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# A FOUR-PARAMETER NON-LINEAR RECURRENCE IDENTITY CLASS FOR TERMS OF A QUASI FIBONACCI SEQUENCE

Peter J. Larcombe and Eric J. Fennessey

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**Abstract**. Using an existing matrix method developed by the authors, a non-linear recurrence identity class is formulated for terms of a quasi Fibonacci sequence which—in addition to fundamental sequence variables—is characterised by four arbitrary parameters. The possibility for further systematic re-application of the technique is also discussed and illustrated.

## 1 Introduction

Denote by $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a,b;p,q)\}_0^{\infty}$, in standard format, the four-parameter Horadam sequence produced by the second order linear recursion

$$w_{n+2} = pw_{n+1} - qw_n, \qquad n \geq 0, \tag{1.1}$$

for which $w_0 = a$ and $w_1 = b$ are initial values. In this short paper we extend a matrix method devised previously (to generate a non-linear recurrence identity class for Horadam sequence terms, see [1]), and produce a new class that necessarily applies to terms of a so called quasi Fibonacci sequence $\{w_n(a,b;p,-1)\}_0^{\infty} = \{a, b, bp+a, bp^2+ap+b, bp^3+ap^2+2bp+a, bp^4+ap^3+3bp^2+2ap+b, \ldots\}$. We then detail briefly how the approach can be re-applied to produce other classes of identities with the same essential structure (but containing additional free parameters), and illustrate the idea accordingly.

## 2 A Result and Proof

### 2.1 Result

Writing

$$\begin{aligned}
\zeta_1(\beta,\gamma,\delta,\epsilon) &= \gamma\epsilon + \beta\delta,\\
\zeta_2(p;\beta,\gamma,\delta,\epsilon) &= \gamma\delta + \beta(\epsilon - \delta p),\\
\zeta_3(p;\beta,\gamma,\delta,\epsilon) &= \beta\delta + (\gamma - \beta p)(\epsilon - \delta p),
\end{aligned} \tag{2.1}$$

we establish the following result which describes a class of identities characterised by the four parameters $\beta,\gamma,\delta,\epsilon$ (over and above $a,b$ that appear explicitly) through $\zeta_1,\zeta_2$ and $\zeta_3$ as defined.

**Identity.** For $r,t \geq 0$, and arbitrary parameters $\beta,\gamma,\delta,\epsilon$,

$$w^*_{r+1}(\zeta_1 w^*_{t+1} + \zeta_2 w^*_t) + w^*_r(\zeta_2 w^*_{t+1} + \zeta_3 w^*_t) = (b\zeta_1 + a\zeta_2)w^*_{r+t+1} + (b\zeta_2 + a\zeta_3)w^*_{r+t},$$

where $\{w^*_n\}_0^{\infty} = \{w_n(a,b;p,-1)\}_0^{\infty}$ is a quasi Fibonacci sequence.

### 2.2 Proof

*Proof.* Let

$$\mathbf{H}(p,q) = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \tag{I.1}$$

from which the recursion (1.1) readily delivers the matrix power relation

$$\begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \mathbf{H}^{n-1}(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \tag{I.2}$$

that holds for $n \geq 1$.

For arbitrary $\beta, \gamma$, the symmetric matrix

$$\mathbf{S}^{[\beta,\gamma]}(p,q) = \begin{pmatrix} \gamma & \beta \\ \beta & -(\beta p + \gamma q) \end{pmatrix} \tag{I.3}$$

is the most general one that *quasi-commutes* with $\mathbf{H}(p,q)$—what we mean by this [1] is that

$$\mathbf{S}^{[\beta,\gamma]}(p,q)\mathbf{H}(p,q) = [\mathbf{H}(p,q)]^T \mathbf{S}^{[\beta,\gamma]}(p,q), \tag{I.4}$$

writing $T$ to denote transposition. With $\delta, \epsilon$ also arbitrary, we introduce a similar (that is, quasi-commuting) matrix

$$\mathbf{T}^{[\delta,\epsilon]}(p,q) = \begin{pmatrix} \epsilon & \delta \\ \delta & -(\delta p + \epsilon q) \end{pmatrix}. \tag{I.5}$$

We will need the product matrix $\mathbf{S}^{[\beta,\gamma]}(p,q)\mathbf{T}^{[\delta,\epsilon]}(p,q)$ to quasi-commute with $\mathbf{H}(p,q)$ in order to formulate our identity class, and it is easily seen that this occurs iff $\mathbf{H}(p,q)$ is symmetric. For the sufficiency element of the argument we assume such symmetry and argue as follows: $\mathbf{H}^T = \mathbf{H} \Rightarrow \mathbf{S}\mathbf{H}^T = \mathbf{S}\mathbf{H} = \mathbf{H}^T\mathbf{S}$ (by quasi-commutativity of $\mathbf{S}$), so that $(\mathbf{H}^T\mathbf{S})\mathbf{T} = (\mathbf{S}\mathbf{H}^T)\mathbf{T} \Rightarrow \mathbf{H}^T(\mathbf{S}\mathbf{T}) = \mathbf{S}(\mathbf{H}^T\mathbf{T}) = \mathbf{S}(\mathbf{T}\mathbf{H})$ (by quasi-commutativity of $\mathbf{T}$) $= (\mathbf{S}\mathbf{T})\mathbf{H}$, as required. The necessary element is similar ($\mathbf{S}^{[\beta,\gamma]}(p,q), \mathbf{T}^{[\delta,\epsilon]}(p,q)$ are taken as invertible), and is left as a routine reader exercise. The matrix $\mathbf{H}(p,q)$ is symmetric *only* for $q = -1$, so our forthcoming result applies in fact to the sequence $\{w_n(a,b;p,-1)\}_0^\infty = \{w_n^*\}_0^\infty$, say, that is generated (given $w_0^* = a$ and $w_1^* = b$) by the recurrence

$$w_{n+2}^* = p w_{n+1}^* + w_n^*, \qquad n \geq 0, \tag{I.6}$$

and which we call here a *quasi Fibonacci sequence* (possessing only one governing recursion variable $p$).

**Remark 2.1.** With $q = -1$ we observe that each of $\mathbf{S}^{[\beta,\gamma]}(p,-1)$ and $\mathbf{T}^{[\delta,\epsilon]}(p,-1)$ commute with $\mathbf{H}(p,-1)$, from which it is elementary to show that the product matrix $\mathbf{S}^{[\beta,\gamma]}(p,-1)\mathbf{T}^{[\delta,\epsilon]}(p,-1)$ also commutes with $\mathbf{H}(p,-1) = [\mathbf{H}(p,-1)]^T$; in other words, $\mathbf{S}^{[\beta,\gamma]}(p,-1)\mathbf{T}^{[\delta,\epsilon]}(p,-1)$ quasi-commutes with $\mathbf{H}(p,-1)$.

Define, using (I.3) and (I.5), $\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)$ to be the (quasi-commuting) product matrix

$$\begin{aligned} \mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p) &= \mathbf{S}^{[\beta,\gamma]}(p,-1)\mathbf{T}^{[\delta,\epsilon]}(p,-1) \\ &= \begin{pmatrix} \gamma & \beta \\ \beta & \gamma - \beta p \end{pmatrix} \begin{pmatrix} \epsilon & \delta \\ \delta & \epsilon - \delta p \end{pmatrix} \\ &= \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix}, \end{aligned} \tag{I.7}$$

whose functional entries are given in (2.1). Then, we consider the construct

$$\mathbf{T}_n(p;\beta,\gamma,\delta,\epsilon) = \mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\mathbf{H}^{n-1}(p,-1) \begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}. \tag{I.8}$$

First, we write

$$\mathbf{T}_n(p;\beta,\gamma,\delta,\epsilon) = \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix} \begin{pmatrix} w_n^* \\ w_{n-1}^* \end{pmatrix} = \begin{pmatrix} \zeta_1 w_n^* + \zeta_2 w_{n-1}^* \\ \zeta_2 w_n^* + \zeta_3 w_{n-1}^* \end{pmatrix} \tag{I.9}$$

by (I.2)[1] and (I.7), so that

$$\begin{aligned}(w_1^*, w_0^*)\mathbf{T}_n(p;\beta,\gamma,\delta,\epsilon) &= w_1^*(\zeta_1 w_n^* + \zeta_2 w_{n-1}^*) + w_0^*(\zeta_2 w_n^* + \zeta_3 w_{n-1}^*)\\ &= (b\zeta_1 + a\zeta_2)w_n^* + (b\zeta_2 + a\zeta_3)w_{n-1}^* \qquad \text{(I.10)}\end{aligned}$$

and, in particular,

$$(w_1^*, w_0^*)\mathbf{T}_{r+t+1}(p;\beta,\gamma,\delta,\epsilon) = (b\zeta_1 + a\zeta_2)w_{r+t+1}^* + (b\zeta_2 + a\zeta_3)w_{r+t}^*. \qquad \text{(I.11)}$$

From (I.8) we may also write (appealing to the quasi-commutivity of $\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)$ as needed, along with (I.2) once more)

$$\begin{aligned}(w_1^*, w_0^*)\mathbf{T}_{r+t+1}(p;\beta,\gamma,\delta,\epsilon) &= (w_1^*, w_0^*)\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\mathbf{H}^{r+t}(p,-1)\begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}\\
&= (w_1^*, w_0^*)\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\mathbf{H}^{t+r}(p,-1)\begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}\\
&= (w_1^*, w_0^*)[\mathbf{H}^{t+r}(p,-1)]^T\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}\\
&= (w_1^*, w_0^*)[\mathbf{H}^{t}(p,-1)\mathbf{H}^{r}(p,-1)]^T\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}\\
&= (w_1^*, w_0^*)[\mathbf{H}^{r}(p,-1)]^T[\mathbf{H}^{t}(p,-1)]^T\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}\\
&= \begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}^T[\mathbf{H}^{r}(p,-1)]^T\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\mathbf{H}^{t}(p,-1)\begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}\\
&= \left[\mathbf{H}^{r}(p,-1)\begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}\right]^T\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\mathbf{H}^{t}(p,-1)\begin{pmatrix} w_1^* \\ w_0^* \end{pmatrix}\\
&= \begin{pmatrix} w_{r+1}^* \\ w_r^* \end{pmatrix}^T\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\begin{pmatrix} w_{t+1}^* \\ w_t^* \end{pmatrix}\\
&= (w_{r+1}^*, w_r^*)\begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix}\begin{pmatrix} w_{t+1}^* \\ w_t^* \end{pmatrix}\\
&= w_{r+1}^*(\zeta_1 w_{t+1}^* + \zeta_2 w_t^*) + w_r^*(\zeta_2 w_{t+1}^* + \zeta_3 w_t^*); \qquad \text{(I.12)}\end{aligned}$$

the proof is concluded upon equating the r.h.s. expressions of (I.11) and (I.12). □

**Remark 2.2.** While $\mathbf{S}^{[\beta,\gamma]}(p,q)$ (I.3) and $\mathbf{T}^{[\delta,\epsilon]}(p,q)$ (I.5) are non-commutative, the matrices $\mathbf{S}^{[\beta,\gamma]}(p,-1)$ and $\mathbf{T}^{[\delta,\epsilon]}(p,-1)$ do commute (this is reflected by the invariance of the parameters $\zeta_1,\zeta_2,\zeta_3$ (2.1) under the interchange $\beta\leftrightarrow\delta$, $\gamma\leftrightarrow\epsilon$), so that no additional result is offered by swapping the order of the latter matrices and re-running the proof methodology using a reverse product matrix $\hat{\mathbf{P}}^{[\beta,\gamma,\delta,\epsilon]}(p) = \mathbf{T}^{[\delta,\epsilon]}(p,-1)\mathbf{S}^{[\beta,\gamma]}(p,-1)$ (since $\hat{\mathbf{P}}^{[\beta,\gamma,\delta,\epsilon]}(p) = \mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)$).

We finish by noting that, on choosing $\delta = 0$, $\epsilon = 1$, then $\mathbf{T}^{[0,1]}(p,-1)$ contracts to the $2\times 2$ identity matrix $\mathbf{I}_2$, $\mathbf{P}^{[\beta,\gamma,0,1]}(p) = \mathbf{S}^{[\beta,\gamma]}(p,-1)$, and our result reads (with $\zeta_1(\beta,\gamma,0,1) = \gamma$, $\zeta_2(p;\beta,\gamma,0,1) = \beta$ and $\zeta_3(p;\beta,\gamma,0,1) = \gamma - \beta p$)

$$\begin{aligned}w_{r+1}^*(\gamma w_{t+1}^* + \beta w_t^*) &+ w_r^*(\beta w_{t+1}^* + [\gamma - \beta p]w_t^*)\\ &= (b\gamma + a\beta)w_{r+t+1}^* + (b\beta + a[\gamma - \beta p])w_{r+t}^*. \qquad \text{(2.2)}\end{aligned}$$

[1]For clarity, with $q = -1$ (I.2) becomes $(w_n^*, w_{n-1}^*)^T = \mathbf{H}^{n-1}(p,-1)(w_1^*, w_0^*)^T$.

Using (I.6) as appropriate, this is seen to match a special case of the result [1, Identity (Generalised), p.408]

$$\begin{aligned} & w_{r+1}(\gamma w_{t+1}+\beta w_t)-qw_r(\gamma w_t+\beta w_{t-1}) \\ & \quad = \gamma b w_{r+t+1}+(\beta b-\gamma q a)w_{r+t}-\beta q a w_{r+t-1}, \end{aligned} \tag{2.3}$$

for the fully general Horadam sequence $\{w_n(a,b;p,q)\}_0^\infty$, on setting $q=-1$ in the latter (and replacing $w$ with $w^*$ throughout—we leave the simple algebra involved to the interested reader). As a point of interest, we note that (2.2) offers a route directly back to the $q=-1$ version of an original Horadam 1965 identity for his sequence if we set $\beta=-p$, $\gamma=1$ (see Appendix A).

## 3 Extensions

The approach taken lends itself to extension that we discuss briefly here. We introduce a matrix

$$\mathbf{U}^{[\mu,\sigma]}(p)=\begin{pmatrix} \sigma & \mu \\ \mu & \sigma-\mu p \end{pmatrix}, \tag{3.1}$$

and, writing $\mathbf{S}^{[\beta,\gamma]}(p,-1),\mathbf{T}^{[\delta,\epsilon]}(p,-1)$ as, resp., $\mathbf{S}^{[\beta,\gamma]}(p),\mathbf{T}^{[\delta,\epsilon]}(p)$, form the (triple) product

$$\begin{aligned} \mathbf{Q}^{[\beta,\gamma,\delta,\epsilon,\mu,\sigma]}(p) &= \mathbf{S}^{[\beta,\gamma]}(p)\mathbf{T}^{[\delta,\epsilon]}(p)\mathbf{U}^{[\mu,\sigma]}(p) \\ &= \mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)\mathbf{U}^{[\mu,\sigma]}(p) \\ &= \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 \end{pmatrix}\begin{pmatrix} \sigma & \mu \\ \mu & \sigma-\mu p \end{pmatrix} \\ &= \begin{pmatrix} \zeta_1\sigma+\zeta_2\mu & \zeta_1\mu+\zeta_2(\sigma-\mu p) \\ \zeta_2\sigma+\zeta_3\mu & \zeta_2\mu+\zeta_3(\sigma-\mu p) \end{pmatrix} \\ &= \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{pmatrix}, \end{aligned} \tag{3.2}$$

where (recall that any two commuting symmetric matrices produce a symmetric product)[2]

$$\begin{aligned} \phi_1(p;\beta,\gamma,\delta,\epsilon,\mu,\sigma) &= \zeta_1\sigma+\zeta_2\mu, \\ \phi_2(p;\beta,\gamma,\delta,\epsilon,\mu,\sigma) &= \zeta_2\sigma+\zeta_3\mu, \\ \phi_3(p;\beta,\gamma,\delta,\epsilon,\mu,\sigma) &= \zeta_2\mu+\zeta_3(\sigma-\mu p); \end{aligned} \tag{3.3}$$

it then follows immediately that our previous identity (which is the $\mu=0$, $\sigma=1$ instance, in which case $\mathbf{U}^{[0,1]}(p)=\mathbf{I}_2$ and $\phi_1=\zeta_1$, $\phi_2=\zeta_2$, $\phi_3=\zeta_3$) becomes $w^*_{r+1}(\phi_1 w^*_{t+1}+\phi_2 w^*_t)+w^*_r(\phi_2 w^*_{t+1}+\phi_3 w^*_t)=(b\phi_1+a\phi_2)w^*_{r+t+1}+(b\phi_2+a\phi_3)w^*_{r+t}$, characterised $(a,b$ aside) by the six arbitrary parameters $\beta,\gamma,\delta,\epsilon,\mu,\sigma$ creating the combinations $\phi_1,\phi_2,\phi_3$. The order of the matrices $\mathbf{S}^{[\beta,\gamma]}(p),\mathbf{T}^{[\delta,\epsilon]}(p)$ and $\mathbf{U}^{[\mu,\sigma]}(p)$ within $\mathbf{Q}^{[\beta,\gamma,\delta,\epsilon,\mu,\sigma]}(p)$ has no bearing on the resulting identity as they are pairwise commuting—this also applies to those matrices within quadruple and quintuple products, and so on, should the procedure be continued.

## 4 Summary

In this paper we have taken an established technique, and extended it to produce a new non-linear recurrence identity class for terms of a quasi Fibonacci sequence; further re-application of the methodology has also been discussed and illustrated. It is worth emphasising that the type of approach adopted would appear to be restricted to sequence generating linear recurrences of degree two only, for those natural analogues of $\mathbf{H}(p,q)$ (I.1) capturing the essence of recursions

[2] We clarify the expected symmetry of $\mathbf{Q}^{[\beta,\gamma,\delta,\epsilon,\mu,\sigma]}(p)$ (as exhibited by $\mathbf{P}^{[\beta,\gamma,\delta,\epsilon]}(p)$ (I.7)) by noting that $\zeta_1\mu+\zeta_2(\sigma-\mu p)=\zeta_2\sigma+\zeta_3\mu$ since, from (2.1), it is seen that the relation $\zeta_1-p\zeta_2=\zeta_3$ holds.

of degree three and beyond—in higher dimensional versions of (I.2)—can evidently never be symmetric matrices.

All results have been verified using known Horadam sequence term closed forms for $w_n(a,b;p,q)$ which have been modified to accommodate the designated recurrence value $q=-1$; the reader is referred to Appendix B, in the interest of completeness, for some details.

## Appendix A

Consider (2.2). With $\beta=-p$, $\gamma=1$, then (appealing only, but repeatedly, to (I.6)) its l.h.s. $= w^*_{r+1}(w^*_{t+1}-pw^*_t)+w^*_r(-pw^*_{t+1}+[1+p^2]w^*_t) = w^*_{t+1}(w^*_{r+1}-pw^*_r)+w^*_t(-p[w^*_{r+1}-pw^*_r]+w^*_r) = w^*_{t+1}w^*_{r-1}+w^*_t(-pw^*_{r-1}+w^*_r) = w^*_{t+1}w^*_{r-1}+w^*_t(w^*_r-pw^*_{r-1}) = w^*_{t+1}w^*_{r-1}+w^*_tw^*_{r-2}$, while the r.h.s. $=(b-ap)w^*_{r+t+1}+(-bp+a[1+p^2])w^*_{r+t} = b(w^*_{r+t+1}-pw^*_{r+t})-ap(w^*_{r+t+1}-pw^*_{r+t})+aw^*_{r+t} = bw^*_{r+t-1}-apw^*_{r+t-1}+aw^*_{r+t} = bw^*_{r+t-1}+a(w^*_{r+t}-pw^*_{r+t-1}) = bw^*_{r+t-1}+aw^*_{r+t-2}$. In other words,

$$w^*_{t+1}w^*_{r-1}+w^*_tw^*_{r-2}=bw^*_{r+t-1}+aw^*_{r+t-2}, \tag{A.1}$$

or

$$w^*_{t+1}w^*_{r+1}+w^*_tw^*_r=bw^*_{r+t+1}+aw^*_{r+t}, \tag{A.2}$$

which is the $q=-1$ instance of Horadam's identity that dates back to one of his seminal 1965 articles and is given (in an equivalent form) as [1, Identity (Horadam), p. 406].

## Appendix B

The general Horadam recurrence (1.1) has associated characteristic equation

$$\lambda^2-p\lambda+q=0, \tag{B.1}$$

with (i) distinct (non-degenerate case; $p^2\neq 4q$) solutions $\alpha(p,q)=(p+\sqrt{p^2-4q})/2$, $\beta(p,q)=(p-\sqrt{p^2-4q})/2$ as the basis of a closed form

$$w_n(a,b;p,q)=w_n(\alpha(p,q),\beta(p,q),a,b)=\frac{(b-a\beta)\alpha^n-(b-a\alpha)\beta^n}{\alpha-\beta}, \qquad n\geq 0, \tag{B.2}$$

and (ii) non-distinct (degenerate case; $p^2=4q$) solutions $\alpha(p)=\beta(p)=p/2$ which yield a corresponding closed form

$$w_n(a,b;p,p^2/4)=w_n(\alpha(p),a,b)=bn\alpha^{n-1}-a(n-1)\alpha^n, \qquad n\geq 0. \tag{B.3}$$

When $q=-1$, $w^*_n=w_n(a,b;p,-1)$ follows immediately in either case; the non-degenerate roots closed form solution (B.2) holds with $\alpha,\beta=\alpha(p,-1),\beta(p,-1)=(p\pm\sqrt{p^2+4})/2$ in the case $p^2\neq -4$, while the degenerate ($p^2=-4$) roots closed form solution (B.3) has two variants since $\alpha(p)=\alpha(\pm 2i)=(\pm 2i)/2=\pm i$.

## References

[1] P. J. Larcombe and E. J. Fennessey, A new non-linear recurrence identity class for Horadam sequence terms, *Palest. J. Math.* **7**, 406–409 (2018).

### Author information

Peter J. Larcombe, Department of Computing and Mathematics, College of Engineering and Technology, University of Derby, Kedleston Road, Derby DE22 1GB, U.K.
E-mail: `p.j.larcombe@derby.ac.uk`

Eric J. Fennessey, BAE Systems Surface Ships Ltd., HM Naval Base, Portsmouth PO1 3NJ, U.K.
E-mail: `eric.fennessey@baesystems.com`

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