Palestine Journal of Mathematics

Vol. 8(2)(2019) , 137–145

Some properties of singular integral operator containing extended Mittag-Leffler Function

P. K. Banerji, R. Desai, D. Saxena and A. K. Shukla

Communicated by P.K. Banerji

MSC2010 Classifications:33E12;45D05; 34A12; 26A33
Keywords: Integral equations Singular integral equations Generalized Mittag-Leffler functions Fractional calculus

This work was partially supported by the DST-SERB, India, Major Research Project No. SR/S4/MS : 836/13, sanctioned to the first author (PKB).

Abstract. The purpose of this paper is an attempt to present the singular integral operator $\mathscr{G}_L(\alpha,\beta,\gamma,\delta)$ associated with singular integral equation involving extended Mittag-Leffler function $\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}(z)$ including its existence and composition with Riemann-Liouville fractional integral operator. Further, some properties are also discussed.

1 Introduction

Fractional integral operators play an important role in the solution of several problems of science and engineering. Many fractional integral operators like Riemann-Liouville, Weyl, kober, Erdelyi-Kober and Saigo operators have been discussed by many researchers due to their application in physical, engineering and technological sciences such as reaction, diffusion, viscoelasticity etc. A detailed account of these operators can be found in the survey paper by Srivastava and Saxena [12]. Various properties of family of Mittag-Leffler functions using fractional integral operators have been obtained by [1], [4], [5], [11], [13], [14] and so forth.

Recently, Desai et al. [[2],[3]] studied the integral operator and integral equation containing generalized Mittag-Leffler function as the kernel. In this paper, the operator is extended to singular integral operator. The Mittag-Leffler function is a direct generalization of exponential function. The classical Mittag-Leffler function [7] is defined by

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{1.1}$$

where z is a complex variable and $\alpha\geq 0$ that occurs as the solution of fractional order differential equation or fractional order integral equations. For $\alpha=1, E_1(z)=e^z$. For $0<\alpha<1$ and $|z|<1$, it interpolates between the exponential function e^z and a geometric function $\frac{1}{(1-z)}=\sum\limits_{k=0}^{\infty}z^k$.

Wiman [16] suggested the generalization of $E_\alpha(z)$ for $\alpha,\beta\in C$, $\mathrm{Re}(\alpha), \mathrm{Re}(\beta)>0$ as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{1.2}$$

which is known as Wiman's function or the generalized Mittag-Leffler function with two parameter.

Prabhakar [8] further extended the Mittag-Leffler function for $\alpha,\beta,\gamma\in C$, $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma)>0$ as

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{1.3}$$

where $(\gamma)_n$ is a Pochhammer symbol, $(\gamma)_n = \gamma(\gamma+1)\ldots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}, n \geq 1, (\gamma)_0 = 1, \gamma \neq 0$.

Shukla and Prajapati [10] introduced the function $E_{\alpha,\beta}^{\gamma,q}(z)$, defined for $\alpha,\beta,\gamma \in C$, $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma) > 0$, and $q \in (0,1) \cup N$, as

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \tag{1.4}$$

where $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol which in particular reduces to $q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma+r-1}{q}\right)_n$ if $q \in N$.

A new generalization of Mittag-Leffler function was defined by Salim [9] as

$$E_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_n}, \tag{1.5}$$

where $\alpha,\beta,\gamma,\delta \in C; Re(\alpha), Re(\beta), Re(\gamma), Re(\delta) > 0$.

Should similar notations in (1.4) and (1.5) may arise confusion, we, henceforth, replace the notation in (1.5) by $\mathscr{E}_{\alpha,\beta}^{\gamma,\delta}(z)$.

Equation (1.5) is the generalization of exponential function. Equations (1.1)–(1.3) can reduce to $\mathscr{E}_{1,1}^{1,1}(z) = \exp(z), \mathscr{E}_{\alpha,1}^{1,1}(z) = \mathscr{E}_{\alpha}(z), \mathscr{E}_{\alpha,\beta}^{1,1}(z) = \mathscr{E}_{\alpha,\beta}(z)$ and $\mathscr{E}_{\alpha,\beta}^{\gamma,1}(z) = \mathscr{E}_{\alpha,\beta}^{\gamma}(z)$. Further, on setting $\gamma = \delta$, we get

$$\mathscr{E}_{\alpha,\beta}^{\delta,\delta}(z) = \mathscr{E}_{\alpha,\beta}(z). \tag{1.6}$$

2 Preliminary Notes

Definition 1. We consider real (or complex) valued function on a real interval $[a,d)$, where $0 \leq a < d \leq b < \infty$. Often they are locally integrable on (a,d), i.e., L-integrable on $[a,l)$ for each $l < d$.

Definition 2. L denotes the linear space of real (or complex) valued functions $f(x)$ which are L - integrable on a finite $[a,b]$, i.e.

$$L(a,b) = \left\{ f : \|f\|_1 \equiv \int_a^b |f(t)|\, dt < \infty \right\}. \tag{2.1}$$

Definition 3. Riemann-Liouville fractional integrals of order μ (Miller and Ross [6]): Let $f(x) \in L(a,b)$, $\mu \in C; \mathrm{Re}(\mu) > 0$. Then $I^{\mu} : L \to L$ is a linear operator defined by the fractional integral

$$I^{\mu} f(x) = {}_aI_x^{\mu} f(x) = I_{a+}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}} dt, \tag{2.2}$$

for almost all $x \in (a,b)$.

It is well known that if I^{μ} is bounded and f is locally integrable. Then

$$I^{\mu} f = 0 \Rightarrow f = 0. \tag{2.3}$$

Hence, inverse operator exists on subspace L_{μ} of L. If $0 < \mathrm{Re}(\mu) < \mathrm{Re}(\nu)$, then it can be proved that $L_{\nu} \subset L_{\mu} \subset L$ and the inclusion is proper. For $\mathrm{Re}(\mu) < 0$, I^{μ} is defined as the inverse of $I^{-\mu}$. If $\mathrm{Re}(\mu) \neq 0, \mathrm{Re}(\nu) \neq 0$, then $I^{\mu} I^{\nu} f = I^{\mu+\nu} f$ for locally integrable functions f. Similarly, for $x < b < \infty$,

$$J^{\mu} f(x) = {}_xI_b^{\mu} f(x) = I_{b-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \frac{f(t)}{(t-x)^{1-\mu}} dt. \tag{2.4}$$

3 The operator $\mathscr{G}_L$ and its properties

The operator $\mathscr{G}_L$ having singularity in the kernel, is given by

$$\mathscr{G}_L(\alpha,\beta,\gamma,\delta)f(x)=\int_a^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}\left(1-\frac{x}{t}\right)f(t)dt=g(x), \tag{3.1}$$

where $\alpha,\beta,\gamma,\delta,\mu\in C$ and $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\gamma),\mathrm{Re}(\delta),\mathrm{Re}(\mu)>0$.

Theorem 4. (Existence of the integral) If $\mathrm{Re}(k)>0$; $q\le\mathrm{Re}(h+k)$; $q<\min(\mathrm{Re}(\alpha),\mathrm{Re}(\beta)),\delta\neq 0,-1,-2...,a>0$ and $x^q f(x)$ is integrable on a finite interval (a,l), then

$$x^h\int_a^x \frac{(x-t)^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}(1-\frac{x}{t})f(t)dt \tag{3.2}$$

exists almost everywhere in (a,l) and integrable on it. The same is true for the function obtained by replacing the integrand by its modulus.

Proof: Let the integrand in (3.2) is a measurable function of t and also a measurable function of (x,t).
Therefore,

$$\left|x^h\right|\int_a^x\left|\frac{(x-t)^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}\left(1-\frac{x}{t}\right)f(t)\right|dt \tag{3.3}$$

is a measurable function of x on (a,l), by Fubini's theorem. In order to prove that (3.3) is integrable on (a,l), it is enough to prove the finiteness of

$$\int_a^l\left|x^h\right|dx\int_a^x\left|\frac{(x-t)^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}(1-\frac{x}{t})f(t)\right|dt.$$

$$=\int_a^l|f(t)|dt\int_t^l\left|x^h\frac{(x-t)^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}(1-\frac{x}{t})\right|dx.$$

For $x=t(1+s)$, we get the estimate

$$\int_a^l|f(t)|t^{h+k}dt\int_0^{\frac{l}{t}-1}\left|\frac{(1+s)^h s^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}(-s)\right|ds$$

$$\le\int_a^l|f(t)|t^{h+k}dt\int_0^3\left|\frac{(1+s)^h s^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}(-s)\right|ds+\int_a^{\frac{l}{4}}|f(t)|t^{h+k}dt\int_3^{\frac{l}{t}-1}\left|\frac{(1+s)^h s^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}(-s)\right|ds.$$

Since $t^q f(t)$ is integrable, it is obvious that the first integral is finite and the second integral can be proved finite. Hence, the above representation is finite.

4 Properties of the operator $\mathscr{G}_L$

Following lemma and its comprehensible proof, is invoked in proving Theorem 6 that follows.

Lemma 5. If $Re(\lambda),Re(\delta)>0$ and z is in the complex plane cut along $z\ge1$, then

$$\int_0^1\frac{(1-u)^{\lambda-1}}{\Gamma(\lambda)}\frac{u^{\delta-1}}{\Gamma(\delta)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}(zu)du=\frac{1}{\Gamma(\lambda+\delta)}\mathscr{E}^{\gamma,\delta+\lambda}_{\alpha,\beta}(z). \tag{4.1}$$

In particular, for $0 < t < x$,

$$\int_t^x \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)} \frac{(s-t)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}(1-{}^{s}\!/_{t})ds = \frac{(x-t)^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta+\lambda}(1-{}^{x}\!/_{t}), \tag{4.2}$$

and

$$\int_t^x \frac{(x-s)^{\delta-1}}{\Gamma(\lambda)} \frac{(s-t)^{\lambda-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}(1-{}^{s}\!/_{x})ds = \frac{(x-t)^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta+\lambda}\left(1-{}^{t}\!/_{x}\right). \tag{4.3}$$

Proof: Substituting value of $\mathscr{E}_{\alpha,\beta}^{\gamma,\delta}$ from (1.5) into the left hand side of (4.1), we have

$$\int_0^1 \frac{(1-u)^{\lambda-1}}{\Gamma(\lambda)} \frac{u^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}(zu)\,du = \int_0^1 \frac{(1-u)^{\lambda-1}}{\Gamma(\lambda)} \frac{u^{\delta-1}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{(zu)^n}{(\delta)_n} du,$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)\Gamma(\lambda)} \frac{z^n}{\Gamma(\delta+n)} B(\lambda, n+\delta),$$

where $B(a,b)$ is a beta function. An accessible calculations leads to the proof of (4.1), and thus, details are avoided.

In particular, (4.2) can be obtained from (4.1) directly by changing the scale. In fact, by substitution $s-t=(x-t)u$, we obtain

$$\int_t^x \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)} \frac{(s-t)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}(1-{}^{s}\!/_{t})ds$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma)_n}{\Gamma(\alpha n+\beta)\Gamma(\lambda)\Gamma(\delta+n)} \int_0^1 \frac{(x-t)^{\delta+n-1+\lambda} u^{\delta-n-1}(1-u)^{\lambda-1}}{t^n} du,$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma)_n(x-t)^{\delta+n-1+\lambda}}{\Gamma(\alpha n+\beta)\Gamma(\lambda)\Gamma(\delta+n)t^n} B(\delta+n,\lambda),$$

where $B(\delta+n,\lambda)$ is the beta function. Further simplification yields

$$\int_t^x \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)} \frac{(s-t)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}(1-{}^{s}\!/_{t})ds$$

$$= (x-t)^{\delta+\lambda-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n(t-x)^n}{\Gamma(\alpha n+\beta)\Gamma(\lambda+\delta+n)t^n}$$

$$= \frac{(x-t)^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta+\lambda}(1-{}^{x}\!/_{t}),$$

which proves (4.2).

Similarly, substituting $\frac{(x-t)}{x}$ for z and $\frac{x-s}{x-t}$ for u in (4.1), we arrive at the desired relation (4.3).

Remark: It would be incorrect to say that (4.3) is derived from (4.2) by interchanging x and t, because both formule assume that $0 < t < x$.

Theorem 6. (Composition with Riemann-Liouville fractional integral operator) If $Re(\lambda), Re(\delta) > 0, q \leq \mathrm{Re}(\delta), q < \min(\mathrm{Re}\alpha, \mathrm{Re}\beta)$ and $x^q f(x)$ is locally integrable (cf. Vulikh [15], p.152) on $[a,d)$, then $\mathscr{G}_L(\alpha,\beta,\gamma,\delta) f(x)$ is also locally integrable and

$$I^{\lambda}\mathscr{G}_L(\alpha,\beta,\gamma,\delta) f(x) = \mathscr{G}_L(\alpha,\beta,\gamma,\delta+\lambda) f(x). \tag{4.4}$$

That is to say, for almost all x in $[a,d)$,

$$\int_a^x \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)} ds \int_a^s \frac{(s-t)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}(1-{}^{s}\!/_{t}) ds = \int_a^x \frac{(x-t)^{\delta+\lambda-1}}{\Gamma(\delta+\lambda)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta+\lambda}(1-{}^{x}\!/_{t}) f(t) dt. \quad (4.5)$$

Proof: It follows from (2.2) and (3.1) that

$$I^{\lambda}\mathscr{G}_L(\alpha,\beta,\gamma,\delta) f(x)$$

$$= \int_a^x \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)} \int_t^s \frac{(s-t)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}\left(1-{}^{s}\!/_{t}\right) f(t) dt ds \quad (4.6)$$

$$= \int_a^x \int_t^s \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)} \frac{(s-t)^{\delta-1}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)(\delta)_n} \left(1-{}^{s}\!/_{t}\right)^n f(t) dt ds.$$

By substitution $s-t=(x-t)u$, we obtain

$$I^{\lambda}\mathscr{G}_L(\alpha,\beta,\gamma,\delta) f(x)$$

$$= \sum_{n=0}^{\infty} \int_a^x \frac{(\gamma)_n}{\Gamma(\lambda)\Gamma(\alpha n+\beta)\Gamma(\delta+n)} f(t) \int_0^1 (x-t)^{\delta+\lambda-1+n} u^{\delta+n-1} (1-u)^{\lambda-1} (-1)^n du dt.$$

Further simplification leads to

$$= \sum_{n=0}^{\infty} \int_a^x f(t) \frac{(-1)^n (\gamma)_n (x-t)^{\delta+n-1+\lambda}}{\Gamma(\alpha n+\beta)\Gamma(\lambda)\Gamma(\delta+n) t^n} B(\delta+n,\lambda) dt,$$

which can be written as

$$= \int_a^x (x-t)^{\delta-1+\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_n (x-t)^n}{\Gamma(\delta+n+\lambda)\Gamma(\alpha n+\beta) t^n} f(t) dt.$$

Here, changing the order of integration and summation is justified by the dominated convergence theorem, conditions for which are prescribed in the theorem.
Therefore,

$$I^{\lambda}\mathscr{G}_L(\alpha,\beta,\gamma,\delta) f(x) = \int_a^x \frac{(x-t)^{\delta-1+\lambda}}{\Gamma(\delta+\lambda)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta+\lambda}(1-{}^{x}\!/_{t}) f(t) dt,$$
$$= \mathscr{G}_L(\alpha,\beta,\gamma,\delta+\lambda) f(x).$$

This completely proves (4.4). This property can also be refered to as a shifting property.

Now, fixing l such that $a<l<d$, we have to prove (4.5) for almost all x in (a,l) and integrability of $\mathscr{G}_L(\alpha,\beta,\gamma,\delta) f$ on (a,l). The integrability follows from Theorem 4 with $h=0$ and $k=\delta$ and incidentally the double integral in (4.5) is absolutely convergent for almost all x in (a,l), because

$$\int_a^x \left|\frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)}\right| \int_t^s \left|\frac{(s-t)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}\left(1-{}^{s}\!/_{t}\right) f(t)\right| dt ds,$$

is a convolution of integrable functions $\left|\frac{(x)^{\lambda-1}}{\Gamma(\lambda)}\right|$ and the function in (3.3) with $h=0$ and $k=\delta$, which is also integrable in the light of Theorem 4. The convolution is finite almost everywhere, and as a result, double integral in (4.5) is absolutely convergent almost everywhere.

Inverting the order of integration, the left side of (4.4) becomes

$$\int_a^x f(t)dt \int_t^s \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)} \frac{(s-t)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}\left(1-{}^{s}\!/_{t}\right) ds$$

for almost all x, which on invoking Lemma 5, establishes (4.4).

Theorem 7. If $Re(\lambda), Re(\delta) > 0, q \leq \mathrm{Re}(\delta+\lambda); q < \min(\mathrm{Re}(\alpha), \mathrm{Re}(\beta)); 0 < a < d < b$ and $x^q f(x)$ is locally integrable on $[a,d)$, then

$$\mathscr{G}_L(\alpha,\beta,\gamma,\delta)\, x^{-(\delta+\lambda)} I^\lambda x^\delta f(x) = \frac{1}{x^\lambda} \mathscr{G}_L(\alpha,\beta,\gamma,\delta+\lambda) f(x) \tag{4.7}$$

Proof: The left hand side

$$\mathscr{G}_L(\alpha,\beta,\gamma,\delta)\, x^{-(\delta+\lambda)} I^\lambda x^\delta f(x)$$

$$= \int_a^x \frac{(x-s)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}(1-{}^{x}\!/_{s}) s^{-(\delta+\lambda)} \int_a^s \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} t^\delta f(t) dt ds.$$

i.e. on changing the order of integration, this gives

$$\mathscr{G}_L(\alpha,\beta,\gamma,\delta)\, x^{-(\delta+\lambda)} I^\lambda x^\delta f(x)$$

$$= \int_a^x \int_t^x \sum_{n=0}^\infty \frac{(x-s)^{\delta-1}}{\Gamma(\delta)} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)(\delta)_n} \frac{(s-x)^n}{s^{n+\delta+\lambda}} \frac{(s-t)^{\lambda-1}}{\Gamma(\lambda)} t^\delta f(t) dt ds.$$

$$= \int_a^x \sum_{n=0}^\infty \frac{(-1)^n \Gamma(\gamma+n)}{\Gamma(\alpha n+\beta)\Gamma(\gamma)\Gamma(\delta+n)\Gamma(\lambda)} \left[\int_t^x \frac{(x-s)^{\delta+n-1}(s-t)^{\lambda-1}}{s^{n+\delta+\lambda}} ds \right] t^\delta f(t) dt. \tag{4.8}$$

Let us consider

$$I = \int_t^x \frac{(x-s)^{\delta+n-1}(s-t)^{\lambda-1}}{s^{n+\delta+\lambda}} ds.$$

Substituting $s = t+(x-t)u$, we get

$$I = \frac{(x-t)^{\delta+n-1+\lambda}}{x^\lambda t^{\delta+n}} \frac{\Gamma(\lambda)\Gamma(\delta+n)}{\Gamma(\lambda+\delta+n)}.$$

Therefore,

$$\mathscr{G}_L(\alpha,\beta,\gamma,\delta)\, x^{-(\delta+\lambda)} I^\lambda x^\delta f(x)$$

$$= \int_a^x \sum_{n=0}^\infty \frac{(-1)^n \Gamma(\gamma+n)}{\Gamma(\alpha n+\beta)\Gamma(\gamma)\Gamma(\delta+n)\Gamma(\lambda)} \left[\frac{(x-t)^{\delta+n-1+\lambda}}{x^\lambda t^{\delta+n}} \frac{\Gamma(\lambda)\Gamma(\delta+n)}{\Gamma(\lambda+\delta+n)} \right] t^\delta f(t) dt,$$

$$= \frac{1}{x^\lambda} \int_a^x \frac{(x-t)^{\delta-1+\lambda}}{\Gamma(\delta+\lambda)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta+\lambda}\left(1-{}^{x}\!/_{t}\right) f(t) dt,$$

Hence, we arrive at

$$= \frac{1}{x^\lambda} \mathscr{G}_L(\alpha,\beta,\gamma,\delta+\lambda) f(x).$$

Following theorem justifies the uniqueness of solution of $\int_a^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}\left(1-\frac{x}{t}\right) f(t) dt = g(x)$ for $0 < a < x < d < \infty$, if exists.

Theorem 8. If $Re(\alpha), Re(\beta), Re(\gamma), Re(\delta) > 0, q \leq \min(\mathrm{Re}(\delta), \mathrm{Re}(\alpha+\beta))$, $q < \min(Re(\alpha), Re(\beta))$, $x^q f(x)$ is locally integrable on $[a,d)$ and

$$\int_a^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} \mathscr{E}_{\alpha,\beta}^{\gamma,\delta}\left(1-\frac{x}{t}\right) f(t)dt = 0, \tag{4.9}$$

then $f(x) = 0$.

Proof: From Theorem 6, we have

$$I^{\lambda} \mathscr{G}_L(\alpha,\beta,\gamma,\delta) f(x) = \mathscr{G}_L(\alpha,\beta,\gamma,\delta+\lambda) f(x),$$

$$= I^{\delta} \mathscr{G}_L(\alpha,\beta,\gamma,\lambda) f(x).$$

On setting $\lambda = \gamma$, it yields

$$I^{\gamma} \mathscr{G}_L(\alpha,\beta,\gamma,\delta) f(x) = \mathscr{G}_L(\alpha,\beta,\gamma,\delta+\gamma) f(x),$$

$$= I^{\delta} \mathscr{G}_L(\alpha,\beta,\gamma,\gamma) f(x).$$

Invoking (4.9), we get

$$\mathscr{G}_L(\alpha,\beta,\gamma,\delta) f(x) = 0.$$

Therefore,

$$I^{\gamma} \mathscr{G}_L(\alpha,\beta,\gamma,\delta) f(x) = I^{\delta} \mathscr{G}_L(\alpha,\beta,\gamma,\gamma) f(x) = 0.$$

Hence, by (2.3)

$$\mathscr{G}_L(\alpha,\beta,\gamma,\gamma) f(x) = 0.$$

Now,

$$\mathscr{G}_L(\alpha,\beta,\gamma,\gamma) f(x) = \int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \mathscr{E}_{\alpha,\beta}^{\gamma,\gamma}\left(1-{}^{x}\!/_{t}\right) f(t)dt,$$

On considering (1.6), we get

$$\mathscr{G}_L(\alpha,\beta,\gamma,\gamma) f(x) = \int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \mathscr{E}_{\alpha,\beta}\left(1-{}^{x}\!/_{t}\right) f(t)dt,$$

$$= I^{\gamma} \mathscr{E}_{\alpha,\beta}\left(1-{}^{x}\!/_{t}\right) f(x) = 0.$$

Therefore, by (2.3), we conclude that $\mathscr{E}_{\alpha,\beta}\left(1-{}^{x}\!/_{t}\right) f(x) = 0$ provided $\mathscr{E}_{\alpha,\beta}\left(1-{}^{x}\!/_{t}\right) f(x)$ is locally integrable.

Theorem 9. If $\mathrm{Re}(\delta) > 0$ and z is in the complex plane cut along $z \geq 1$, then

$$\left[\mathscr{E}_{\alpha,\beta}^{\gamma,\delta+1}(z) - \frac{1}{\Gamma(\beta-\alpha)}\right] = \gamma\delta z \int_0^1 u(1-u)^{\delta-1} \mathscr{E}_{\alpha,\beta}^{\gamma+1,2}(zu)\, du. \tag{4.10}$$

In particular,

$$(x-t)^{\delta}\left[\mathscr{E}_{\alpha,\beta-\alpha}^{\gamma,\delta+1}\left(1-{}^{x}\!/_{t}\right) - \frac{1}{\Gamma(\beta-\alpha)}\right] = \delta\gamma \int_t^x \left(1-{}^{s}\!/_{t}\right)(x-s)^{\delta-1} \mathscr{E}_{\alpha,\beta}^{\gamma+1,2}\left(1-{}^{s}\!/_{t}\right) du. \tag{4.11}$$

Now, we prove the property in which one parameter can be completely replaced by any newly introduced parameter.

Theorem 10. If $\alpha,\beta,\gamma,\delta,k\in C$; $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\gamma)>0$; $\mathrm{Re}(k)>\mathrm{Re}(\delta)>0$, then for $x>0$,

$$\int_0^t \frac{(t-s)^{k-\delta-1}s^{\delta-1}}{\Gamma(\delta)\Gamma(k-\delta)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}\left(-\frac{s}{x}\right)ds = \frac{t^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,k}_{\alpha,\beta}\left(-\frac{t}{x}\right). \tag{4.12}$$

Proof: If z is any complex number such that $|z|>t$, then

$$\int_0^t \frac{(t-s)^{k-\delta-1}s^{\delta-1}}{\Gamma(\delta)\Gamma(k-\delta)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}\left(\frac{s}{z}\right)ds = \int_0^t \frac{(t-s)^{k-\delta-1}s^{\delta-1}}{\Gamma(\delta)\Gamma(k-\delta)}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{\left({}^{s}\!/_{z}\right)^n}{(\delta)_n}ds.$$

On changing order of integration and summation, justified for the absolute convergence of integrals under given conditions, we get

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)\Gamma(\delta)\Gamma(k-\delta)}\frac{\left({}^{1}\!/_{z}\right)^n}{(\delta)_n}\int_0^t (t-s)^{k-\delta-1}s^{\delta+n-1}ds.$$

Substituting $s=t\,u$ and using beta function, above integral reduces to

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)\Gamma(\delta)\Gamma(k-\delta)}\frac{t^{k+n-1}}{(\delta)_n}B(k-\delta,\delta+n).$$

Using relation between Beta and Gamma function and putting $z=-x$, we get the required result. It can be observed from (4.12) that the fourth parameter δ is completely replaced by the newly introduced parameter k.

Inwhat follow is the result that prescribes the transformation of $\mathscr{E}^{\gamma,k}_{\alpha,\beta}$ under integral sign into the fractional integral with randomly chosen parameter.

Theorem 11. If $\alpha,\beta,\gamma,\delta,k\in C$, $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\gamma)>0$; $\mathrm{Re}(k)>\mathrm{Re}(\delta)>0$, then for $x>0$ and $f\in S$,

$$\int_0^\infty \frac{t^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,k}_{\alpha,\beta}\left(-\frac{t}{x}\right)f(t)dt = \int_0^\infty \frac{t^{\delta-1}}{\Gamma(\delta)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}\left(-\frac{t}{x}\right)J^{k-\delta}f(t)dt.$$

Proof: Using (4.12), we obtain

$$\int_0^\infty \frac{t^{k-1}}{\Gamma(k)}\mathscr{E}^{\gamma,k}_{\alpha,\beta}\left(-\frac{t}{x}\right)f(t)dt = \int_0^\infty f(t)\int_0^t \frac{(t-s)^{k-\delta-1}s^{\delta-1}}{\Gamma(\delta)\Gamma(k-\delta)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}\left(-\frac{s}{x}\right)dsdt.$$

Changing the order of integration, which is justified by Fubini's theorem, we get

$$\int_0^\infty \frac{s^{\delta-1}}{\Gamma(\delta)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}\left(-\frac{s}{x}\right)ds\int_s^\infty \frac{(t-s)^{k-\delta-1}}{\Gamma(k-\delta)}f(t)dt$$
$$=\int_0^\infty \frac{s^{\delta-1}}{\Gamma(\delta)}\mathscr{E}^{\gamma,\delta}_{\alpha,\beta}\left(-\frac{s}{x}\right)J^{k-\delta}f(s)ds.$$

5 Conclusion

Owing to the occurrence of four parameter generalized Mittag-Leffler function in the kernel of the integral equations and having studied its properties, it is believed that results of this paper can be extended to study singular integral equation on distribution spaces. Further, the function in Theorems 6, 7 and 8 being locally integrable, may support investigations in wavelet analysis.

References

[1] Desai, R., Salehbhai, I.A. and Shukla, A.K., Note on generalized Mittag-Leffler function, SpringerPlus, 5:683, (2016).

[2] Desai, R., Banerji, P.K. and Shukla, A.K., Singular Integral Equations Containing Extended Mittag-Leffler Function, communicated for publication

[3] Desai, R., Salehbhai, I.A. and Shukla A. K. , Integral equations involving Generalized Mittag-Leffler function, Ukr. Math. J., (accepted for publications)

[4] Kilbas, A.A. and Saigo, M., On Mittag-Leffler type function, fractional calculas operators and solutions of integral equations, Integr. Transf. Spec. F. 4 (4), (1996), 355-370.

[5] Kilbas, A.A., Saigo, M. and Saxena, R.K., Generalized Mittag-Leffler function and generalized fractional calculus operators, Integr. Transf. Spec. F. 15 (1), (2004), 31-49.

[6] Miller, K.S. and Ross, B., An introduction to fractional calculus and fractional differential equations, Wiley, New York (1993).

[7] Mittag- Leffler G., Sur la nouvelle fonction $E_\alpha(x)$. C.R. Acad. Sci. Paris 137 (1903), 554-558.

[8] Prabhakar T.R., A singular equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19 (1971), 7-15.

[9] Salim T.O., Some properties relating to the generalized Mittag-Leffler function, Advances in Applied Mathematical Analysis, 4(1) (2009), 21-30.

[10] Shukla A.K., and Prajapati J.C., On a generalization of Mittag-Leffler function and its properties. J. Math. Anal. Appl. 336(2) (2007), 797-811.

[11] Shukla A.K and Prajapati J.C., On a generalization of Mittag-Leffler function and generalized integral operator, Math. Sci. Rec. J 12(12) (2008), 283-290.

[12] Srivastava, H.M. and Saxena, R.K., Operators of fractional integration and their applications, Appl. Math. Comut., 118 (2001), 1-52.

[13] Srivastava, H.M. and Tomovski, Z., Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, Applied Mathematics and Computation 211 (1), (2009), 198-210.

[14] Tomovski, Z., Hilfer, R. and Srivastava, H.M., Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, Integr. Transf. Spec. F. 21 (11), (2010), 797-814.

[15] Vulikh, B.Z., A Brief Course in the Theory of Functions of Real Variable, Mir Publishers, Moscow (1976).

[16] Wiman A., Uber de Fundamental satz in der theorie der funktionen. Acta Math. 29 (1905), 191-201.

Author information

P. K. Banerji, Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur-342011., India.
E-mail: banerjipk@yahoo.com

R. Desai, Department of Science and Humanities, K.J.Somaiya College of Engineering, University of Mumbai, Mumbai- 400077., India.
E-mail: rachanadesai@somaiya.edu

D. Saxena, Department of Mathematics, University of Jazan KSA, Jazan., Saudi Arabia.
E-mail: deepali.saxena@rediffmail.com

A. K. Shukla, Department of Applied Mathematics, S.V. National Institute of Technology, Surat-395007, India.
E-mail: ajayshukla2@rediffmail.com

Received: March 3, 2018.
Accepted: May 12, 2018.