Palestine Journal of Mathematics

Vol. 8(2)(2019) , 114–126

COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS SATISFYING A GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE

Rashwan A. Rashwan and Hasanen A. Hammad

Communicated by Ayman Badawi

MSC 2010 Classifications: 47H10.

Keywords and phrases: Complete metric space, Lebesgue integrable mapping, monotonically decreasing function, weakly compatible mappings.

Abstract. In this paper, we establish common fixed point theorems for weakly compatible mappings satisfying a general contractive condition of integral form in rational setting. Some examples to justify our results are given. The mapping involved here generalized various type of contractive mapping of integral inequality.

1 Introduction

The first important result on fixed points for contractive-type mapping was the well known Banach's contraction principle appeared in explicit form in Banach's thesis in 1922, where it was used to establish the existence of a solution for an integral equation [1]. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. It widely considered as the source of metric fixed point theory and its significance lies in its vast applicability in a number of branches of mathematics. There are many generalizations of Banach's contraction mapping principle in the literature [3, 6, 8]. In the general setting of complete metric space this theorem runs as follows:

Theorem 1.1. *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$d(Tx, Ty) \leq cd(x, y). \tag{1.1}$$

where $c \in [0, 1)$ and $x, y \in X$, Then T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

After this classical result, many theorems dealing with maps satisfying various types of contractive inequalities have been established (see [4, 5, 7], [15, 16, 17, 18]). The intrested reader who wants to read about this matter is recommended to go deep into the survey articles by Rhoades [13, 12, 14].

In 2002, A. Branciari [2] analyzed the existence of fixed point for mapping defined on a complete metric space satisfying a general contractive condition of integral type in the following theorem:

Theorem 1.2. *Let (X, d) be a complete metric space, $c \in [0, 1)$ and let $T : X \to X$ be a mapping such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt \ \text{ for all } x, y \in X, \tag{1.2}$$

where $\varphi : R^+ \to R^+$ be a Lebesgue-integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. Then T has a unique fixed point $a \in X$, such that for each $x \in X$, $lim_{n\to\infty} T^n x = a$.

After Theorem 1.2, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties (see [10, 11]). Affine work has been done by Rhoades [14] extending the result of Theorem 1.2 by replacing the condition (1.2) by the following condition:

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{m(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\}} \varphi(t)dt, \tag{1.3}$$

for each $c \in [0,1)$ and $x, y \in X$.

In 1982, Sessa [16] introduced the notion of weak commutativity which generalized the notion of commutativity as follows:

Definition 1.3. The self mappings f and g of a metric space X are said to be weakly commuting if

$$d(fgx, gfy) \le d(gx, fy) \text{ for all } x, y \in X,$$

Jungck [9] introduced more a generalized commuting mappings, called compatible mappings as the following:

Definition 1.4. Two self mappings f and g of a metric space X are called compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 1.5. [9] Let f and g are two mappings from a metric space (X, d) into itself, f and g are called weakly compatible if they commute at there coincidence point, i.e., $fx = gx$ for some $x \in X \Rightarrow fgx = gfx$.

Definition 1.6. [9] The mappings f and g of a metric space X are called commuting if

$$fgx = gfx \; \forall x \in X.$$

Definition 1.7. [6] Let f and g are two mappings on a set X, if $fx = gx$ for some $x \in X$, then x is called coincidence point of f and g.

Definition 1.8. A function is called increasing on any interval if the function value increases as the independent value increases. That is if $x_1 > x_2$, then $f(x_1) > f(x_2)$, on the other hand, a function is called decreasing on an interval if the function value decreases as the independent value increases. That is if $x_1 > x_2$, then $f(x_1) < f(x_2)$. A function increasing or decreasing is called monotonicity on its domain.

The aim of this paper is to generalize some mixed type of contraction conditions to the mapping and then two mappings and then four compatible mappings satisfying a general contractive condition of integral type satisfying a rational inequality.

2 Main Results

We begin with the following theorem:

Theorem 2.1. *Let f be a self mapping of complete metric space X satisfying the following condition:*

$$\begin{aligned}\int_0^{d(fx,fy)} \varphi(t)dt \le\; & \alpha_1 \int_0^{d(x,y)} \varphi(t)dt + \alpha_2 \int_0^{d(x,fx)} \varphi(t)dt + \alpha_3 \int_0^{d(y,fy)} \varphi(t)dt + \alpha_4 \int_0^{d(x,fy)} \varphi(t)dt \\ & + \alpha_5 \int_0^{\max\{d(y,fx),d(x,fy)\}} \varphi(t)dt + \alpha_6 \int_0^{\frac{d^3(x,fx)+d^3(y,fy)}{1+d^2(x,fx)+d^2(y,fy)}} \varphi(t)dt + \alpha_7 \int_0^{\frac{d^2(x,fy)+d^2(y,fx)}{1+d(x,fy)+d(y,fx)}} \varphi(t)dt,\end{aligned} \tag{2.1}$$

for each $x, y \in X$, $x \neq y$ with $\alpha_i : (0,1) \to [0,1)$ is monotonically decreasing functions, satisfying $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7 < 1$, where $\varphi : R^+ \to R^+$ be a Lebesgue-integrable mapping which is summable on each compact subset of R^+ such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$, then f has a unique fixed point $z \in X$, such that for each $x \in X$, $lim_{n\to\infty} T^n x = z$.

Proof. For any arbitrary $x_\circ \in X$ there is x_1 in X such that $x_1 = fx_\circ$. Proceeding the same way, we construct a sequence $\{x_n\}$ such that $x_{n+1} = fx_n$, for each integer $n = 0, 1, 2, ..$. From (2.1), we have

$$\begin{aligned}
\int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt &= \int_0^{d(fx_n,fx_{n+1})} \varphi(t)dt \leq \alpha_1 \int_0^{d(x_n,x_{n+1})} \varphi(t)dt + \alpha_2 \int_0^{d(x_n,fx_n)} \varphi(t)dt \\
&\quad + \alpha_3 \int_0^{d(x_{n+1},fx_{n+1})} \varphi(t)dt + \alpha_4 \int_0^{d(x_n,fx_{n+1})} \varphi(t)dt \\
&\quad + \alpha_5 \int_0^{\max\{d(x_{n+1},fx_n),d(x_n,fx_{n+1})\}} \varphi(t)dt + \alpha_6 \int_0^{\frac{d^3(x_n,fx_n)+d^3(x_{n+1},fx_{n+1})}{1+d^2(x_n,fx_n)+d^2(x_{n+1},fx_{n+1})}} \varphi(t)dt \\
&\quad + \alpha_7 \int_0^{\frac{d^2(x_n,fx_{n+1})+d^2(x_{n+1},fx_n)}{1+d(x_n,fx_{n+1})+d(x_{n+1},fx_n)}} \varphi(t)dt \\
&\leq \alpha_1 \int_0^{d(x_n,x_{n+1})} \varphi(t)dt + \alpha_2 \int_0^{d(x_n,x_{n+1})} \varphi(t)dt + \alpha_3 \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt \\
&\quad + \alpha_4 \int_0^{d(x_n,x_{n+2})} \varphi(t)dt + \alpha_5 \int_0^{\max\{d(x_{n+1},x_{n+1}),d(x_n,x_{n+2})\}} \varphi(t)dt \\
&\quad + \alpha_6 \int_0^{\frac{d^3(x_n,x_{n+1})+d^3(x_{n+1},x_{n+2})}{1+d^2(x_n,x_{n+1})+d^2(x_{n+1},x_{n+2})}} \varphi(t) + \alpha_7 \int_0^{\frac{d^2(x_n,x_{n+2})+d^2(x_{n+1},x_{n+1})}{1+d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})}} \varphi(t)dt \\
&\leq (\alpha_1+\alpha_2+\alpha_4+\alpha_5) \int_0^{d(x_n,x_{n+1})} \varphi(t)dt + (\alpha_3+\alpha_4+\alpha_5) \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt \\
&\quad + \alpha_6 \int_0^{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})} \varphi(t)dt + \alpha_7 \int_0^{d(x_n,x_{n+2})} \varphi(t)dt.
\end{aligned}$$

This leads to

$$\int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt \leq \left(\frac{\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7}{1-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}\right) \int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq q(t) \int_0^{d(x_n,x_{n+1})} \varphi(t)dt,$$

where $q(t)$ is monotonically decreasing functions such $0 \leq q(t) < 1$.
Thus by continuing this way, we have

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq q^n(t) \int_0^{d(x_\circ,x_1)} \varphi(t)dt,$$

taking the limit as $n \to \infty$, we get $\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0$, hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.2}$$

Now we show that $\{x_n\}$ is a Cauchy sequence. Suppose the contrary. Then there exists $\epsilon > 0$ and subsequences $\{m(p)\}$ and $\{n(p)\}$ such that, $m(p) < n(p) \leq m(p+1)$, for each $p \in \mathbb{N}$,

$$d(x_{m(p)}, x_{n(p)}) > \epsilon, \;\; d(x_{m(p)}, x_{n(p)-1}) \leq \epsilon. \tag{2.3}$$

Now,

$$d(x_{m(p)-1}, x_{n(p)-1}) \leq d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1}) < d(x_{m(p)-1}, x_{m(p)}) + \epsilon. \tag{2.4}$$

Hence from (2.2) and (2.4), we can write

$$\lim_{n\to\infty} \int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)dt \leq \int_0^{\epsilon} \varphi(t)dt, \tag{2.5}$$

using (2.1), (2.3), (2.4) and (2.5), we get

$$\int_0^{\epsilon} \varphi(t)dt \leq \int_0^{d(x_{m(p)},x_{n(p)})} \varphi(t)dt \leq q(t) \int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)dt \leq q(t) \int_0^{\epsilon} \varphi(t)dt,$$

which is a contradiction since $q(t) < 1$. Therefore $\{x_n\}$ is a Cauchy sequence, hence it convergent to the point z or $\lim_{n\to\infty} x_n = z$.
Again from (2.1), we have

$$\begin{aligned}
\int_0^{d(fz,fx_{n+1})} \varphi(t)dt \;\; \leq \;\; & \alpha_1 \int_0^{d(z,x_{n+1})} \varphi(t)dt + \alpha_2 \int_0^{d(z,fz)} \varphi(t)dt + \alpha_3 \int_0^{d(x_{n+1},fx_{n+1})} \varphi(t)dt + \alpha_4 \int_0^{d(z,fx_{n+1})} \varphi(t)dt \\
& +\alpha_5 \int_0^{\max\{d(x_{n+1},fz),d(z,fx_{n+1})\}} \varphi(t)dt + \alpha_6 \int_0^{\frac{d^3(z,fz)+d^3(x_{n+1},fx_{n+1})}{1+d^2(z,fz)+d^2(x_{n+1},fx_{n+1})}} \varphi(t)dt \\
& +\alpha_7 \int_0^{\frac{d^2(z,fx_{n+1})+d^2(x_{n+1},fz)}{1+d(z,fx_{n+1})+d(x_{n+1},fz)}} \varphi(t)dt,
\end{aligned}$$

taking the limit in the both sides as $n \to \infty$, we get

$$\int_0^{d(fz,z)} \varphi(t)dt \leq (\alpha_2 + \alpha_5 + \alpha_6 + \alpha_7) \int_0^{d(z,fz)} \varphi(t)dt = q(t) \int_0^{d(z,fz)} \varphi(t)dt,$$

since $q(t)$ is monotonically decreasing functions satisfies $q(t) \in [0, 1)$, then $\lim_{n\to\infty} \int_0^{d(fz,z)} \varphi(t)dt = 0$ this implies that $\lim_{n\to\infty} d(fz, z) = 0$ or $z = fz$, hence z is a fixed point of f.

Next suppose that $(w \neq z)$ be another fixed point of f, then from (2.1), we get

$$\begin{aligned}
\int_0^{d(z,w)} \varphi(t)dt &= \int_0^{d(fz,fw)} \varphi(t)dt \leq \alpha_1 \int_0^{d(z,w)} \varphi(t)dt + \alpha_2 \int_0^{d(z,fz)} \varphi(t)dt \\
&+\alpha_3 \int_0^{d(w,fw)} \varphi(t)dt + \alpha_4 \int_0^{d(z,fw)} \varphi(t)dt + \alpha_5 \int_0^{\max\{d(w,fz),d(z,fw)\}} \varphi(t)dt \\
&+\alpha_6 \int_0^{\frac{d^3(z,fz)+d^3(w,fw)}{1+d^2(z,fz)+d^2(w,fw)}} \varphi(t)dt + \alpha_7 \int_0^{\frac{d^2(z,fw)+d^2(w,fz)}{1+d(z,fw)+d(w,fz)}} \varphi(t)dt, \\
&\leq (\alpha_1+\alpha_4+\alpha_5+\alpha_7) \int_0^{d(z,w)} \varphi(t)dt \leq q(t) \int_0^{d(z,w)} \varphi(t)dt < \int_0^{d(z,w)} \varphi(t)dt
\end{aligned}$$

which is a contradiction, so $\int_0^{d(z,w)} \varphi(t)dt = 0$, which leads to $d(z,w) = 0$ or $z = w$. Therefore a fixed point is unique □

Remark 2.2. (i) Letting $\varphi(t) = 1$ over R^+, the contractive condition of integral type transforms into a general contractive condition not involving the integral.

(ii) If we take $\alpha_1 = a \in [0,1)$ and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = 0$, gives Branciari mapping of integral type [2].

To generalize Theorem 2.1 in two mappings, we give the following theorem:

Theorem 2.3. *Let f and g be self-mappings on complete metric space X satisfy the following condition:*

$$\begin{aligned}
\int_0^{d(fx,gy)} \varphi(t)dt &\leq \alpha_1 \int_0^{d(x,y)} \varphi(t)dt + \alpha_2 \int_0^{d(x,fx)} \varphi(t)dt + \alpha_3 \int_0^{d(y,gy)} \varphi(t)dt + \alpha_4 \int_0^{d(x,gy)} \varphi(t)dt \\
&+\alpha_5 \int_0^{\max\{d(y,fx),d(x,gy)\}} \varphi(t)dt + \alpha_6 \int_0^{\frac{d^3(x,fx)+d^3(y,gy)}{1+d^2(x,fx)+d^2(y,gy)}} \varphi(t)dt + \alpha_7 \int_0^{\frac{d^2(x,gy)+d^2(y,fx)}{1+d(x,gy)+d(y,fx)}} \varphi(t)dt,
\end{aligned} \tag{2.6}$$

for each $x, y \in X$, $x \neq y$ with $\alpha_i : (0,1) \to [0,1)$ is monotonically decreasing functions, satisfying $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 < 1$, where $\varphi : R^+ \to R^+$ be a Lebesgue-integrable mapping which is summable on each compact subset of R^+ such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$, then f and g have a unique common fixed point $z \in X$.

Proof. Let $x_\circ$ be an arbitrary point of X. Define $x_{2n+1} = fx_n$ and $x_{2n+2} = gx_{n+1}$ then from

(2.6), we have

$$\begin{aligned}
\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt &= \int_0^{d(fx_{2n},gx_{2n+1})} \varphi(t)dt \le \alpha_1 \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt + \alpha_2 \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt \\
&+\alpha_3 \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt + \alpha_4 \int_0^{d(x_{2n},x_{2n+2})} \varphi(t)dt \\
&+\alpha_5 \int_0^{\max\{d(x_{2n+1},x_{2n+1}),d(x_{2n},x_{2n+2})\}} \varphi(t)dt \\
&+\alpha_6 \int_0^{\frac{d^3(x_{2n},x_{2n+1})+d^3(x_{2n+1},x_{2n+2})}{1+d^2(x_{2n},x_{2n+1})+d^2(x_{2n+1},x_{2n+2})}} \varphi(t)dt + \alpha_7 \int_0^{\frac{d^2(x_{2n},x_{2n+2})+d^2(x_{2n+1},x_{2n+1})}{1+d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}} \varphi(t)dt \\
&\le (\alpha_1+\alpha_2+\alpha_4+\alpha_5) \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt + (\alpha_3+\alpha_4+\alpha_5) \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \\
&+\alpha_6 \int_0^{d(x_{2n},x_{2n+1})+d(x_{2n+2},x_{2n+2})} \varphi(t)dt + \alpha_7 \int_0^{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})} \varphi(t)dt.
\end{aligned}$$

This yields,

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \le \left(\frac{\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7}{1-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}\right) \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt \le q(t) \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt,$$

thus in general, for all $n = 0, 1, 2, ..$

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le q^n(t) \int_0^{d(x_\circ,x_1)} \varphi(t)dt,$$

taking the limit as $n \to \infty$ and definition $q(t)$, we have $\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0$, hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

By the same manner proof of Theorem 2.1, we can show that $\{x_n\}$ is a Cauchy sequence, hence it convergent to the point z i.e., $\lim_{n\to\infty} x_n = z$.
Again from (2.6), we get

$$\begin{aligned}
\int_0^{d(fz,fx_{2n+1})} \varphi(t)dt &\le \alpha_1 \int_0^{d(z,x_{2n+1})} \varphi(t)dt + \alpha_2 \int_0^{d(z,fz)} \varphi(t)dt + \alpha_3 \int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt + \alpha_4 \int_0^{d(z,x_{2n+2})} \varphi(t)dt \\
&+\alpha_5 \int_0^{\max\{d(x_{2n+1},fz),d(z,x_{2n+2})\}} \varphi(t)dt + \alpha_6 \int_0^{\frac{d^3(z,fz)+d^3(x_{2n+1},x_{2n+2})}{1+d^2(z,fz)+d^2(x_{2n+1},x_{2n+2})}} \varphi(t)dt \\
&+\alpha_7 \int_0^{\frac{d^2(z,x_{2n+2})+d^2(x_{2n+1},fz)}{1+d(z,x_{2n+2})+d(x_{n+1},fz)}} \varphi(t)dt,
\end{aligned}$$

taking the limit in the above inequality, one can write

$$\int_0^{d(fz,z)} \varphi(t)dt \leq (\alpha_2 + \alpha_5 + \alpha_6 + \alpha_7) \int_0^{d(z,fz)} \varphi(t)dt,$$

since $\sum_{i=1}^{5} \alpha_i < 1$, then we have $\lim_{n\to\infty} \int_0^{d(fz,z)} \varphi(t)dt = 0$, implies that $\lim_{n\to\infty} d(fz,z) = 0$ or $z = fz$. Similarly it can be show that $gz = z$, so f and g have a common fixed point and the uniqueness is easy. □

In the following theorem we obtain common fixed point result for four mappings by using weakly compatible concept:

Theorem 2.4. *Let (X,d) be a complete metric space and $\alpha_i : (0,\infty) \to [0,1)$ is monotonically decreasing functions satisfying $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 < 1$ and f, g, Q and P are four self mappings in X satisfies the following conditions:*
(i) $f(X) \subseteq P(X)$ and $g(X) \subseteq Q(X)$,
(ii) the pairs (g,P) and (f,Q) are weakly compatible,
(iii)

$$\begin{aligned} \int_0^{d(fx,gy)} \varphi(t)dt \leq\ & \alpha_1 \int_0^{d(Px,Qy)} \varphi(t)dt + \alpha_2 \int_0^{d(Px,fx)} \varphi(t)dt + \alpha_3 \int_0^{d(Qy,gy)} \varphi(t)dt + \alpha_4 \int_0^{d(fx,Qy)} \varphi(t)dt \\ & + \alpha_5 \int_0^{\max\{d(Qy,fx),d(Px,gy)\}} \varphi(t)dt + \alpha_6 \int_0^{\frac{d^3(Px,fx)+d^3(Qy,gy)}{1+d^2(Px,fx)+d^2(Qy,gy)}} \varphi(t)dt \\ & + \alpha_7 \int_0^{\frac{d^2(Px,gy)+d^2(Qy,fx)}{1+d(Px,gy)+d(Qy,fx)}} \varphi(t)dt, \end{aligned} \tag{2.7}$$

for each $x, y \in X$, $x \neq y$ with $\alpha_i : (0,1) \to [0,1)$ is monotonically decreasing functions, satisfying $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 < 1$, where $\varphi : R^+ \to R^+$ be a Lebesgue-integrable mapping which is summable on each compact subset of R^+ such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$, then f, g, Q and P have a unique common fixed point $z \in X$.

Proof. Let $x_\circ$ be an arbitrary point of X and define the sequence $\{y_n\}$ in X such that

$$y_n = gx_n = Qx_{n+1} \text{ and } y_{n+1} = fx_{n+1} = Px_{n+2}. \tag{2.8}$$

Applying (2.7) and (2.8), we get

$$\begin{aligned}
\int_0^{d(y_{n+1},y_n)} \varphi(t)dt &= \int_0^{d(fx_{n+1},gx_n)} \varphi(t)dt \leq \alpha_1 \int_0^{d(Px_{n+1},Qx_n)} \varphi(t)dt + \alpha_2 \int_0^{d(Px_{n+1},fx_{n+1})} \varphi(t)dt \\
&+\alpha_3 \int_0^{d(Qx_n,gx_n)} \varphi(t)dt + \alpha_4 \int_0^{d(Qx_n,fx_{n+1})} \varphi(t)dt \\
&+\alpha_5 \int_0^{\max\{d(Px_{n+1},gx_n),d(Qx_n,fx_{n+1})\}} \varphi(t)dt \\
&+\alpha_6 \int_0^{\frac{d^3(Px_{n+1},fx_{n+1})+d^3(Qx_n,gx_n)}{1+d^2(Px_{n+1},fx_{n+1})+d^2(Qx_n,gx_n)}} \varphi(t)dt + \alpha_7 \int_0^{\frac{d^2(Px_{n+1},gx_n)+d^2(Qx_n,fx_{n+1})}{1+d(Px_{n+1},gx_n)+d(Qx_n,fx_{n+1})}} \varphi(t)dt \\
&\leq \alpha_1 \int_0^{d(y_n,y_{n-1})} \varphi(t)dt + \alpha_2 \int_0^{d(y_n,y_{n+1})} \varphi(t)dt + \alpha_3 \int_0^{d(y_n,y_{n-1})} \varphi(t)dt \\
&+\alpha_4 \int_0^{d(y_{n-1},y_{n+1})} \varphi(t)dt + \alpha_5 \int_0^{\max\{d(y_n,y_n),d(y_{n-1},y_{n+1})\}} \varphi(t)dt \\
&+\alpha_6 \int_0^{\frac{[d(y_n,y_{n+1})+d(y_{n-1},y_n)].[d^2(y_n,y_{n+1})+d^2(y_{n-1},y_n)]}{1+d^2(y_n,y_{n+1})+d^2(y_{n-1},y_n)}} \varphi(t) + \alpha_7 \int_0^{\frac{d^2(y_n,y_n)+d^2(y_{n-1},y_{n+1})}{1+d(y_n,y_n)+d(y_{n-1},y_{n+1})}} \varphi(t)dt.
\end{aligned}$$

By the same calculations above, we get

$$\begin{aligned}
\int_0^{d(y_{n+1},y_n)} \varphi(t)dt &\leq \left(\frac{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}{1-\alpha_2-\alpha_4-\alpha_5-\alpha_6-\alpha_7}\right) \int_0^{d(y_n,y_{n-1})} \varphi(t)dt \\
&\leq q(t) \int_0^{d(y_n,y_{n-1})} \varphi(t)dt \leq q^n(t) \int_0^{d(y_1,y_\circ)} \varphi(t)dt \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} d(y_{n+1},y_n) = 0. \tag{2.9}$$

Now we show that $\{y_n\}$ is a Cauchy sequence in X. Let $m > n$ where $m,n \in N$, from (2.7) and (2.8), we have

$$\begin{aligned}
\int_0^{d(y_n,y_m)} \varphi(t)dt &= \int_0^{d(fx_n,gx_m)} \varphi(t)dt \leq \alpha_1 \int_0^{d(y_{n-1},y_{m-1})} \varphi(t)dt + \alpha_2 \int_0^{d(y_{n-1},y_n)} \varphi(t)dt + \alpha_3 \int_0^{d(y_{m-1},y_m)} \varphi(t)dt \\
&+\alpha_4 \int_0^{d(y_{m-1},y_n)} \varphi(t)dt + \alpha_5 \int_0^{\max\{d(y_{n-1},y_m),d(y_{m-1},y_n)\}} \varphi(t)dt \\
&+\alpha_6 \int_0^{\frac{d^3(y_{n-1},y_n)+d^3(y_{m-1},y_m)]}{1+d^2(y_{n-1},y_n)+d^2(y_{m-1},y_m)}} \varphi(t) + \alpha_7 \int_0^{\frac{d^2(y_{n-1},y_m)+d^2(y_{m-1},y_n)}{1+d(y_{n-1},y_m)+d(y_{m-1},y_n)}} \varphi(t)dt.
\end{aligned}$$

By the same way, for $m > n$ there are two cases:
Case (i). When $d(y_{n-1}, y_m) > d(y_{m-1}, y_n)$, we can write

$$\int_0^{d(y_n,y_m)} \varphi(t)dt \le \left(\frac{\alpha_1+\alpha_2+\alpha_5+\alpha_6+\alpha_7}{1-\alpha_1-\alpha_4-\alpha_5-2\alpha_7}\right) \int_0^{d(y_{n-1},y_n)} \varphi(t)dt + \left(\frac{\alpha_1+\alpha_3+\alpha_4+\alpha_6+\alpha_7}{1-\alpha_1-\alpha_4-\alpha_5-2\alpha_7}\right) \int_0^{d(y_{m-1},y_m)} \varphi(t)dt.$$

Case (ii). When $d(y_{n-1}, y_m) < d(y_{m-1}, y_n)$, one can write

$$\int_0^{d(y_n,y_m)} \varphi(t)dt \le \left(\frac{\alpha_1+\alpha_2+\alpha_6+\alpha_7}{1-\alpha_1-\alpha_4-\alpha_5-2\alpha_7}\right) \int_0^{d(y_{n-1},y_n)} \varphi(t)dt + \left(\frac{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}{1-\alpha_1-\alpha_4-\alpha_5-2\alpha_7}\right) \int_0^{d(y_{m-1},y_m)} \varphi(t)dt.$$

From two cases and taking the limit as $n, m \to \infty$, we obtain

$$\int_0^{d(y_n,y_m)} \varphi(t)dt \le (q^{n-1}(t) + q^{m-1}(t)) \int_0^{d(y_\circ,y_1)} \varphi(t)dt \to 0.$$

Hence $\{y_n\}$ is a Cauchy sequence in complete metric space X, so it is convergent to the point z i.e., $\lim_{n\to\infty} y_n = z$,

$$z = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Qx_{n+1} = \lim_{n\to\infty} fx_{n+1} = \lim_{n\to\infty} Px_{n+2}. \tag{2.10}$$

Since $f(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$. If $z \neq gu$, then from (2.7), we get

$$\begin{aligned}\int_0^{d(y_{n+1},gu)} \varphi(t)dt &= \int_0^{d(fx_{n+1},gu)} \varphi(t)dt \le \alpha_1 \int_0^{d(Px_{n+1},Qu)} \varphi(t)dt + \alpha_2 \int_0^{d(Px_{n+1},fx_{n+1})} \varphi(t)dt \\ &+\alpha_3 \int_0^{d(Qu,gu)} \varphi(t)dt + \alpha_4 \int_0^{d(Qu,fx_{n+1})} \varphi(t)dt + \alpha_5 \int_0^{\max\{d(Px_{n+1},gu),d(Qu,fx_{n+1})\}} \varphi(t)dt \\ &+\alpha_6 \int_0^{\frac{d^3(Px_{n+1},fx_{n+1})+d^3(Qu,gu)}{1+d^2(Px_{n+1},fx_{n+1})+d^2(Qu,gu)}} \varphi(t)dt + \alpha_7 \int_0^{\frac{d^2(Px_{n+1},gu)+d^2(Qu,fx_{n+1})}{1+d(Px_{n+1},gu)+d(Qu,fx_{n+1})}} \varphi(t)dt.\end{aligned}$$

Taking the limit in the above inequality and using (2.10), we have

$$\begin{aligned}\int_0^{d(z,gu)} \varphi(t)dt &\le \alpha_1 \int_0^{d(z,Qu)} \varphi(t)dt + \alpha_2 \int_0^{d(z,z)} \varphi(t)dt + \alpha_3 \int_0^{d(Qu,gu)} \varphi(t)dt + \alpha_4 \int_0^{d(Qu,z)} \varphi(t)dt \\ &+\alpha_5 \int_0^{\max\{d(z,gu),d(Qu,z)\}} \varphi(t)dt + \alpha_6 \int_0^{\frac{d^3(z,z)+d^3(Qu,gu)}{1+d^2(z,z)+d^2(Qu,gu)}} \varphi(t)dt + \alpha_7 \int_0^{\frac{d^2(z,gu)+d^2(Qu,z)}{1+d(z,gu)+d(Qu,z)}} \varphi(t)dt. \\ &\le \left(\frac{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}{1-\alpha_3-\alpha_6-\alpha_7}\right) \int_0^{d(z,Qu)} \varphi(t)dt = q(t) \int_0^{d(z,Qu)} \varphi(t)dt.\end{aligned}$$

We have a contradiction again, therefore $z = gu$. So$z = Pu = gu$.
Hence u is a coincidence point of P and g. Since (g, P) is weakly compatible, then

$$Pgu = gPu \Rightarrow Pz = gz. \tag{2.11}$$

Similarly, $g(X) \subseteq Q(X)$, there exists a point $v \in X$, such that $z = Qv$. Then from (2.7) and applied the same above steps, we can find that f$v = z$, so $z = fv = Qv$.
Hence v is a coincidence point of f and Q. Also the pair (f, Q) are weakly compatible, then

$$fQu = Qfu \Rightarrow fz = Qz. \tag{2.12}$$

Now we show that z is a fixed point of g, by using (2.7), we get

$$\begin{aligned}\int_0^{d(z,gz)} \varphi(t)dt &= \int_0^{d(fx_{n+1},gz)} \varphi(t)dt \le \alpha_1 \int_0^{d(z,Qz)} \varphi(t)dt + \alpha_2 \int_0^{d(z,z)} \varphi(t)dt + \alpha_3 \int_0^{d(Qz,gz)} \varphi(t)dt \\ &\quad + \alpha_4 \int_0^{d(Qz,z)} \varphi(t)dt + \alpha_5 \int_0^{\max\{d(z,gz),d(Qz,z)\}} \varphi(t)dt \\ &\quad + \alpha_6 \int_0^{\frac{d^3(z,z)+d^3(Qz,gz)}{1+d^2(z,z)+d^2(z,z)}} \varphi(t)dt + \alpha_7 \int_0^{\frac{d^2(z,gz)+d^2(Qz,z)}{1+d(z,gz)+d(Qz,z)}} \varphi(t)dt. \\ &\le \left(\frac{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7}{1 - \alpha_3 - \alpha_6 - \alpha_7}\right) \int_0^{d(z,Qz)} \varphi(t)dt = q(t) \int_0^{d(z,Qz)} \varphi(t)dt,\end{aligned}$$

also by the notion on $q(t)$, we get a contradiction a gain. So $\int_0^{d(z,gz)} \varphi(t)dt = 0$ and $d(z, gz) = 0$ or $z = gz$, also from (2.11), we get

$$Pz = gz = z. \tag{2.13}$$

By the same way we can show that z is a fixed point of f, so from (2.12), we have

$$Qz = fz = z. \tag{2.14}$$

From (2.13) and (2.14), we obtain that $Pz = gz = fz = Qz = z$. Therefore z is a common fixed point of f, g, Q and P. For uniqueness, it is simple. □

If we put $f = g$ in Theorem 2.4, we have the following result:

Corollary 2.5. *Let (X, d) be a complete metric space and $\alpha_i : (0, \infty) \to [0, 1)$ is monotonically decreasing functions satisfying $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 < 1$ and f, Q, P are self mappings in X satisfies the following conditions:*
(i) $f(X) \subseteq P(X)$ and $f(X) \subseteq Q(X)$,
(ii) the pairs (f, P) and (f, Q) are weakly compatible,
(iii)

$$\begin{aligned}\int_0^{d(fx,fy)} \varphi(t)dt &\le \alpha_1 \int_0^{d(Px,Qy)} \varphi(t)dt + \alpha_2 \int_0^{d(Px,fx)} \varphi(t)dt + \alpha_3 \int_0^{d(Qy,fy)} \varphi(t)dt + \alpha_4 \int_0^{d(Px,fy)} \varphi(t)dt \\ &\quad + \alpha_5 \int_0^{\max\{d(Qy,fx),d(Px,fy)\}} \varphi(t)dt + \alpha_6 \int_0^{\frac{d^3(Px,fx)+d^3(Qy,fy)}{1+d^2(Px,fx)+d^2(Qy,fy)}} \varphi(t)dt \\ &\quad + \alpha_7 \int_0^{\frac{d^2(Px,fy)+d^2(Qy,fx)}{1+d(Px,fy)+d(Qy,fx)}} \varphi(t)dt,\end{aligned}$$

for each $x, y \in X$, $x \neq y$ with $\alpha_i : (0,1) \to [0,1)$ is monotonically decreasing functions, satisfying $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 < 1$, where $\varphi : R^+ \to R^+$ be a Lebesgue-integrable mapping which is summable on each compact subset of R^+ such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$, then f, Q and P have a unique common fixed point $z \in X$.

The following examples justify all requirements of our theorems.

Example 2.6. Let $X = [0,1]$ and d is usual metric on X. Define a self mapping f such that $fx = \frac{1}{2}x$, $x \in X$. Let us define $\varphi(t) = 2t \ \forall t \in R^+$, $x = \frac{1}{2}$ and $y = 0$ then for every $\epsilon > 0$,

$$\int_0^\epsilon \varphi(t)dt = \int_0^\epsilon 2tdt = \epsilon^2 > 0.$$

Since d is usual metric, for all $x = \frac{1}{2}$ and $y = 0$ in X

$$L.H.S = \int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(\frac{x}{2},\frac{y}{2})} 2tdt = \int_0^{\left|\frac{x-y}{2}\right|} 2tdt = \left(\left|\frac{x-y}{2}\right|\right)^2 = \frac{1}{16}.$$

also by the same calculations, we have

$$\alpha_1 \int_0^{d(x,y)} \varphi(t)dt = \frac{\alpha_1}{4} \qquad \alpha_2 \int_0^{d(x,fx)} \varphi(t)dt = \frac{\alpha_2}{16} \qquad \alpha_3 \int_0^{d(y,fy)} \varphi(t)dt = 0$$

$$\alpha_4 \int_0^{d(x,fy)} \varphi(t)dt = \frac{\alpha_4}{4} \qquad \alpha_5 \int_0^{\max\{d(y,fx),d(x,fy)\}} \varphi(t)dt = \frac{\alpha_5}{4} \qquad \alpha_6 \int_0^{\frac{d^3(x,fx)+d^3(y,fy)}{1+d^2(x,fx)+d^2(y,fy)}} \varphi(t)dt = \frac{\alpha_6}{(68)^2}$$

$$\alpha_7 \int_0^{\frac{d^2(x,fy)+d^2(y,fx)}{1+d(x,fy)+d(y,fx)}} \varphi(t)dt = \left(\frac{5}{28}\right)^2 \alpha_7.$$

So,

$$R.H.S = \frac{\alpha_1}{4} + \frac{\alpha_2}{16} + \frac{\alpha_4}{4} + \frac{\alpha_5}{4} + \frac{\alpha_6}{(68)^2} + \left(\frac{5}{28}\right)^2 \alpha_7 \geq \frac{1}{16} = L.H.S. \tag{2.15}$$

The inequality (2.15) is satisfies if we take any monotonically decreasing function $\alpha_i : (0,1) \to [0,1)$, therefore the inequality of Theorem 2.1 is verified and 0 is a unique fixed point of f.

Example 2.7. By regarding all requirements of Example 2.6, define a self mapping g such that $gx = x$, $x \in X$ and by same calculations, we get

$$R.H.S = \frac{\alpha_1}{4} + \frac{\alpha_2}{16} + \frac{\alpha_4}{4} + \frac{\alpha_5}{4} + \frac{\alpha_6}{(68)^2} + \left(\frac{5}{28}\right)^2 \alpha_7 \geq \frac{1}{16} = L.H.S. \tag{2.16}$$

The inequality (2.16) is satisfied if we take any monotonically decreasing function $\alpha_i : (0,1) \to [0,1)$, therefore the inequality of Theorem 2.3 is verified and 0 is a unique common fixed point of f and g.

Example 2.8. Let $X = [0,1)$ with the usual metric on X, we define self mappings f, g, P and Q on X by

$$fx = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{2}) \\ \frac{1}{8}, & \text{if } x \in [\frac{1}{2},1) \end{cases}, \quad Px = \begin{cases} \frac{1}{2}, & \text{if } x \in [0,\frac{1}{2}) \\ \frac{3}{4}, & \text{if } x \in [\frac{1}{2},1) \end{cases},$$

$$gx = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{2}) \\ \frac{3}{4}, & \text{if } x \in [\frac{1}{2},1) \end{cases}, \quad Qx = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{2}) \\ \frac{1}{3}, & \text{if } x \in [\frac{1}{2},1) \end{cases}.$$

Let $\varphi(t) = 2t \; \forall t \in R^+$, $x = \frac{1}{2}$ and $y = 0$ then for every $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt = \int_0^{\epsilon} 2tdt = \epsilon^2 > 0.$$

It's clearly that $f(X) \subseteq P(X)$ and $g(X) \subseteq Q(X)$, so at the points $x = \frac{1}{2}$ and $y = 0$ the pairs (g, P) and (f, Q) are weakly compatible.

$$L.H.S = \int_0^{d(fx,gy)} \varphi(t)dt = \int_0^{d(f(\frac{1}{2}),g(0))} 2tdt = \int_0^{d(\frac{1}{8},0)} 2tdt = \left(\left|\frac{1}{8}\right|\right)^2 = \frac{1}{64}.$$

Also,

$$\alpha_1 \int_0^{d(Px,Qy)} \varphi(t)dt = \tfrac{9\alpha_1}{16} \qquad \alpha_2 \int_0^{d(Px,fx)} \varphi(t)dt = \tfrac{25\alpha_2}{64} \qquad \alpha_3 \int_0^{d(Qy,fy)} \varphi(t)dt = 0$$

$$\alpha_4 \int_0^{d(fx,Qy)} \varphi(t)dt = \tfrac{\alpha_4}{64} \qquad \alpha_5 \int_0^{\max\{d(Qy,fx),d(Px,fy)\}} \varphi(t)dt = \tfrac{9\alpha_5}{16}$$

$$\alpha_6 \int_0^{\frac{d^3(Px,fx)+d^3(Qy,fy)}{1+d^2(Px,fx)+d^2(Qy,fy)}} \varphi(t)dt = \tfrac{\alpha_6}{(68)^2} \qquad \alpha_7 \int_0^{\frac{d^2(Px,gy)+d^2(Qy,fx)}{1+d(Px,gy)+d(Qy,fx)}} \varphi(t)dt = \left(\tfrac{37}{120}\right)^2 \alpha_7$$

So,

$$R.H.S = \frac{9\alpha_1}{16} + \frac{25\alpha_2}{64} + \frac{\alpha_4}{64} + \frac{9\alpha_5}{16} + \frac{\alpha_6}{(68)^2} + \left(\frac{37}{120}\right)^2 \alpha_7 \geq \frac{1}{64} = L.H.S. \tag{2.17}$$

The inequality (2.17) is satisfied if we take any monotonically decreasing function $\alpha_i : (0,1) \to [0,1)$, therefore the inequality (2.7) is verified and all axioms of Theorem 2.4 are hold and 0 is a unique common fixed point of f, g, P and Q.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, *Fundamenta Math.,* **3**, 133–181, (1922).

[2] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.,* **29**, 531-536, (2002).

[3] R. Caccioppoli, Un teorema generale sull'esistenza de elemente uniti in una transformazione funzionale, *Rend. Acad. Naz. Linzei,* **11**, 31–49, (1930).

[4] D. Dey, A. Ganguly and M. Saha, Fixed point theorems for mappings under general contractive condition of integral type, *Bull. Math. Anal. Appl.,* **3**, 27-34, (2011).

[5] K. Geobel and W.A. Kirk, Topics in metric fixed point theory, *Combridge University Press*, New York, (1990).

[6] V. Gupta, A. Saini and R. Kumar, Common fixed points for weakly compatible maps in 2-metric spaces, *Int. J. Math. Archive*, **3**, 3670-3675, (2012).

[7] V. Gupta, N. Mani and N. Gulati, A common fixed point theorem satisfying contractive condition of integral type, *Int. J. Engineering & Appl. Sci.,* **2**, 1-7, (2012).

[8] V. R. Hosseini, Common fixed point theorems for maps altering distance under a contractive condition of integral type, *Int. J. Contemporary Math. Sci.*, **5**, 1615-1620, (2010).

[9] G. Jungck, Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.,* **9**, 771-779, (1986).

[10] R. A. Rashwan and H. A. Hammad, A common fixed point theorems for a pair of self maps satisfying a general contractive condition of integral type, *International Journal of Research in Engineering & Applied Sciences*, **4**, 1-12, (2014).

[11] R. A. Rashwan and H. A. Hammad, Common fixed point theorems for weak contraction conditions of integral type, *Proceedings of the* 1st *International Conference on New Horizons in Basic and Applied Science, Hurghada-Egypt, Anglisticum Journal, Tetovo, Macedonia,* **1**, 133-148, (2013).

[12] B. E. Rhoades, A comparison on various definitions of contractive mapping, *Transactions of the American Math. Soc.,* **226**, 257-290, (1977).

[13] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. Theory Method Appl., **74**, 2683-2693, (2001).

[14] B. E. Rhoades, Two fixed point theorems for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Sci.,* **3**, 4007-4013, (2003).

[15] K. Sanjay, R. Chugh and R. Kumar, Fixed point theorem for compatible mappings satisfying a contractive condition of integral type, *Soochow J. Math.,* **33**, 181-185, (2007).

[16] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.*, **32**, 149-153, (1982).

[17] O. R. Smart, Fixed point theorems, Cambridge University Press, London, (1974).

[18] P. Vijayraju, B. E. Rhoades and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, **15** (2005), 2364-2005.

Author information

Rashwan A. Rashwan, Department of Mathematics, Faculty of Science, Assuit University, Assuit, 82524, Egypt.
E-mail: rr_rashwan54@yahoo.com

Hasanen A. Hammad, Department of Mathematics, Faculty of Science, Sohag University, Sohag, 82524, Egypt.
E-mail: h_elmagd89@yahoo.com
Received: August 22, 2017.
Accepted: July 20, 2018.