameter $\text{diam}(G)$. Then

$$(\text{diam}(G) + 1) \text{diam}(G) \leq N_k(G) \leq n(n-1) \text{diam}(G).$$

In a connected graph $G$, a cut edge is an edge $e \in E(G)$ that when removed (the vertices stay in place) from a graph creates more components than previously in $G$ or an if $G - e$ results in a disconnected graph.

**Theorem 4.4.** Let $G$ be a connected graph and let $e$ be not a cut edge of $G$. Then

$$N_k(G) \leq N_k(G-e).$$

**Proof.** The proof is immediately consequences of the result $\text{diam}(G-e) \geq \text{diam}(G)$ and Corollary 4.3. \hfill \qed

**Corollary 4.5.** Let $G$ be a connected graph with $n$ vertices such that $G \neq K_n$. Then

$$N_k(K_n) < N_k(G).$$

**Corollary 4.6.** Let $G$ be a connected graph and let $H$ be a connected spanning subgraph of $G$. Then

$$N_k(G) \leq N_k(H).$$

5 Cartesian product

**Definition 5.1.** [4] For given graphs $G$ and $H$, their Cartesian product, denoted by $G \square H$, is defined as the graph on the vertex set $V(G) \times V(H)$, and vertices $u = (u_1, v_1)$ and $v = (u_2, v_2)$ of $V(G) \times V(H)$ are connected by an edge if and only if either $(u_1 = u_2$ and $v_1v_2 \in E(H))$ or $(v_1 = v_2$ and $u_1u_2 \in E(G))$.

It is a well known fact that the Cartesian product of graphs is commutative and associative up to isomorphism, $|V(G \square H)| = |V(G)||V(H)|$, the distance between any two vertices $u = (u_1, v_1)$ and $v = (u_2, v_2)$ in $G \square H$ is given by $d_{G \square H}(u, v) = d_G(u_1, u_2) + d_H(v_1, v_2)$. The eccentricity $e(u, v)$ is obtained in the same way. Also, $\text{diam}(G \square H) = \text{diam}(G) + \text{diam}(H)$. Let $\text{diam}(G) \leq \text{diam}(H)$. If $1 \leq i \leq \text{diam}(H) - \text{diam}(G) - 1$ and $1 \leq j \leq \text{diam}(G) - 1$, then $d_{\text{diam}(G)+i(u/G)} = 0$ and $d_{\text{diam}(H)+j(v/H)} = 0$. For more details on cartesian product properties, see [4].

The following result is required to prove the next our main result.
**Theorem 5.2.** [16] Let $G$ and $H$ be connected graphs of orders $n_G$ and $n_H$, respectively. Then for any vertex $w = (u, v) \in G \square H$,

$$d_k(w/G \square H) = \sum_{i=1}^{k} d_i(u/G) d_{k-i}(v/H).$$

**Theorem 5.3.** Let $G$ and $H$ be nontrivial connected graphs. Then

$$N_k(G \square H) = |V(H)|^2 N_k(G) + |V(G)|^2 N_k(H).$$

**Proof.** Let $G$ and $H$ be connected graphs of orders $|V(G)| \geq 2$ and $|V(H)| \geq 2$, respectively and let $D_1 = \text{diam}(G)$ and $D_2 = \text{diam}(H)$. Then $G \square H$ is connected graph with $|V(G)||V(H)|$ vertices. Let $w = (u, v) \in V(G \square H)$ and suppose, without loss of generality, that $D_1 \leq D_2$. Then by Theorem 5.2, and properties of summation notion, we get

$$N_k(G \square H) = \sum_{k=1}^{\text{diam}(G \square H)} \left( \sum_{w \in V(G \square H)} d_k(w/G \square H) \right) k$$

$$= \sum_{k=1}^{D_1+D_2} \left( \sum_{(u, v) \in V(G \square H)} d_k((u, v)/G \square H) \right) k$$

$$= \sum_{k=1}^{D_1+D_2} \left( \sum_{(u, v) \in V(G \square H)} \sum_{i=0}^{k} d_i(u/G) d_{k-i}(v/H) \right) k$$

$$= \sum_{(u, v) \in G \square H} \left[ (d_0(u/G)d_1(v/H) + d_1(u/G)d_0(v/H)) + (d_0(u/G)d_2(v/H) + d_1(u/G)d_1(v/H) + d_2(u/G)d_0(v/H)). 2 + ... 
+ (d_0(u/G)d_{D_1}(v/H) + d_1(u/G)d_{D_1-1}(v/H) + ... + d_{D_1}(u/G)d_0(v/H)). D_1 + ... 
+ (d_0(u/G)d_{D_1+i}(v/H) + d_1(u/G)d_{D_1+i-1}(v/H) + ... + d_{D_1+i}(u/G)d_0(v/H)). (D_1 + i) + ... 
+ (d_0(u/G)d_{D_2}(v/H) + d_1(u/G)d_{D_2-1}(v/H) + ... + d_{D_2}(u/G)d_0(v/H)). D_2 + ... 
+ (d_0(u/G)d_{D_2+i}(v/H) + d_1(u/G)d_{D_2+i-1}(v/H) + ... + d_{D_2+i}(u/G)d_0(v/H)). (D_2 + i) + ... 
+ (d_0(u/G)d_{D_1+D_2}(v/H) + d_1(u/G)d_{D_1+D_2-1}(v/H) + ... + d_{D_1+D_2}(u/G)d_0(v/H)). (D_1 + D_2) \right].$$
\[ N_k(G \square H) = \sum_{(u,v) \in V(G \square H)} \left[ \left( d_0(u/G)d_1(v/H) + d_1(u/G)d_0(v/H) \right) \right] \\
+ 2 \left( d_0(u/G)d_2(v/H) + d_1(u/G)d_1(v/H) + d_2(u/G)d_0(v/H) \right) + ... \\
+ D_1 \left( d_0(u/G)d_{D_1}(v/H) + d_1(u/G)d_{D_1-1}(v/H) + ... + d_{D_1}(u/G)d_0(v/H) \right) + ... \\
+ (D_1 + i) \left( d_0(u/G)d_{D_1+i}(v/H) + d_1(u/G)d_{D_1+i-1}(v/H) + ... \\
+ d_{D_1}(u/G)d_i(v/H) \right) + ... \\
+ D_2 \left( d_0(u/G)d_{D_2}(v/H) + d_1(u/G)d_{D_2-1}(v/H) + ... \\
+ d_{D_1}(u/G)d_{D_2-D_1}(v/H) \right) \\
+ (D_2 + j) \left( d_j(u/G)d_{D_2}(v/H) + d_{j+1}(u/G)d_{D_2-1}(v/H) + ... \\
+ d_{D_1}(u/G)d_{D_2+j}(v/H) \right) + ... \\
+ (D_1 + D_2 - 1) \left( d_{D_1-1}(u/G)d_{D_2}(v/H) + d_{D_1}(u/G)d_{D_2-1}(v/H) \right) \\
+ (D_1 + D_2)(d_{D_1+1}(u/G)d_{D_2}(v/H)) \right] \\
= \sum_{(u,v)} \left[ \left( d_0(u/G)d_1(v/H) + 2d_0(u/G)d_2(v/H) + ... + D_2d_0(u/G)d_{D_2}(v/H) \right) \\
+ (d_1(u/G)d_0(v/H) + 2d_1(u/G)d_1(v/H) + ... + (D_2 + 1)d_1(u/G)d_{D_2}(v/H)) + \\
(2d_2(u/G)d_0(v/H) + 3d_2(u/G)d_1(v/H) + ... + (D_2 + 2)d_2(u/G)d_{D_2}(v/H)) + ... \\
+ (jD_1(u/G)d_0(v/H) + (j + 1)d_2(u/G)d_1(v/H) + ... \\
+ (D_2 + j)d_j(u/G)d_{D_2}(v/H) \right) + ... \\
+ (D_1D_2(u/G)d_0(v/H) + (D_1 + 1)d_{D_1}(u/G)d_1(v/H) + ... \\
+ (D_2 + D_1)d_{D_2}(u/G)d_{D_2}(v/H)) \right] \\
= \sum_{(u,v)} \left[ \sum_{k=0}^{D_2} d_0(u/G)d_0(v/H) \right] \right] \\
= \sum_{(u,v)} \left[ \sum_{k=0}^{D_2} d_1(u/G)d_0(v/H) \right] \right] \\
= \sum_{(u,v)} \left[ \sum_{k=0}^{D_2} d_1(u/G)d_0(v/H) \right] \right] \\
= |V(G)|^2 N_k(H) + |V(H)|^2 N_k(G). \]
The Cartesian product of more than two graphs is defined inductively, 
\[ G_1 \square G_2 \square \ldots \square G_k = G_1 \square (G_2 \square \ldots \square G_k). \]
We denote by \( \prod_{i=1}^{k} G_i \) to \( G_1 \square G_2 \square \ldots \square G_k \). It is clear that \( |V(\prod_{i=1}^{k} G_i)| = \prod_{i=1}^{k} |V(G_i)| \).

**Theorem 5.4.** Let \( G_1, G_2, \ldots, G_t \), for \( t \geq 2 \) be nontrivial connected graphs with \( n_1, n_2, \ldots, n_t \) vertices, respectively. Then
\[ N_k(\prod_{i=1}^{t} G_i) = \sum_{i=1}^{t} \left( \prod_{j=1}^{t} n_j^{\alpha_j} \right) N_k(G_i). \]

**Proof.** Let \( G_1, G_2, \ldots, G_t \), for \( t \geq 2 \), be connected graphs with \( n_1, n_2, \ldots, n_t \) vertices, respectively. Then we set \( \prod_{i=1}^{t} n_i = n_1 n_2 \ldots n_t \) is a usual product of integer numbers. We prove this result by mathematical induction.

(i) The result is true for \( t = 2 \), by Theorem 5.3.

(ii) Assume there is a \( t \geq 2 \) such that \( N_k(\prod_{i=1}^{t} G_i) = \sum_{i=1}^{t} \left( \prod_{j=1}^{t} n_j^{\alpha_j} \right) N_k(G_i) \).

(iii) Now we have to prove that the result is true for \( t + 1 \). So let \( \prod_{i=1}^{t+1} G_i = (\prod_{i=1}^{t} G_i) \square G_{t+1} \), where \( G_{t+1} \) is a connected graph of order \( n_{t+1} \). Then
\[ N_k(\prod_{i=1}^{t+1} G_i) = N_k \left( (\prod_{i=1}^{t} G_i) \square G_{t+1} \right) \]
\[ = (n_{t+1}^2) N_k(\prod_{i=1}^{t} G_i) + \left( \prod_{j=1}^{t+1} n_j \right)^2 N_k(G_{t+1}) \]
\[ = (\prod_{j=2}^{t+1} n_j^2) N_k(G_1) + (\prod_{j=1}^{t} n_j^2) N_k(G_2) + \ldots \]
\[ + (\prod_{j \neq t}^{t+1} n_j^2) N_k(G_t) + (\prod_{j=1}^{t+1} n_j^2) N_k(G_{t+1}) \]
\[ = \sum_{i=1}^{t+1} \left( \prod_{j=1}^{t+1} n_j^{\alpha_j} \right) N_k(G_i). \]
Therefore, the result is true for every positive integer \( t \geq 2 \).

**Corollary 5.5.** Let \( G \) be a connected graph with \( n \geq 2 \) vertices. Then for \( t \geq 1 \)
\[ N_k(\prod_{i=1}^{t} G) = t n^{2t-2} N_k(G). \]
By Theorem 5.4 and Corollary 5.5, we can compute the $N_k$-index for several classes of graphs which defined as a cartesian product of graphs. For examples, hypercube graph, Hamming graphs, $(n \times m)$-grid graphs, $n$-prism graph and nanotube graphs. etc. see[3, 7]. Such graphs appear in many applications, for instance in the theory of communication networks and in chemistry.

**Definition 5.6.** [7]

(i) A Hypercube graph $Q_d$ is the Cartesian product of $d$ copies of $K_2$.

(ii) The Hamming graph $H(d, n)$ is, equivalently, the Cartesian product of $d$ complete graphs $K_n$.

**Example 5.7.** For $d \geq 1$,

(i) $N_k(Q_d) = d2^{2d-1}$.

(ii) $N_k(H(d, n)) = dn^{2d-1} \left(1 - \frac{1}{n}\right)$.

**Definition 5.8.** [3]

(i) The $(n \times m)$-grid graphs $G(n, m)$ is the cartesian product of the path $P_n$ by the path $P_m$.

(ii) A prism graph $Y_n$ is the Cartesian product of a cycle $C_n$ by $K_2$.

(iii) The $C_4$ nanotube graph $R$ is the Cartesian product of a cycle $C_n$ by a path $P_m$.

(iv) The nanotori graph $S$ is the Cartesian product of a cycle $C_b$ by a cycle $C_m$.

**Example 5.9.** For $n \geq 3$ and $m \geq 2$,

(i) $N_k(G(n, m)) = \frac{nm(n+m)(nm-1)}{3}$.

(ii) $N_k(Y_n) = \begin{cases} n^3 + 2n^2, & \text{if } n \text{ is even;} \\ n^3 + 2n^2 - n, & \text{if } n \text{ is odd.} \end{cases}$

(iii) $N_k(R) = N_k(C_n \square P_m) = \begin{cases} \frac{n^2m(3nm+4m^2-4)}{12}, & \text{if } n \text{ is even;} \\ \frac{nm(3n^2m-3nm+4n^2-4n)}{12}, & \text{if } n \text{ is odd.} \end{cases}$

(iv) $N_k(S) = N_k(C_n \square C_m) = \begin{cases} \frac{n^2m^2(n+m)}{4}, & \text{if } n \text{ and } m \text{ are even;} \\ \frac{n^2m^2(n+m-1)}{4}, & \text{if } n \text{ is even and } m \text{ is odd;} \\ \frac{nm^2(n^2+nm-1)}{4}, & \text{if } n \text{ is odd and } m \text{ is even;} \\ \frac{nm(n+m)(nm-1)}{4}, & \text{if } n \text{ and } m \text{ are odd.} \end{cases}$

6 Conclusions

In this paper, the new distance-based topological index, called a $k$-distance degree index (Shortly, $N_k$-index), of graphs is introduced. It is shown that the $N_k$-index of a graph is even integer number. Bounds and interesting result for $N_k$-index are obtained. Exact formulaes of the $N_k$-index for some well-known graphs are presented. Finally, the exact formulaes of the $N_k$-index for Cartesian product of graphs are computed.

**Open Problems**

- Compute the values of $N_k$-index of some others families of graphs.
- Compute the values of $N_k$-index of some others operations on graphs, as line graph, complement of graph, corona product of graphs, etc.
- Find the relationships between $N_k$-index with other indices of a graph.
- Find the relationships between $N_k$-index of a graph with other parameters of a graph, such as maximum degree $\Delta(G)$, minimum degree $\delta(G)$, clique number $\omega(G)$, chromatic number $\chi(G)$ and etc.
- Find the relationships between $N_k$-index of a graph with other distance-based topological indices of a graph.
References


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