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SOME MULTIPLE INTEGRAL RELATIONS INVOLVING GENERAL CLASS OF POLYNOMIALS AND I-FUNCTION

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Abstract In the present paper.we obtain a finite integral involving general class of polynomials and I-function.Next with the application of this and the lamma due to Srivastava et.al. [12], we obtain two general multiple integrals relation involving general class of polynomials.I-function and two arbitrary function f and g.By suitably specializing the functions f and g occuring in the main integral relation,a number of multiple integrals are evaluated which are new and quit general in nature.

1. Introduction :

The I-function introduced and studied by Saxena occuring in this paper is defined as follows[10];

$$I[z] = I^{m,n}_{p_i,q_i;r}[z] = I^{m,n}_{p_i,q_i;r}\left[z \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,n}, (a_{ji},\alpha_{ji})_{n+1,p_i} \\ \\ (b_j,\beta_j)_{1,m}, (b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right]$$

$$= \frac{1}{2\pi\omega}\int_L \chi(\xi) z^{\xi}\, d\xi \tag{1.1}$$

where $\omega = \sqrt{-1}$,

$$\chi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\{\prod_{j=m+1}^{q_i}\{\Gamma(1-b_{ji}+\beta_{ji}\xi)\}\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\}} \tag{1.2}$$

$p_i(i=1,\cdots,r),\;\; q_i(i=1,\cdots,r),\;\; m,n$ are integers satisfying $0 \le n \le p_i,\;\; 0 \le m \le q_i, (i = 1,\cdots,r), r$ is finite,$\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive and a_j,b_j,a_{ji},b_{ji} are complex numbers such that

$$\alpha_j(b_h+\nu) \ne \beta_h(a_j-v-k); h=1,\cdots,m; j=1,\cdots,n.$$

Srivastava [11]has introduced the general class of polynomial

$$\mathbf{S}^m_n[x] = \sum_{k=0}^{[n/m]}\frac{(-n)_{m_k}}{k!}\quad A_{n,k}x^k, n=0,1,2,\cdots \tag{1.3}$$

where m is arbitrary positive integer and coefficients$A[n,k(n,k>0)$ are arbitrary constant, real or complex.

we shall use the following notations;$A^* = (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}$; $\quad B^*(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}$.

Agarwal and Chaubey [2], (see also Srivastava and Manocha [13], p.447, eq(16)) studied the following general sequence of function:

$$\mathbf{R}_n^{(\alpha,s)}[x;a,b,s,d;p,q;\gamma,\delta;\omega(x)] = \frac{(ax^p+b)^{-\alpha}(sx^q+d)^{-s}}{k_n\omega(x)}$$

$$\times\mathbf{T}_{k,l}^n[(ax^p+b)^{\alpha+\gamma_n}(sx^q+d)^{s+\delta_n}\omega(x)] \tag{1.4}$$

with the differential operator $T_{k,l}$, being defined as

$$\mathbf{T}_{k,l} = x^l(k+xD_x) \quad where\ D_x = \frac{d}{dx} \tag{1.5}$$

In (1.4), $\{k_n\}$, $(n = 0,1,\cdots,\infty)$ is a sequence of constants, $\omega(x)$ is independent of n and differentiable an arbitrary number of times. The definition (1.4) was motivated essentially by an earlier work of srivastava and Panda [14] (see also srivastava and Manocha [13]). On taking $\omega(x) = 1,\ p = d = 1,\ s = -\tau$ in (1.4) and replaying s by $\frac{s}{\tau}$ therein we arrive at the following polynomial set after making some obvious changes in parameters (see Gupta et. al. [4]):

$$\mathbf{S}_n^{(\alpha,s,\tau)}[x;r,s,q,A,B,k,l] = (Ax+b)^{-\alpha}(1-\tau x^l)^{-s/\tau}$$

$$\times\mathbf{T}_{k,l}^n[(Ax+B)^{\alpha+q_n}(1-\tau x^l)^{s/r+sn}] \tag{1.6}$$

The explicit form of the generalized polynomial set given by (1.6) is

$$\mathbf{S}_n^{(\alpha,s,\tau)}[x;r,s,q,A,B,k,l] = B^{q_n}x^{l_n}(1-\tau x^l)^{sn}l^n\sum_{e,p,u,v}\frac{(-1)^p(-v)_u(-p)_e(-\alpha-qn)_e}{u!v!e!p!}$$

$$\frac{(-s/\tau-sn)_v}{(1-\alpha-p)_e}((e+k+ru)/l)_n\{\tau x^r/(1-\tau x^r)\}^v(Ax/B)^p \tag{1.7}$$

where

$$\sum_{e,p,u,v} = \sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{p=0}^{n}\sum_{e=0}^{p} \tag{1.8}$$

Taking $\tau\to 0$ in (1.7), we get the following

$$\mathbf{S}_n^{\alpha,s,0}[x:r,q,A,B,k,l] = (Ax+b)^{-\alpha}exp(sx^r)\mathbf{T}_{k,l}^n[(Ax+B)^{\alpha+qn}exp(-sx^r)]$$

$$= \sum_{e,p,u,v}\phi(e,p,u,v)x^L \tag{1.9}$$

where

$$\phi(e,p,u,v) = B^{qn-l}l^n\frac{(-1)^p(-v)_u(-p)_e(\alpha)_p}{u!v!e!p!}\frac{(-\alpha-qn)_l}{(1-\alpha-p)_e}\left(\frac{(e+k+ru)}{l}\right)_n A^pB^q \tag{1.10}$$

and

$$L = ln+p+rv,\ \ (p,v = 0,1,\cdots,n) \tag{1.11}$$

It may be pointed out here that through the polynomial set defined by (1.6) is a special case of the general sequence of functions (1.4), yet this polynomial set is sufficiently general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [3], Gould and Hopper [5], Krall and Frink [7], Singh and Srivastava [15] etc. Some of special cases of (1.6) are given by Saigo, Goyal and Saxena [9] and Agarwal, Pareek and Saigo [1]. Moreover, the explicit series form similar to (1.7) may not be easily obtainable for the sequence of functions defined by (1.4).

2. Results Required :

The following results will be required in establishing our main integral relations:

(i) Lemma (Srivastava et.al. [12]): Let the function $f(x)$ and $g(x)$ be integrable over the semi infinite integral $(0,\infty)$ and define

$$F(R)=\int_0^{\pi/2} h(R,\theta)d\theta \tag{2.1}$$

where $h(R,\theta)$ is an integrable function of two variables in the rectangular region $0\leq R<\infty,\ 0\leq\theta\leq$ Then

$$\int_0^\infty\int_0^\infty f(x^2+y^2)h\{(x^2+y^2)^{1/2},tan^{-1}(y/x)\}dxdy=\frac{1}{2}\int_0^\infty f(t)F(\sqrt{t})dt \tag{2.2}$$

and

$$\int_0^\infty\int_0^\infty\int_0^\infty(x^2+y^2)^{-1/2}f(x^2+y^2+z^2)g[tan^{-1}\{(x^2+y^2)^{1/2}/z\}]h\{(x^2+y^2+z^2)^{1/2}tan^{-1}(y/x)\}dxdydz$$

$$=\int_0^\infty\int_0^\infty f(u^2+v^2)g[tan^{-1}(u/v)]F[(u^2+v^2)^{1/2}]dudv \tag{2.3}$$

Provided that the various integral involved are absolutely convergent.

(ii) A result due to Kalla et.al. [6]:

$$H'[at,bt]=\mathbf{I}^{0,n_1'\ ;1,n_2';1,n_3'}_{p_1'q_1'\ ;p_2',q_2';p_3',q_3':r}\left[\begin{array}{c|c} at & (a_j';A_j',A_J'')_{1,p'},(e_j,E_j)_{1,p_1'},(g_i,G_i)_{1,p_1'} \\ : & \\ bt & (b_j';B_j',B_J'')_{1,q'},(0,1),(f_j,F_j)_{1,q_2'},(0,1),(h_j,H_j)_{1,q_3'} \end{array}\right]$$

$$=\sum_{M'=0}^{\infty}\phi(M')\frac{(-at)^{M'}}{M'!} \tag{2.4}$$

where

$$\phi(M')=\sum_{N'=0}^{M'}\phi'(M'-N',N')\theta_1'(M'-N')\theta_2'(N')(b/a)^{N'}\begin{pmatrix} M' \\ N' \end{pmatrix} \tag{2.5}$$

3. A Useful Integral :

We obtain the following integral, which will be required in the next section:

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta}(sin\theta)^{\alpha-1}(cos\theta)^{\beta-1}e^{-\mu R}S_n^m[y_1R_1^{2\omega'}e^{i(\lambda+\omega)\theta}(sin\theta)^{\lambda}(cos\theta)^{\omega}]\mathbf{S}_n^{\alpha,\beta,0}[y_2R^{2\rho}e^{i(\zeta+\eta)\theta}$$

$$(sin\theta)^{\zeta}(cos\theta)^{\eta}:r',q,A,B,k,l]\mathbf{H}'[ae^{i(\sigma+\rho)\theta}(sin\theta)^{\sigma}(cos\theta)^{\rho},be^{i(\sigma+\rho)\theta}(sin\theta)^{\sigma}(cos\theta)^{\rho}]$$

$$\mathbf{I}[gR^{2\rho_1}e^{i(\sigma_1+\delta_1)\theta}(sin\theta)^{\sigma_1}(cos\theta)^{\delta_1}]d\theta=\sum_{k=0}^{[\frac{n}{m}]}\sum_{M'=0}^{\infty}\sum_{i,\delta,t',p}\frac{(-n)_{mk}}{k!M'!}y_1^ky_2^LR^{z(\omega'k+L)}$$

$$e^{-\mu R}(-a)^{M'}\phi(i,\delta,t',p)\phi(M')A_{n,k}\ exp[\frac{i\pi(\alpha+\lambda k+\zeta L+\sigma M'}{2}]$$

$$I^{m,n+2}_{p_i+2,q_i+2;r}\left[zR^{2\rho_1}exp(i\pi\sigma_1/2) \;\middle|\; \begin{matrix}(1-\alpha-\lambda k-\zeta L-\sigma M',\sigma_1)\\ \\ B^*\end{matrix} \right.$$

$$\left. \begin{matrix}(1-\beta-\omega k-\eta L-\rho M',\delta_1), \quad A^*\\ \\ (1-\alpha-(\lambda+\omega)k-(\zeta+\eta)L-(\sigma+\rho)M',\sigma_1+\delta_1)\end{matrix} \right] \tag{3.1}$$

The integral (3.1)is valid under the following set of conditions:

(i) $\sigma_1 > 0, \delta_1 > 0, Re(\alpha) > 0, Re(\beta) > 0$

(ii) $[Re(\alpha) + \sigma_1 \min\limits_{1\leq j\leq m} Re(b_j/\beta_j)] > 0, [Re(\beta) + \delta_1 \min\limits_{1\leq j\leq m} Re(b_j/\beta_j)] > 0.$

(iii) m is an arbitrary positive integer and the coefficients $n, k(n, k \geq 0)$ are arbitrary constants, real or complex.

(iv) The series occuring on the right hand side of (3.1) is absolutely convergent.

Proof:

To prove (3.1), we first express the general classes of polynomials and the I-function of two variable occurring on the left hand side of (3.1) in series form given by (1.3), (1.8) and (2.4) interchange the orders of summations and integration (which is permissible under the conditions stated). Now, we express the I-function in Mellin-Barnes contour representation given by (1.1), interchange the orders of θ and ξ, integrals and evaluate the θ integral with the help of the following known result Mac Robert [8]).

$$\int_0^{\pi/2} e^{i(\alpha+\beta)\theta}(sin\theta)^{\alpha-1}(cos\theta)^{\beta-1}d\theta = exp(i\pi\alpha/2)\frac{\Gamma\alpha\Gamma\beta}{\Gamma\alpha+\beta} \tag{3.2}$$

Finally we interpret the result thus obtained with the help of (1.1) and easily arrive at the desired result (3.1)

4. The Main Integral

The following double and triple integral relations will he established in this section.

$$\int_0^\infty\int_0^\infty x^{\beta-1}y^{\alpha-1}(x^2+y^2)^{1-(\alpha+\beta)/2}exp\{i(\alpha+\beta)tan^{-1}\left(\frac{y}{x}\right) - \mu(x^2+y^2)^{1/2}\}f(x^2+y^2)$$

$$.\mathbf{S}^m_n[y_1\; exp\{i(\lambda+\omega)tan^{-1}\left(\frac{y}{x}\right)\}y^\lambda x^\omega(x^2+y^2)^{\omega'-(\omega+\lambda)/2}]$$

$$.\mathbf{S}^{\alpha,\beta,0}_n[y_2\; exp\{i(\zeta+\eta)tan^{-1}\left(\frac{y}{x}\right)\}y^\zeta x^\eta(x^2+y^2)^{\rho-(\zeta+\eta)/2}; r', q, A, B, k, l]$$

$$.\mathbf{H}'[a\,exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}y^\sigma x^\rho(x^2+y^2)^{-(\sigma+\rho)/2}, b\,exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}y^\sigma x^\rho(x^2+y^2)^{-(\sigma+\rho)/2}]$$

$$.\mathbf{I}[z_1\; exp\{i(\sigma_1+\delta_1)tan^{-1}\left(\frac{y}{x}\right)\}\; x^{\delta_1}y^{\sigma_1}(x^2+y^2)^{\rho_1-(\sigma_1+\delta_1)/2}]$$

$$= \frac{1}{2}\sum_{k=0}^{[\frac{n}{m}]}\sum_{M'=0}^{\infty}\sum_{i,\delta,\alpha,p}\frac{(-n)_{mk}}{k!M'!}y_1^{k_1}y_2^L(-a)^{M'}\phi(i,\delta,t',p)\phi(M')A_{n,k}\,exp\,\{\frac{i\pi(\alpha+\lambda k+\zeta L+\sigma M'}{2}\}$$

$$\int_0^\infty t^{\omega' k+\rho L}e^{-\mu vt}I^{m,n+2}_{p_i+2,q_i+2;r}\left[z_1\; t^{\rho_1}exp(i\pi\sigma_1/2) \;\middle|\; \begin{matrix}(1-\alpha-\lambda k-\zeta L-\sigma M',\sigma_1)\\ \\ B^*\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\beta-\omega k-\eta L-\rho M:\delta_1)A^* \\ \\ (1-\alpha-\beta-(\lambda+\omega)k-(\zeta+\eta)L-(\sigma+\rho)M',\sigma_1+\delta_1)\end{matrix}\right] f(t)dt \qquad (4.1)$$

and

$$\int_0^\infty\int_0^\infty\int_0^\infty x^{\beta-1}y^{\alpha-1}(x^2+y^2)^{(1-\alpha+\beta)/2}exp\{i(\alpha+\beta)tan^{-1}\left(\frac{y}{x}\right)-\mu(x^2+y^2+z^2)^{1/2}\}f(x^2+y^2+z^2)$$

$$.g[tan^{-1}\frac{(x^2+y^2)^{1/2}}{z}]\mathbf{S}_n^m[y_1\ (x^2+y^2+z^2)^{\omega'}exp\{i(\lambda+\omega)tan^{-1}\left(\frac{y}{x}\right)\}y^\lambda x^\omega(x^2+y^2)^{-(\lambda+\omega)/2}$$

$$.\mathbf{S}_n^{\alpha,\beta,0}[y_2\ (x^2+y^2+z^2)^\rho\ exp\{i(\zeta+\eta)tan^{-1}\left(\frac{y}{x}\right)\}y^\zeta x^\eta(x^2+y^2)^{-(\zeta+\eta)/2};r',q,A,B,k,l]$$

$$.\mathbf{H}'[a\,exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}y^\sigma x^\rho(x^2+y^2)^{-(\sigma+\rho)/2},b\,exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}y^\sigma x^\rho(x^2+y^2)^{-(\sigma+\rho)/2}]$$

$$.\mathbf{I}[z_1\ \ (x^2+y^2+z^2)^{\rho_1}exp\{i(\sigma_1+\delta_1)tan^{-1}\left(\frac{y}{x}\right)\}\ x^{\delta_1}y^{\sigma_1}(x^2+y^2)^{-(\sigma_1+\delta_1)/2}]$$

$$=\frac{1}{2}\sum_{k=0}^{[\frac{n}{m}]}\sum_{M'=0}^{\infty}\sum_{i,\delta,\alpha,p}\frac{(-n)_{mk}}{k!M'!}y_1^k y_2^L(-a)^{M'}\phi(i,\delta,t',p)\phi(M')A_{n,k}\,exp\,\{\frac{i\pi(\alpha+\lambda k+\zeta L+\sigma M'}{2}\}$$

$$\int_0^\infty\int_0^\infty f(u^2+v^2)g\{tan^{-1}\left(\frac{u}{v}\right)\}(u^2+v^2)^{\omega' k+\rho L}\ exp\{-\mu(u^2+v^2)\}$$

$$I_{p_i+2,q_i+1:r}^{m,n+2}\left[\ z_1\ (u^2+v^2)^{\rho_1}exp(i\pi\sigma_1/2)\ \middle|\ \begin{matrix}(1-\alpha-\lambda k-\zeta L-\sigma M',\sigma_1) \\ \\ B^*\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\beta-\omega k-\eta L-\rho M',\delta_1)\ \ A^* \\ \\ (1-\alpha-\beta-(\lambda+\omega)k-(\zeta+\eta)L-(\sigma+\rho)M',\sigma_1+\delta_1)\end{matrix}\right] du\,dv \qquad (4.2)$$

where sets of conditions (1), (ii), (iii) and (iv) mentioned with (3.1) are satisfied.

Proof :

To establish the integral relations (4.1) and (4.2) we take in (2.1)

$$\int_0^{\pi/2} h(R,\theta)d\theta = L.H.S.of(3.1) \qquad (4.3)$$

we obtain F(R) = R.H.S. of (3.1)

Now substituting the values of h(R,θ) and F(R) in (2.2) and (2.3) in succession, we obtain the integral relations (4.1) and (4.2) after little simplification.

5. Special Cases :

The integral relations (4.1) and (4.2) are quite general in nature on account of the arbitrary nature of the functions f and g and also on account of the presence of H-function of two variables, the I-function and general classes of polynomials. A very large number of (known and new) integrals can be derived as special cases. We mention below a special case of our result.

If we set $\omega = \lambda = \zeta = \eta = 0$ and max $(\sigma_1, \delta_1) = 0$ in (4.1),we get

$$\int_0^\infty\int_0^\infty x^{\beta-1}y^{\alpha-1}(x^2+y^2)^{1-(\alpha+\beta)/2}exp\{i(\alpha+\beta)tan^{-1}\left(\frac{y}{x}\right)-\mu(x^2+y^2)^{1/2}\}f(x^2+y^2)\mathbf{S}_n^m[y_1\ (x^2+y^2)^{\omega'}]$$

$$\mathbf{S}_n^{\alpha,\beta,0}[y_2\ (x^2+y^2)^{\rho}; r', q, A, B, k, l]\mathbf{H}'[a\ exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}y^{\sigma}x^{\rho}(x^2+y^2)^{-(\sigma+\rho)/2},$$

$$b\ exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}y^{\sigma}x^{\rho}(x^2+y^2)^{-(\sigma+\rho)/2}]\mathbf{I}[z_1\ (x^2+y^2)^{\rho_1}exp\{(i\pi+\sigma_1)/2]dxdy$$

$$=\frac{1}{2}\sum_{k=0}^{[\frac{n}{m}]}\sum_{M'=0}^{\infty}\sum_{i,\delta,\alpha,p}\frac{(-n)_{mk}}{k!M'!}y_1^k y_2^L(-a)^{M'}\phi(i,\delta,t',p)\phi(M')A_{n,k}\ exp\ \{\frac{i\pi(\alpha+\omega M')}{2}\}$$

$$\frac{\Gamma(\alpha+\sigma M')\Gamma(\beta+\rho M')}{\Gamma(\alpha+\beta+\sigma M'+\rho M')}\int_0^\infty t^{\omega' k+\rho L}e^{-\mu v t}I[z_1\ t^{\rho_1}exp(i\pi\sigma_1/2)]\ f(t)dt \qquad (5.1)$$

If we further put $M = 1, N = 0, K = 0, A_{0,0} = 1$ and $A = 1, B = 0, q = k = 0, l = r' = 1$ and $n = 0$ in (4.1)and (4.2) then both the polynomials reduce to unity and we arrive at a number of results similar to those considered by Srivastava et.al. [12].

6. Applications :

By suitable choosing the functions f and g in the main integral relations, a large number of interesting double and triple integrals can be evaluated. We shall however, obtain here only one double and one triple integral by way of illustration.
Thus if in (4.2), we set

$$g(t) = e^{i(\alpha+\beta)\theta}(sint)^{\alpha-1}(cost)^{\beta-1}H'[ae^{i(\sigma+\rho)t}(sint)^{\sigma}(cost)^{\rho}, b\ e^{i(\sigma+\rho)t}(sint)^{\sigma}(cost)^{\rho}] \quad (6.1)$$

We arrive at the following integral relation on making use of (5.1)

$$\int_0^\infty\int_0^\infty\int_0^\infty(xz)^{\beta-1}y^{\alpha-1}(x^2+y^2)^{\beta)/2}(x^2+y^2+x^2)^{1-(\alpha+\beta)/2}f(x^2+y^2+x^2)exp\{i(\alpha+\beta)tan^{-1}\left(\frac{y}{x}\right)$$

$$+tan^{-1}\left(\frac{(x^2+y^2)^{1/2}}{z}\right)\}\mathbf{S}_n^m[y_1(x^2+y^2+z^2)^{\omega'}\ exp\{i(\lambda+\omega)tan^{-1}\left(\frac{y}{x}\right)\}x^{\omega}y^{\lambda}(x^2+y^2)^{-(\lambda+\omega)/2}]$$

$$.\mathbf{S}_n^{\alpha,\beta,0}[y_2(x^2+y^2+z^2)^{\rho}\ exp\{i(\eta+\zeta)tan^{-1}\left(\frac{y}{x}\right)\}x^{\eta}y^{\zeta}(x^2+y^2)^{-(\zeta+\eta)/2}; r', q, A, B, k, l]$$

$$.\mathbf{H}'[a\ exp\{i(\eta+\rho)tan^{-1}\left(\frac{y}{x}\right)\}x^{\rho}y^{\sigma}(x^2+y^2)^{-(\sigma+\rho)/2}, b\ exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}x^{\rho}y^{\sigma}(x^2+y^2)^{-(\sigma+\rho)/2}]$$

$$.\mathbf{I}[z_1(x^2+y^2+z^2)^{\rho'}\ exp\{i(\sigma_1+\delta_1)tan^{-1}\left(\frac{y}{x}\right)\}\ x^{\delta_1}y^{\sigma_1}(x^2+y^2)^{\rho_1-(\sigma_1+\delta_1)/2}]dx\ dy\ dz$$

$$=\frac{1}{2}\sum_{k=0}^{[\frac{n}{m}]}\sum_{M',M''=0}^{\infty}\sum_{i,\delta,t',p}\frac{(-n)_{mk}}{k!M'!M''!}y_1^k y_2^L(-a)^{M'+M''}\phi(i,\delta,t',p)A_{n,k}\phi(M')\phi(M'')$$

$$.exp\ \{\frac{i\pi(2\alpha+\lambda k+\zeta L+\sigma M'+\sigma M''}{2}\}\frac{\Gamma(\alpha+\sigma M'')\Gamma(\beta+\rho M'')}{\Gamma(\alpha+\beta+\sigma M''+\rho M'')}$$

$$\int_0^\infty t^{\omega' k+\rho L} e^{-\mu vt} \bar{H}^{M,N+2}_{P+2,Q+1}\left[z_1\, t^{\rho_1} exp(i\pi\sigma_1/2) \;\middle|\; \begin{matrix} (1-\alpha-\lambda k-\zeta L-\sigma M',\sigma_1) \\ B^* \end{matrix}\right.$$

$$\left.\begin{matrix} (1-\beta-\omega k-\eta L-\rho M',\delta_1)\;\; A^* \\ (1-\alpha-\beta-(\lambda+\omega)k-(\zeta+\eta)L-(\sigma+\rho)M',\sigma_1+\delta_1) \end{matrix}\right] f(t)dt \qquad (6.2)$$

The conditions of validity of above result can easily be derived frome those mentioned with(3.1).

Next we take

$$f(t) = H'[ct, dt] \qquad (6.3)$$

and evaluate the t-integral occuring on the R.H.S. of (4.1) and (6.2) with the help of Gamma functions and arrive at the following multiple integrals after simplifications:

$$\int_0^\infty\int_0^\infty x^{\beta-1}y^{\alpha-1}(x^2+y^2)^{1-(\alpha+\beta)/2} exp\{i(\alpha+\beta)tan^{-1}\left(\frac{y}{x}\right)-\mu(x^2+y^2)^{1/2}\}$$

$$.\mathbf{S}_n^m[y_1\; exp\{i(\lambda+\omega)tan^{-1}\left(\frac{y}{x}\right)\}x^\omega y^\lambda (x^2+y^2)^{\omega'-(\omega+\lambda)/2}]$$

$$.\mathbf{S}_n^{\alpha,\beta,0}[y_2\; exp\{i(\eta+\zeta)tan^{-1}\left(\frac{y}{x}\right)\}x^\eta y^\zeta (x^2+y^2)^{\rho-(\zeta+\eta)/2}; r', q, A, B, k, l]$$

$$.\mathbf{H}'[a\, exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}y^\sigma x^\rho (x^2+y^2)^{-(\sigma+\rho)/2}, b\, exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}y^\sigma x^\rho (x^2+y^2)^{-(\sigma+\rho)/2}]$$

$$.\mathbf{H}'[c\,(x^2+y^2)\; d(x^2+y^2)]\mathbf{I}[z_1\; exp\{i(\sigma_1+\delta_1)tan^{-1}\left(\frac{y}{x}\right)\}\; x^{\delta_1}y^{\sigma_1}(x^2+y^2)^{\rho_1(\sigma_1+\delta_1)/2}]dxdy$$

$$=\frac{1}{2}\sum_{k=0}^{[\frac{n}{m}]}\sum_{M',M'''=0}^{\infty}\sum_{i,\delta,\alpha,p}\frac{(-n)_{mk}}{k!M'!M'''!}y_1^{k_1}y_2^L(-a)^{M'}\phi(i,\delta,t',p)\phi(M')\phi(M''')A_{n,k}(-a)^{M'}(-c)^{M'''}$$

$$(2/\mu^{2\omega_1 k+2\rho L+2M'''+2})\; exp\,\{\frac{i\pi(\alpha+\lambda k+\zeta L+\sigma M'}{2}\}$$

$$I^{m,n+3}_{p_i+3,q_i+1}\left[z_1\,\mu^{-2\rho_1} exp(i\pi\sigma_1/2) \;\middle|\; \begin{matrix} (1-\alpha-\lambda k-\zeta L-\omega M',\sigma_1), \\ B^* \end{matrix}\right.$$

$$\left.\begin{matrix} (1-\beta-\omega k-\eta L-\rho M',\sigma_1),(1-2\omega' k-2\rho L-2M''',2\rho_1),\; A^* \\ (1-\alpha-\beta-(\lambda+\omega)k-(\zeta+\eta)L-(\sigma+\rho)M',\sigma_1+\delta_1) \end{matrix}\right] \qquad (6.4)$$

and

$$\int_0^\infty\int_0^\infty\int_0^\infty (xz)^{\beta-1}y^{\alpha-1}(x^2+y^2)^{-\beta/2}(x^2+y^2+z^2)^{1-(\alpha+\beta)/2} exp\{i(\alpha+\beta)$$

$$[tan^{-1}\left(\frac{y}{x}\right)+tan^{-1}\left(\frac{(x^2+y^2)^{1/2}}{z}\right)]-\mu(x^2+y^2+z^2)^{1/2}\}$$

$$.\mathbf{S}_n^m[y_1\;(x^2+y^2+z^2)^{\omega'} exp\{i(\lambda+\omega)tan^{-1}\left(\frac{y}{x}\right)\}x^\omega y^\lambda (x^2+y^2)^{-(\lambda+\omega)/2}]$$

$$.\mathbf{S}_n^{\alpha,\beta,0}[y_2\;(x^2+y^2+z^2)^\rho\; exp\{i(\eta+\zeta)tan^{-1}\left(\frac{y}{x}\right)\}x^\eta y^\zeta (x^2+y^2)^{-(\zeta+\eta)/2}; r', q, A, B, k, l]$$

$$.\mathbf{H}'[c\,(x^2+y^2+z^2), d(x^2+y^2+z^2)].\mathbf{H}'[a\; exp\{i(\sigma+\rho)tan^{-1}\left(\frac{(x^2+y^2)^{1/2}}{z}\right)\}]$$

$$(x^2+y^2)^{\rho/2}(x^2+y^2+z^2)^{-(\sigma+\rho)}z^\rho, b\, exp\{i(\sigma+\rho)tan^{-1}\left(\frac{(x^2+y^2)^{1/2}}{z}\right)\}(x^2+y^2)^{\rho/2}(x^2+y^2+z^2)^{-(\sigma+\rho)}z^\rho].$$

$$.\mathbf{H}'[a\, exp\{i(\sigma+\rho)tan^{-1}\left(\frac{y}{x}\right)\}\, x^{\delta_1}y^{\sigma_1}(x^2+y^2)^{-(\sigma_1+\delta_1)/2}]dxdydz$$

$$= \frac{1}{2}\sum_{k=0}^{[\frac{n}{m}]}\sum_{M',M'',M'''=0}^{\infty}\sum_{i,\delta,\alpha,p}\frac{(-n)_{mk}}{k!M'!M''!M'''!}y_1^k y_2^L(-a)^{M'+M''}(-c)^{M''}\phi(i,\delta,t',p)A_{n,k}\phi(M')\phi(M'')\phi(M''')$$

$$exp\{\frac{i\pi(\alpha+\lambda k+\zeta L+\sigma M'+\sigma M''}{2}\}\frac{\Gamma(\alpha+\sigma M'')\Gamma(\beta+\rho M'')}{\Gamma(\alpha+\beta+\sigma M''+\rho M'')}(2\mu^{2/\omega_1 k+2\rho L+2M'''+2})$$

$$\bar{H}^{m,n+3}_{p_i+3,q_i+1}\left[z_1\,\mu^{-2\rho_1}exp(i\pi\sigma_1/2)\ \middle|\ \begin{matrix}(1-\alpha-\lambda k-\zeta L-\sigma M',\sigma_1),\delta_1)\\ B^*\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\beta-\omega k-\eta L-\rho M',(-1-\omega' k-2\rho L-2M''',2\rho_1)\ A^*\\ (1-\alpha-(\lambda+\omega)k-(\zeta+\eta)L-(\sigma+\rho)M',\sigma_1+\delta_1)\end{matrix}\right] \tag{6.5}$$

The result (6.4)and (6.5)are valid under the following conditions:

(i) $Re(\alpha)>0, Re(\beta)>0, \sigma_1>0, \delta_1>0, \rho_1>0$

(ii) $Re(2\omega' k+2\rho L+2M'''+2\rho_1\xi_1+2j)>0$ and sets of conditions (ii), (iii) and (iv) mentioned with (3.1) are satisfied.

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