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SOME NEW RESULTS FOR HUMBERT'S DOUBLE HYPERGEOMETRIC SERIES ψ_2 AND ϕ_2

Ghazala Yasmin and Abdulghani Muhyi

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Abstract. This article aims to establish new expressions of Humbert's functions ψ_2 and ϕ_2 with the help of the generalization of Kummer's summation theorem. Some special cases are also obtained. Further, we present some interesting integrals involving the Humbert's functions, which are expressed in terms of generalized (Wright) hypergeometric function. Also, some special cases were considered as an application of the presented integral formulas.

1 Introduction

The theory of hypergeometric functions has a vast popularity and great interest. The usefulness of this functions lies in the fact that the solutions of many applied problems involving partial differential equations are obtainable with the help of such hypergeometric functions. For instance, the energy absorbed by some non ferromagnetic conductor sphere contained in an internal magnetic field can be calculated with the help of such functions [12, 15]. Multi-variable hypergeometric functions are used in physical and quantum chemical applications as well [14, 19]. Especially, many problems in gas dynamics lead to those of degenerate second order partial differential equations, which are then solvable in terms of multiple hypergeometric functions. Further, in the investigation of the boundary value problems for these partial differential equations, we need decompositions for multivariate hypergeometric functions in terms of simpler hypergeometric functions of the Gauss and Humbert types.

The generalized hypergeometric function (GHF) is defined by [16, p.73 (2)] (see also [1]) :

$$\begin{aligned} {}_pF_q\left[\begin{array}{lr} \alpha_1,\alpha_2,..,\alpha_p; & \\ & z \\ \beta_1,\beta_2,...,\beta_q; & \end{array}\right] &= {}_pF_q[\alpha_1,\alpha_2,..,\alpha_p;\beta_1,\beta_2,...,\beta_q;z] \\ &= 1+\sum_{m=1}^{\infty}\frac{\prod\limits_{i=1}^{p}(\alpha_i)_m}{\prod\limits_{j=1}^{q}(\beta_j)_m}\frac{z^m}{m!}, \quad (\beta_j>0, j=0,1,2,...,q), \end{aligned} \tag{1.1}$$

where $(\lambda)_m$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_m=m!$) defined, for any complex number α, by

$$\begin{aligned} (\lambda)_m &:= \begin{cases} \alpha(\alpha+1)...(\alpha+m-1), & m\in\mathbb{N}=1,2,3,... \\ 1, & m=0 \end{cases} \\ &= \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}, \quad (\lambda\in\mathbb{C}\backslash\mathbb{Z}_0^-) \end{aligned} \tag{1.2}$$

where $\Gamma(r)$ is the well known gamma function [16, 20] defined by

$$\Gamma(r)=\int_0^\infty e^{-x}x^{r-1}dx, \qquad (\Re(r)>0). \tag{1.3}$$

The Beta function $B(\upsilon,\tau)$, υ, $\tau \in \mathbb{C}$, is defined by [16, 20]:

$$B(\upsilon,\tau)=\begin{cases}\int_0^1 x^{\upsilon-1}\,(1-x)^{\tau-1}\,dx, & \Re(\upsilon)>0, \Re(\tau)>0\\ \frac{\Gamma(\upsilon)\Gamma(\tau)}{\Gamma(\upsilon+\tau)}, & \Re(\upsilon)<0, \Re(\tau)<0,\ \upsilon,\tau\neq -1,-2,-3,....\end{cases}. \tag{1.4}$$

The Wright hypergeometric function ${}_p\Psi_q$ is defined by [13, 16]

$${}_p\Psi_q\left[\begin{array}{c}(\alpha_1,\eta_1),(\alpha_2,\eta_2),..,(\alpha_p,\eta_p);\\ \\ (\beta_1,\xi_1),(\beta_2,\xi_2),...,(\beta_q,\xi_q);\end{array} z\right]=\sum_{m=1}^{\infty}\frac{\prod\limits_{i=1}^{p}\Gamma(\alpha_i+\eta_i m)}{\prod\limits_{j=1}^{q}\Gamma(\beta_j+\xi_j m)}\frac{z^m}{m!}, \tag{1.5}$$

where the coefficient η_1, η_2,..., η_p and ξ_1, ξ_2, ..., ξ_q are positive real numbers such that

$$(a)\qquad 1+\sum_{j=1}^{q}\xi_j-\sum_{i=1}^{p}\eta_i>0 \quad and \quad 0<|z|<\infty; z\neq 0,$$

$$(b)\qquad 1+\sum_{j=1}^{q}\xi_j-\sum_{i=1}^{p}\eta_i=0 \quad and \quad 0<|z|<\eta_1^{-\eta_1}...\eta_p^{-\eta_p}\xi_1^{\xi_1}...\xi_q^{\xi_q}.$$

The Humbert's functions are defined by [2] (see also[3, 13, 20, 21]) :

$$\psi_2(a_1;a_2,a_3;y,z)=\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\frac{(a_1)_{s+r}}{(a_2)_s(a_3)_r}\frac{y^s z^r}{s!\,r!} \tag{1.6}$$

and

$$\phi_2(a_1,a_2;a_3;y,z)=\sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\frac{(a_1)_s(a_2)_s}{(a_3)_{s+r}}\frac{y^s z^r}{s!\,r!}, \tag{1.7}$$

where the double series (1.6)-(1.7) converge absolutely at any $y, z \in \mathbb{C}$.

Further, we recall some relations between hypergeometric series [3] (see also [13, 20]) and Humbert's functions as :

$$\psi_2(a_1;a_1,a_1;y,z)=exp(y+z)\,{}_0F_1\left[\begin{array}{c}-\,;\\ \\ a_1\,;\end{array} yz\right], \tag{1.8}$$

$$\psi_2(a_1;a_3,a_3;-y,y)={}_2F_3\left[\begin{array}{c}\frac{1}{2}a_1,\frac{1}{2}a_1+\frac{1}{2};\\ \\ a_3,\frac{1}{2}a_3,\frac{1}{2}a_3+\frac{1}{2};\end{array} -y^2\right], \tag{1.9}$$

$$\psi_2(a_1;a_2,a_3;y,y)={}_3F_3\left[\begin{array}{c}a_1,\frac{1}{2}a_2+\frac{1}{2}a_3-\frac{1}{2},\frac{1}{2}a_2+\frac{1}{2}a_3;\\ \\ a_2,a_3,a_2+a_3-1;\end{array} 4y\right], \tag{1.10}$$

$$\phi_2(a_1,a_2-a_1;a_2;y,z)=exp(z)\,{}_1F_1\left[\begin{array}{c}a_1\,;\\ \\ a_2\,;\end{array} y-z\right], \tag{1.11}$$

$$\phi_2(a_1,a_1';a_2;y,y)={}_0F_1\left[\begin{array}{c}a_1+a_1';\\ \\ a_2\,;\end{array} y\right], \tag{1.12}$$

$$\phi_2(a_1,a_1;a_3;-y,y) = {}_1F_2\left[\begin{matrix} a_1; \\ \\ \frac{1}{2}a_3, \frac{1}{2}a_3+\frac{1}{2}; \end{matrix} \quad \frac{y^2}{4}\right]. \tag{1.13}$$

The two identities (1.6) and (1.7) may be written as [13, 17]

$$\psi_2(a_1;a_2,a_3;y,z) = \sum_{s=1}^{\infty}\frac{(a_1)_s}{(a_2)_s}{}_2F_1\left[\begin{matrix} -s,-s-a_2+1; \\ \\ a_3; \end{matrix} \quad \frac{z}{y}\right]\frac{y^s}{s!}, \tag{1.14}$$

where $a_2, a_3 \neq 0,-1,-2,...$ and $|y| \neq 0$.

$$\phi_2(a_1,a_2;a_3;y,z) = \sum_{r=1}^{\infty}\frac{(a_1)_r}{(a_3)_r}{}_2F_1\left[\begin{matrix} -r,a_2; \\ \\ 1-a_1-r; \end{matrix} \quad \frac{z}{y}\right]\frac{y^r}{r!}, \tag{1.15}$$

where $a_3 \neq 0,-1,-2,...$ and $|y| \neq 0$.

In the present investigation, the results are obtained with the help of the following generalization of the classical Kummer's theorem [10] (see also [4]):

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a_1,a_2; \\ \\ 1+a_1-a_2+j; \end{matrix} \quad -1\right] &= \frac{\Gamma(\frac{1}{2})\Gamma(1-a_2)\Gamma(1+a_1-a_2+j)}{2^{a_1}\Gamma(1-a_2+\frac{1}{2}(j+|j|))} \\ \times\Bigg\{&\frac{\mu_j}{\Gamma(\frac{1}{2}a_1-a_2+\frac{1}{2}j+1)\Gamma(\frac{1}{2}a_1+\frac{1}{2}j+\frac{1}{2}-[\frac{j+1}{2}])}+\frac{\nu_j}{\Gamma(\frac{1}{2}a_1-a_2+\frac{1}{2}j+\frac{1}{2})\Gamma(\frac{1}{2}a_1+\frac{1}{2}j-[\frac{j}{2}])}\Bigg\}, \end{aligned} \tag{1.16}$$

where $j = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Here $[z]$ denotes the greatest integer less than or equal to z.

For $j = 0$ the identity (1.16) reduces to the classical Kummer's theorem [16]:

$${}_2F_1\left[\begin{matrix} a_1,a_2; \\ \\ 1+a_1-a_2; \end{matrix} \quad -1\right] = \frac{2^{-a_1\Gamma(\frac{1}{2})\Gamma(a_1-a_2+1)}}{\Gamma(\frac{1}{2}a_1+\frac{1}{2})\Gamma(\frac{1}{2}a_1-a_2+1)}, \quad (\Re(a_2)<1; a_1-a_2 \in \mathbb{C}\setminus\mathbb{Z}^-). \tag{1.17}$$

Consider the well known identity:

$$\sum_{m=0}^{\infty}\varphi(m) = \sum_{m=0}^{\infty}\varphi(2m) + \sum_{m=0}^{\infty}\varphi(2m+1). \tag{1.18}$$

Also, we note the following identities[16, 20]:

$$(\alpha)_{2m} = 2^{2m}\left(\frac{\alpha}{2}\right)_m\left(\frac{\alpha+1}{2}\right)_m, \tag{1.19}$$

$$(\alpha)_{s+r} = (\alpha)_s(\alpha+s)_r, \tag{1.20}$$

$$(\alpha)_{-s} = \frac{(-1)^s}{(1-\alpha)_s}. \tag{1.21}$$

Further, we recall the well known result of Lavoie and Trottier [11]:

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.22}$$

where $(\Re(\alpha)>0$ and $\Re(\beta)>0)$.

Very recently, Choi et al. [4, 6, 8, 9, 17, 18] have considered the reducibility of some extensively generalized special functions and established many results using generalized Kummer's summation theorem in various ways. In this sequel, using the same technique, we propose to derive certain explicit expressions for Humbert's functions ψ_2 and ϕ_2. Some special cases of the main results are also derived. Some integral formulas involving Humbert's functions ψ_2 and ϕ_2, expanded in terms of the Wright hypergeometric function are also established.

2 Main Results

We establish two generalized formulas for the Humbert's functions ψ_2 and ϕ_2 asserted by the following two theorems.

Theorem 2.1. *For* $a_3\neq 0,-1,-2,...,$ *the following identity for* ψ_2 *holds true:*

$$\begin{aligned}\psi_2(a_1;a_3,a_3+j;-y,y)=&\sum_{s=1}^{\infty}\frac{(a_1)_{2s}\,y^{2s}}{(a_3)_{2s}(2s)!}(\alpha_j^{'}\,A_{2s}+\beta_j^{'}\,B_{2s})\\&-\sum_{s=1}^{\infty}\frac{(a_1)_{2s+1}\,y^{2s+1}}{(a_3)_{2s+1}(2s+1)!}(\alpha_j^{''}\,A_{2s+1}+\beta_j^{''}\,B_{2s+1}),\end{aligned}\tag{2.1}$$

where

$$\begin{aligned}A_{2s}&=\frac{2^{2s}\,\Gamma(\frac{1}{2})\,\Gamma(2s+a_3)\,\Gamma(a_3+j)}{\Gamma(2s+a_3+\frac{1}{2}(j+|j|))\,\Gamma(s+a_3+\frac{j}{2})\,\Gamma(-s+\frac{j}{2}+\frac{1}{2}-[\frac{j+1}{2}])},\\B_{2s}&=\frac{2^{2s}\,\Gamma(\frac{1}{2})\,\Gamma(2s+a_3)\,\Gamma(a_3+j)}{\Gamma(2s+a_3+\frac{1}{2}(j+|j|))\,\Gamma(s+a_3+\frac{j}{2}-\frac{1}{2})\,\Gamma(-s+\frac{j}{2}-[\frac{j}{2}])},\\A_{2s+1}&=\frac{2^{2s+1}\,\Gamma(\frac{1}{2})\,\Gamma(2s+a_3+1)\,\Gamma(a_3+j)}{\Gamma(2s+a_3+1+\frac{1}{2}(j+|j|))\,\Gamma(s+a_3+\frac{j}{2}+\frac{1}{2})\,\Gamma(-s+\frac{j}{2}-[\frac{j+1}{2}])},\\B_{2s+1}&=\frac{2^{2s+1}\,\Gamma(\frac{1}{2})\,\Gamma(2s+a_3+1)\,\Gamma(a_3+j)}{\Gamma(2s+a_3+1+\frac{1}{2}(j+|j|))\,\Gamma(s+a_3+\frac{j}{2})\,\Gamma(-s+\frac{j}{2}-\frac{1}{2}-[\frac{j}{2}])}\end{aligned}\tag{2.2}$$

for $j=0,\pm1,\pm2,\pm3,\pm4,\pm5$. The coefficients $\alpha_j^{'}$ and $\beta_j^{'}$ can be obtained from the tables given by J. Choi [4], by substituting a and b with $-2s$ and $-2s-a_3+1$, respectively, while the coefficients $\alpha_j^{''}$ and $\beta_j^{''}$ can be obtained from the same table by substituting a and b with $-2s-1$ and $-2s-a_3$, respectively. The obtained values of $\alpha_j^{'}$, $\beta_j^{'}$, $\alpha_j^{''}$ and $\beta_j^{''}$ are given in Table 1.

Table 1. Table for $\alpha_j^{'}$, $\alpha_j^{''}$, $\beta_j^{'}$ and $\beta_j^{''}$

j	$\alpha_j^{'}$	$\beta_j^{'}$	$\alpha_j^{''}$	$\beta_j^{''}$
5	$4s^2-5a_3^2+2s-15a_3-12$	$-4s^2+a_3^2-14s-8sa_3+a_3$	$4s^2-5a_3^2+6s-15a_3-10$	$-4s^2+a_3^2-8sa_3-18s-3a_3-8$
4	$-4s^2+a_3^2-4sa_3-6s+a_3$	$-4s-4$	$-4s^2+a_3^2-4sa_3-10s-a_3-4$	$-4a_3-4$
3	$-2s-3a_3-2$	$-2s+a_3$	$-2s-3a_3-3$	$-2s+a_3-1$
2	a_3	-2	a_3	-2

1	-1	1	-1	1
0	1	0	1	0
-1	1	1	1	1
-2	a_3-2	2	-2	2
-3	$2s+3a_3-7$	$-2s+a_3-3$	$2s+3a_3-6$	$-2s+a_3-4$
-4	$-4s^2+a_3^2-4sa_3+10s-7a_3+12$	$4a_3-12$	$-4s_2+a_3^2-4sa_3+6s-9a_3+16$	$4a_3-12$
-5	$-4s^2+5a_3-2s-35a_3+62$	$-4s^2+a_3^2-8sa_3+26s-9a_3+20$	$-4s^2+5a_3^2-6s-35a_3+60$	$-4s^2-8sa_3+22s-13a_3+32$

Proof. In order to derive our result we first replace y by $-y$ and set $z=y$ in (1.14). Further, replacing a_2 and a_3 by a_3 and a_3+j respectively in the resultant equation, we get

$$\psi_2(a_1;a_3,a_3+j;-y,y)=\sum_{s=1}^{\infty}\frac{(a_1)_s}{(a_3)_s}{}_2F_1\left[\begin{array}{c}-s,-s-a_3+1;\\ \\ a_3+j;\end{array} -1\right]\frac{(-y)^s}{s!}. \tag{2.3}$$

Now, in view of (1.18), we separate the r.h.s of (2.3) into even and odd terms as:

$$\begin{aligned}\psi_2(a_1;a_3,a_3+j;-y,y)&=\sum_{s=1}^{\infty}\frac{(a_1)_{2s}y^{2s}}{(a_3)_{2s}(2s)!}{}_2F_1\left[\begin{array}{c}-2s,-2s-a_3+1;\\ \\ a_3+j;\end{array} -1\right]\\ &-\sum_{s=1}^{\infty}\frac{(a_1)_{2s+1}y^{2s+1}}{(a_3)_{2s+1}(2s+1)!}{}_2F_1\left[\begin{array}{c}-2s-1,-2s-a_3;\\ \\ a_3+j;\end{array} -1\right].\end{aligned} \tag{2.4}$$

Finally, evaluating the both resultant ${}_2F_1$ with the help of the generalized Kummer's theorm (1.16) and then, after some simplification by using the identities (1.19) - (1.21), we arrive at the right-hand side of our general formula (2.1). This completes the proof of Theorem 2.1. □

Theorem 2.2. *For* $a_3\neq 0,-1,-2,...$, *the following identity for* ϕ_2 *holds true:*

$$\begin{aligned}\phi_2(a_1,a_1+j;a_3;-y,y)&=\sum_{r=1}^{\infty}\frac{(a_1)_{2r}\,y^{2r}}{(a_3)_{2r}(2r)!}(\gamma_j\,C_{2r}+\delta_j\,D_{2r})\\ &-\sum_{r=1}^{\infty}\frac{(a_1)_{2r+1}\,y^{2r+1}}{(a_3)_{2r+1}(2r+1)!}(\gamma_j^{'}\,C_{2r+1}+\delta_j^{'}\,D_{2r+1}),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}C_{2r}&=\frac{2^{2r}\,\Gamma(\frac{1}{2})\,\Gamma(1-a_1-j)\,\Gamma(1-a_1-2r)}{\Gamma(1-a_1+\frac{1}{2}(|j|-j))\,\Gamma(-r-a_1-\frac{j}{2}+1)\,\Gamma(-r+\frac{j}{2}+\frac{1}{2}-[\frac{j+1}{2}])},\\ D_{2r}&=\frac{2^{2r}\,\Gamma(\frac{1}{2})\,\Gamma(1-a_1-j)\,\Gamma(1-a_1-2r)}{\Gamma(1-a_1+\frac{1}{2}(|j|-j))\,\Gamma(-r-a_1-\frac{j}{2}+\frac{1}{2})\,\Gamma(-r+\frac{j}{2}-[\frac{j}{2}])},\\ C_{2r+1}&=\frac{2^{2r+1}\,\Gamma(\frac{1}{2})\,\Gamma(1-a_1-j)\,\Gamma(-a_1-2r)}{\Gamma(1-a_1+\frac{1}{2}(|j|-j))\,\Gamma(-r-a_1-\frac{j}{2}+\frac{1}{2})\,\Gamma(-r+\frac{j}{2}-[\frac{j+1}{2}])},\\ D_{2r+1}&=\frac{2^{2r+1}\,\Gamma(\frac{1}{2})\,\Gamma(1-a_1-j)\,\Gamma(-a_1-2r)}{\Gamma(1-a_1+\frac{1}{2}(|j|-j))\,\Gamma(-r-a_1-\frac{j}{2})\,\Gamma(-r+\frac{j}{2}-\frac{1}{2}-[\frac{j}{2}])}\end{aligned} \tag{2.6}$$

for $j=0,\pm1,\pm2,\pm3,\pm4,\pm5$. The coefficients γ_j and δ_j can be obtained from the tables given by J. Choi [4], by substituting a and b with $-2r$ and a_1+j, respectively, while the coefficients $\gamma_j^{'}$ and $\delta_j^{'}$ can be obtained from the same table by substituting a and b with $-2r-1$ and a_1+j, respectively. The obtained values of γ_j, δ_j, $\gamma_j^{'}$ and $\delta_j^{'}$ are given in Table 2.

Table 2. Table for γ_j, δ_j, $\gamma_j^{'}$ and $\delta_j^{'}$

j	γ_j	δ_j	$\gamma_j^{'}$	$\delta_j^{'}$
5	$-16r^2 - 5a_1^2 - 20ra_1 - 48r - 25a_1 - 32$	$16r^2 + a_1^2 + 12ra_1 + 32r + 7a_1 + 12$	$-16r^2 - 5a_1^2 - 20ra_1 - 64r - 35a_1 - 60$	$16r^2 + a_1^2 + 12ra_1 + 48r + 13a_1 + 32$
4	$8r^2 + a_1^2 + 8ra_1 + 16r + 5a_1 + 6$	$4a_1 + 8r + 8$	$8r^2 + a_1^2 + 8ra_1 + 24r + 9a_1 + 16$	$8r + 4a_1 + 12$
3	$3a_1 + 4r + 4$	$-4r - a_1 - 2$	$4r + 3a_1 + 6$	$-4r - a_1 - 4$
2	$-2r - a_1 - 1$	-2	$-2r - a_1 - 2$	-2
1	-1	1	-1	1
0	1	0	1	0
-1	1	1	1	1
-2	$-2r - a_1 + 1$	2	$-2r - a_1$	2
-3	$-4r - 3a_1 + 5$	$-4r - a_1 + 1$	$-4r - 3a_1 + 3$	$-4r - a_1 - 1$
-4	$8r^2 + r^2 + 8ra_1 - 16r - 3a_1 + 2$	$-8r - 4a_1 + 8$	$8r^2 + a_1^2 + 8ra_1 - 8r + a_1 - 4$	$-8r - 4a_1 + 4$
-5	$16r^2 + 5r^2 + 20ra_1 - 52r - 25a_1 + 32$	$16r^2 + a_1^2 + 12ra_1 - 28r - 3a_1 + 2$	$16r^2 + 5a_1^2 - 20ra_1 - 36r - 15a_1 + 10$	$16r^2 + a_1^2 + 12ra_1 - 12r + 3a_1 - 8$

Proof. Making use of (1.15) and using a similar argument as in the proof of the Theorem 2.1, we can establish the general formula (2.5). □

Several integral representations of the Humbert's function ϕ_2 is derived in [5], one of which is given as follows:

$$\phi_2(\beta_1, \beta_2; \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\varepsilon_1)\Gamma(\beta_2)\Gamma(\gamma - \varepsilon_1 - \beta_2)} \int_0^1 \int_0^1 e^{x\xi + y(1-\xi)\eta} \xi^{\varepsilon_1 - 1} \eta^{\beta_2 - 1}$$
$$\times (1-\xi)^{\gamma - \varepsilon_1 - 1} (1-\eta)^{\gamma - \varepsilon_1 - \beta_2 - 1} {}_1F_1(\beta_1 - \varepsilon_1; \gamma - \varepsilon_1 - \beta_2; x(1-\xi)(1-\eta) d\xi d\eta, \quad (2.7)$$
$$(\Re(\gamma - \varepsilon_1 - \beta_2) > 0, \Re(\varepsilon_1) > 0, \Re(\beta_2) > 0).$$

Now, by using (2.7) and (2.5), we establish general integral relation which includes eleven identities (i.e, for $j = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$) asserted by the following corollary.

Corollary 2.3. *The following integral relation holds true:*

$$\int_0^1 \int_0^1 e^{-y\xi + y(1-\xi)\eta} \xi^{\varepsilon_1 - 1} \eta^{a_1 + j - 1} (1-\xi)^{a_3 - \varepsilon_1 - 1} (1-\eta)^{a_3 - \varepsilon_1 - a_1 - j - 1}$$
$${}_1F_1(a_1 - \varepsilon_1; a_3 - \varepsilon_1 - a_1 - j; -y(1-\xi)(1-\eta) d\xi d\eta$$
$$= \frac{\Gamma(\varepsilon_1)\Gamma(a_1 + j)\Gamma(\gamma - \varepsilon_1 - a_1 - j)}{\Gamma(a_3)} \left\{ \sum_{r=1}^{\infty} \frac{(a_1)_{2r}\, y^{2r}}{(a_3)_{2r}(2r)!} (\gamma_j\, C_{2r} + \delta_j\, D_{2r}) \right.$$
$$\left. - \sum_{r=1}^{\infty} \frac{(a_1)_{2r+1}\, y^{2r+1}}{(a_3)_{2r+1}(2r+1)!} (\gamma_j^{'}\, C_{2r+1} + \delta_j^{'}\, D_{2r+1}) \right\}, \quad (2.8)$$

where $(\Re(a_3 - \varepsilon_1 - a_1 - j) > 0, \Re(\varepsilon_1) > 0, \Re(a_1 + j) > 0)$ *and* $j = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$*, the coefficients* γ_j*,* δ_j*,* $\gamma_j^{'}$*,* $\delta_j^{'}$ *and other notations are the same as in* (2.5,2.6).

Proof. We can derive our assertion in a straightforward way. Indeed, if we set $\beta_1 = a_1, \beta_2 = a_1 + j$ and $\gamma = a_3$ in (2.7), we obtain

$$\phi_2(a_1, a_1 + j; a_3; x, y) = \frac{\Gamma(a_3)}{\Gamma(\varepsilon_1)\Gamma(a_1 + j)\Gamma(\gamma - \varepsilon_1 - a_1 - j)} \int_0^1 \int_0^1 e^{-y\xi + y(1-\xi)\eta} \xi^{\varepsilon_1 - 1} \eta^{a_1 + j - 1}$$
$$(1-\xi)^{a_3 - \varepsilon_1 - 1} (1-\eta)^{a_3 - \varepsilon_1 - a_1 - j - 1} {}_1F_1(a_1 - \varepsilon_1; a_3 - \varepsilon_1 - a_1 - j; -y(1-\xi)(1-\eta) d\xi d\eta. \quad (2.9)$$

Replacing x by $-y$ in (2.9) and applying result (2.5) in the L.H.S. of the resultant equation we get our desired result. This completes the proof of (2.8). □

3 Special Cases

(i) Setting $j = 0$ in the results (2.1) and (2.5) yield the known results (1.9) and (1.13) respectively.

(ii) Taking $j = 1$ in the results (2.1) and (2.5), we obtain

$$\psi_2(a_1; a_3, a_3+1; -y, y) = {}_2F_3\left[\begin{matrix} \frac{1}{2}a_1, \frac{1}{2}a_1+\frac{1}{2}; \\ a_3+1, \frac{1}{2}a_3+\frac{1}{2}, \frac{1}{2}a_3+1; \end{matrix} - y^2\right] - \frac{a_1 y}{a_3(a_3+1)} {}_2F_3\left[\begin{matrix} \frac{1}{2}a_1+\frac{1}{2}, \frac{1}{2}a_1+1; \\ a_3+1, \frac{1}{2}a_3+1, \frac{1}{2}a_3+\frac{3}{2}; \end{matrix} - y^2\right] \tag{3.1}$$

and

$$\phi_2(a_1, a_1+1; a_3; -y, y) = {}_1F_2\left[\begin{matrix} a_1+1; \\ \frac{1}{2}a_3, \frac{1}{2}a_3+\frac{1}{2}; \end{matrix} \frac{y^2}{4}\right] + \frac{y}{a_3} {}_1F_2\left[\begin{matrix} a_1+1; \\ \frac{1}{2}a_3+\frac{1}{2}, \frac{1}{2}a_3+1; \end{matrix} \frac{y^2}{4}\right], \tag{3.2}$$

respectively.

(iii) Taking $j = -1$ in the results (2.1) and (2.5), we get

$$\psi_2(a_1; a_3, a_3-1; -y, y) = {}_2F_3\left[\begin{matrix} \frac{1}{2}a_1, \frac{1}{2}a_1+\frac{1}{2}; \\ a_3-1, \frac{1}{2}a_3, \frac{1}{2}a_3+\frac{1}{2}; \end{matrix} - y^2\right] + \frac{a_1 y}{a_3(a_3-1)} {}_2F_3\left[\begin{matrix} \frac{1}{2}a_1+\frac{1}{2}, \frac{1}{2}a_1+1; \\ a_3, \frac{1}{2}a_3+\frac{1}{2}, \frac{1}{2}a_3+1; \end{matrix} - y^2\right] \tag{3.3}$$

and

$$\phi_2(a_1, a_1-1; a_3; -y, y) = {}_1F_2\left[\begin{matrix} a_1; \\ \frac{1}{2}a_3, \frac{1}{2}a_3+\frac{1}{2}; \end{matrix} \frac{y^2}{4}\right] - \frac{y}{a_3} {}_1F_2\left[\begin{matrix} a_1; \\ \frac{1}{2}a_3+\frac{1}{2}, \frac{1}{2}a_3+1; \end{matrix} \frac{y^2}{4}\right], \tag{3.4}$$

respectively.

Similarly, we can derive the other special cases (for $j = \pm 2, \pm 3, \pm 4, \pm 5$).

Remark 3.1. Setting $y = -x$ in (3.1)-(3.4), we get the known results ([7];17, 18,21,22).

4 Integral Formulas

This section deals with some integrals involving Humbert's functions, which are expressed in terms of Wright hypergeometric function .

Theorem 4.1. *For $\upsilon, \tau \in \mathbb{C}$ and $x > 0$ with $\Re(\upsilon) > 0$, $\Re(\tau) > 0$, each of the following integral formulas holds true:*

$$(1) \int_0^1 x^{\upsilon-1} (1-x)^{2\tau-1} \left(1 - \frac{x}{3}\right)^{2\upsilon-1} \left(1 - \frac{x}{4}\right)^{\tau-1}$$

$$\times\ \psi_2\big(a_1; a_3, a_3; -y\big(1-x/4\big)^{\frac{1}{2}}(1-x), y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$= \left(\frac{2}{3}\right)^{2\upsilon} \frac{\Gamma(\upsilon)\ (\Gamma(a_3))^2}{\Gamma(a_1)}\ {}_2\Psi_3\left[\begin{matrix}(a_1,2),(\tau,1);\\ (a_3,1),(a_3,2),(\upsilon+\tau,1);\end{matrix}\ -y^2\right], \quad (4.1)$$

(2) $$\int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$

$$\times\ \psi_2\big(a_1; a_3, a_3+1; -y\big(1-x/4\big)^{\frac{1}{2}}(1-x), y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$= \left(\frac{2}{3}\right)^{2\upsilon} \frac{\Gamma(\upsilon)\Gamma(a_3+1)}{\Gamma(a_1)}\left\{\Gamma(a_3+1)\,{}_2\Psi_3\left[\begin{matrix}(a_1,2),(\tau,1);\\ (a_3+1,1),(a_3+1,2),(\upsilon+\tau,1);\end{matrix}\ -y^2\right]\right.$$
$$\left. -\ y\,\Gamma(a_3)\ {}_2\Psi_3\left[\begin{matrix}(a_1+1,2),(\tau+\frac{1}{2},1);\\ (a_3+1,1),(a_3+2,2),(\upsilon+\tau+\frac{1}{2},1);\end{matrix}\ -y^2\right]\right\}, \quad (4.2)$$

(3) $$\int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$

$$\times\ \psi_2\big(a_1; a_3, a_3-1; -y\big(1-x/4\big)^{\frac{1}{2}}(1-x), y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$= \left(\frac{2}{3}\right)^{2\upsilon} \frac{\Gamma(\upsilon)\Gamma(a_3)}{\Gamma(a_1)}\left\{\Gamma(a_3-1)\,{}_2\Psi_3\left[\begin{matrix}(a_1,2),(\tau,1);\\ (a_3-1,1),(a_3,2),(\upsilon+\tau,1);\end{matrix}\ -y^2\right]\right.$$
$$\left. +\ y\,\Gamma(a_3-1)\ {}_2\Psi_3\left[\begin{matrix}(a_1+1,2),(\tau+\frac{1}{2},1);\\ (a_3,1),(a_3+1,2),(\upsilon+\tau+\frac{1}{2},1);\end{matrix}\ -y^2\right]\right\}. \quad (4.3)$$

Proof. By making use of the relation (1.9) in the integrand of (4.1) and then interchanging the order of the integration and summation of the resultant equation which is verified by uniform convergence of the involved series under the given conditions, gives

$$\int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$

$$\times\ \psi_2\big(a_1; a_3, a_3; -y\big(1-x/4\big)^{\frac{1}{2}}(1-x), y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_{2k}\ (-y^2)^k}{(a_3)_k\ (a_3)_{2k}\ k!} \int_0^1 x^{\upsilon-1}\,(1-x)^{2(\tau+k)-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{(\tau+k)-1} dx. \quad (4.4)$$

Now, using the identity (1.22) in the above equation, we get

$$\int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$

$$\times\ \psi_2\big(a_1;a_3,a_3;-y\big(1-x/4\big)^{\frac{1}{2}}(1-x),y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$=\left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\upsilon)\ (\Gamma(a_3))^2}{\Gamma(a_1)}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+2k)\ \Gamma(\tau+k)}{\Gamma(a_3+k)\ \Gamma(a_3+2k)\ \Gamma(\upsilon+\tau+k)}\frac{(-y^2)^k}{k!},\tag{4.5}$$

which on using the definition of Wright hypergeometric function ${}_p\Psi_q$ (1.5) in the above equation yields our desired result (4.1). Further, making use of the similar argument as in the above proof of the result (4.1), we can establish the integral formulas (4.2) and (4.3). □

Corollary 4.2. *For $\upsilon,\tau\in\mathbb{C}$ and $x>0$ with $\Re(\upsilon)>0$, $\Re(\tau)>0$, each of the following integral formulas holds true:*

$$(1)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$

$$\times\ \psi_2\big(a_1;a_3,a_3;-y\big(1-x/4\big)^{\frac{1}{2}}(1-x),y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$=\left(\frac{2}{3}\right)^{2\upsilon}B(\upsilon,\tau)\ {}_3F_4\left[\begin{matrix}\tau,\frac{1}{2}a_1,\frac{1}{2}a_1+\frac{1}{2};\\ \upsilon+\tau,a_3,\frac{1}{2}a_3,\frac{1}{2}a_3+\frac{1}{2};\end{matrix}\ -y^2\right],\tag{4.6}$$

$$(2)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$

$$\times\ \psi_2\big(a_1;a_3,a_3+1;-y\big(1-x/4\big)^{\frac{1}{2}}(1-x),y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$=\left(\frac{2}{3}\right)^{2\upsilon}B(\upsilon,\tau)\Bigg\{{}_3F_4\left[\begin{matrix}\tau,\frac{1}{2}a_1,\frac{1}{2}a_1+\frac{1}{2};\\ \upsilon+\tau,a_3+1,\frac{1}{2}a_3+\frac{1}{2},\frac{1}{2}a_3+1;\end{matrix}\ -y^2\right]$$

$$-\frac{a_1y}{a_3(a_3+1)}B\left(\upsilon,\tau+\frac{1}{2}\right)\ {}_3F_4\left[\begin{matrix}\tau+\frac{1}{2},\frac{1}{2}a_1+\frac{1}{2},\frac{1}{2}a_1+1;\\ \upsilon+\tau+\frac{1}{2},a_3+1,\frac{1}{2}a_3+1,\frac{1}{2}a_3+\frac{3}{2};\end{matrix}\ -y^2\right]\Bigg\},\tag{4.7}$$

$$(3)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$

$$\times\ \psi_2\big(a_1;a_3,a_3-1;-y\big(1-x/4\big)^{\frac{1}{2}}(1-x),y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$=\left(\frac{2}{3}\right)^{2\upsilon}B(\upsilon,\tau)\Bigg\{{}_3F_4\left[\begin{matrix}\tau,\frac{1}{2}a_1,\frac{1}{2}a_1+\frac{1}{2};\\ \upsilon+\tau,a_3-1,\frac{1}{2}a_3,\frac{1}{2}a_3+\frac{1}{2};\end{matrix}\ -y^2\right]$$

$$+\frac{a_1y}{a_3(a_3-1)}B\left(\upsilon,\tau+\frac{1}{2}\right)\ {}_3F_4\left[\begin{matrix}\tau+\frac{1}{2},\frac{1}{2}a_1+\frac{1}{2},\frac{1}{2}a_1+1;\\ \upsilon+\tau+\frac{1}{2},a_3,\frac{1}{2}a_3+\frac{1}{2},\frac{1}{2}a_3+1;\end{matrix}\ -y^2\right]\Bigg\}.\tag{4.8}$$

Proof. By writing the r.h.s. of equation (4.6) in the series form and applying identity (1.19) to the resultant series, then after some simplifications and using identity (1.4) and definition of the generalized hypergeometric function ${}_pF_q$ (1.1), we arrive at our assertion (4.6). Similarly we can derive integral formulas (4.7) and (4.8). □

Remark 4.3. Using a similar argument as in the proof of Theorem 4.1, we can prove the following theorems.

Theorem 4.4. *For $\upsilon, \tau \in \mathbb{C}$ and $x > 0$ with $\Re(\upsilon) > 0$, $\Re(\tau) > 0$, each of the following integral formulas holds true:*

$$(1)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1} \times\ \psi_2\big(a_1; a_3, a_3; -yx^{\frac{1}{2}}(1-x/3), yx^{\frac{1}{2}}(1-x/3)\big)dx$$

$$= \left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\tau)\,(\Gamma(a_3))^2}{\Gamma(a_1)}\,{}_2\Psi_3\left[\begin{matrix}(a_1,2),(\upsilon,1);\\ (a_3,1),(a_3,2),(\upsilon+\tau,1);\end{matrix}\ \frac{-4y^2}{9}\right], \tag{4.9}$$

$$(2)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1} \times\ \psi_2\big(a_1; a_3, a_3+1; -yx^{\frac{1}{2}}(1-x/3), yx^{\frac{1}{2}}(1-x/3)\big)dx$$

$$= \left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\tau)\Gamma(a_3+1)}{\Gamma(a_1)}\left\{\Gamma(a_3+1)\,{}_2\Psi_3\left[\begin{matrix}(a_1,2),(\upsilon,1);\\ (a_3+1,1),(a_3+1,2),(\upsilon+\tau,1);\end{matrix}\ \frac{-4y^2}{9}\right]\right.$$
$$\left. -\ \frac{2y}{3}\,\Gamma(a_3)\,{}_2\Psi_3\left[\begin{matrix}(a_1+1,2),(\upsilon+\frac{1}{2},1);\\ (a_3+1,1),(a_3+2,2),(\upsilon+\tau+\frac{1}{2},1);\end{matrix}\ \frac{-4y^2}{9}\right]\right\}, \tag{4.10}$$

$$(3)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1} \times\ \psi_2\big(a_1; a_3, a_3-1; -yx^{\frac{1}{2}}(1-x/3), yx^{\frac{1}{2}}(1-x/3)\big)dx$$

$$= \left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\tau)\Gamma(a_3)}{\Gamma(a_1)}\left\{\Gamma(a_3-1)\,{}_2\Psi_3\left[\begin{matrix}(a_1,2),(\upsilon,1);\\ (a_3-1,1),(a_3,2),(\upsilon+\tau,1);\end{matrix}\ \frac{-4y^2}{9}\right]\right.$$
$$\left. +\ \frac{2y}{3}\,\Gamma(a_3-1)\,{}_2\Psi_3\left[\begin{matrix}(a_1+1,2),(\upsilon+\frac{1}{2},1);\\ (a_3,1),(a_3+1,2),(\upsilon+\tau+\frac{1}{2},1);\end{matrix}\ \frac{-4y^2}{9}\right]\right\}. \tag{4.11}$$

Theorem 4.5. *For $\upsilon, \tau \in \mathbb{C}$ and $x > 0$ with $\Re(\upsilon) > 0$, $\Re(\tau) > 0$, each of the following integral formulas holds true:*

$$(1)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1} \times\ \phi_2\big(a_1, a_1; a_3; -y(1-x/4)^{\frac{1}{2}}(1-x), y(1-x/4)^{\frac{1}{2}}(1-x)\big)dx$$

$$= \left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\upsilon)\,(\Gamma(a_3))}{\Gamma(a_1)}\,{}_2\Psi_2\left[\begin{matrix}(a_1,1),(\tau,1);\\ (a_3,2),(\upsilon+\tau,1);\end{matrix}\ y^2\right], \tag{4.12}$$

$$(2)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$
$$\times\ \phi_2\big(a_1,a_1+1;a_3,-y\big(1-x/4\big)^{\frac{1}{2}}(1-x),y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$=\left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\upsilon)(\Gamma(a_3))}{\Gamma(a_1+1)}\left\{{}_2\Psi_2\left[\begin{array}{c}(a_1+1,2),(\tau,1);\\ \\(a_3,2),(\upsilon+\tau,1);\end{array}\ y^2\right]\right.$$
$$\left.+\ y\ {}_2\Psi_2\left[\begin{array}{c}(a_1+1,1),(\tau+\frac{1}{2},1);\\ \\(a_3+1,2),(\upsilon+\tau+\frac{1}{2},1);\end{array}\ y^2\right]\right\}, \tag{4.13}$$

$$(3)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$
$$\times\ \phi_2\big(a_1,a_1-1;a_3,y\big(1-x/4\big)^{\frac{1}{2}}(1-x),y\big(1-x/4\big)^{\frac{1}{2}}(1-x)\big)dx$$

$$=\left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\upsilon)(\Gamma(a_3))}{\Gamma(a_1)}\left\{{}_2\Psi_2\left[\begin{array}{c}(a_1,2),(\tau,1);\\ \\(a_3,2),(\upsilon+\tau,1);\end{array}\ y^2\right]\right.$$
$$\left.-\ y\ {}_2\Psi_2\left[\begin{array}{c}(a_1,1),(\tau+\frac{1}{2},1);\\ \\(a_3+1,2),(\upsilon+\tau+\frac{1}{2},1);\end{array}\ y^2\right]\right\}. \tag{4.14}$$

Theorem 4.6. *For $\upsilon,\tau\in\mathbb{C}$ and $x>0$ with $\Re(\upsilon)>0$, $\Re(\tau)>0$, each of the following integral formulas holds true:*

$$(1)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$
$$\times\ \phi_2\big(a_1,a_1;a_3;-yx^{\frac{1}{2}}(1-x/3),yx^{\frac{1}{2}}(1-x/3)\big)dx$$
$$=\left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\tau)\ (\Gamma(a_3)}{\Gamma(a_1)}\ {}_2\Psi_2\left[\begin{array}{c}(a_1,1),(\upsilon,1);\\ \\(a_3,2),(\upsilon+\tau,1);\end{array}\ \frac{4y^2}{9}\right], \tag{4.15}$$

$$(2)\ \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1}$$
$$\times\ \phi_2\big(a_1,a_1+1;a_3,-yx^{\frac{1}{2}}(1-x/3),yx^{\frac{1}{2}}(1-x/3)\big)dx$$

$$=\left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\tau)\Gamma(a_3)}{\Gamma(a_1+1)}\left\{{}_2\Psi_2\left[\begin{array}{c}(a_1+1,1),(\upsilon,1);\\ \\(a_3,2),(\upsilon+\tau,1);\end{array}\ \frac{4y^2}{9}\right]\right.$$
$$\left.+\ \frac{2y}{3}\ {}_2\Psi_2\left[\begin{array}{c}(a_1+1,1),(\upsilon+\frac{1}{2},1);\\ \\(a_3+1,2),(\upsilon+\tau+\frac{1}{2},1);\end{array}\ \frac{4y^2}{9}\right]\right\}, \tag{4.16}$$

$$
\begin{aligned}
(3)\ & \int_0^1 x^{\upsilon-1}\,(1-x)^{2\tau-1}\left(1-\frac{x}{3}\right)^{2\upsilon-1}\left(1-\frac{x}{4}\right)^{\tau-1} \\
& \qquad \times\ \phi_2\big(a_1, a_3-1; a_3, -yx^{\frac{1}{2}}(1-x/3), yx^{\frac{1}{2}}(1-x/3)\big)dx \\
= & \left(\frac{2}{3}\right)^{2\upsilon}\frac{\Gamma(\tau)\Gamma(a_3)}{\Gamma(a_1)}\left\{{}_2\Psi_2\left[\begin{array}{c} (a_1,1),(\upsilon,1); \\ \\ (a_3,2),(\upsilon+\tau,1); \end{array}\ \frac{4y^2}{9}\right]\right. \\
& \left. -\ \frac{2y}{3}\,{}_2\Psi_2\left[\begin{array}{c} (a_1,1),(\upsilon+\frac{1}{2},1); \\ \\ (a_3+1,2),(\upsilon+\tau+\frac{1}{2},1); \end{array}\ \frac{4y^2}{9}\right]\right\}. \qquad (4.17)
\end{aligned}
$$

We conclude this paper by remarking that this approach is general and can be applied to some other multivariate hypergeometric functions and is a problem for further research.

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Author information

Ghazala Yasmin and Abdulghani Muhyi, Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India.
E-mail: ghazala30@gmail.com, muhyi2007@gmail.com

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