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Certain Integral Transforms Involving The Product of Galue Type Struve Function and Jacobi Polynomial

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Abstract. Significant results are obtained in the study of integral transform involving the product of generalized Galue type Struve function(GGTSF) and jacobi polynomial, which are expressed in terms of Srivastava and Daoust and Kampé de Fériet function. A number of known and new results are also considered as a special cases of our main results.

1 Introduction

Integral transforms have been extensively used in various problems of mathematical physics and applied mathematics. Numerous integral transforms(for example Laplace, Fourier, Mellin and Hankel etc.)involving a variety of special functions have been established by many mathematicians(for example: [2], [3], [4], [9], [12], [13], [14], [15], [16], [17]). Such integral transforms play an important role in many diverse field of physics and engineering.

In sequel of such type of works, we present a new integral transform involving the product of GGTSF and Jacobi polynomial, which are expressed in terms of Srivastava and Dauost and Kampé de Fériet function. Various new transformations(involving GST, SF, Bessel function of first kind, Galue type generalization of modified Bessel function, Legendre, Gegnbouer polynomial and Chebcheff polynomial of first and second kind) are also obtained as special cases of our main result.

For the purpose of our present study, we begin by recalling here the definition of GGTSF and its generalizations. The generalized Galue type struve function(GGTSF) is defined [11, p.3] as:

$$ {}_qW^{\gamma,\eta}_{p,b,c,\delta}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n (z/2)^{2n+p+1}}{\Gamma(\gamma n+\eta)\,\Gamma(qn+\frac{p}{\delta}+\frac{b+2}{2})}\ , \quad (q \in N,\ p,b,c \in C) \tag{1.1} $$

where $\gamma > 0$, $\delta > 0$ and η is arbitrary parameter.

We have a number of important special functions, which can be expressed in terms of generalized GTSF for different values of the parameters are:

$$ {}_1W^{1,\frac{3}{2}}_{p,b,c,1}(z) = H_{p,b,c}(z), \tag{1.2} $$

where $H_{p,b,c}(z)$ is generalized Struve function, which is defined by Yagmur and Orhan(see [8]-[11]).

$$ {}_1W^{1,\frac{3}{2}}_{p,1,1,1}(z) = H_p(z), \tag{1.3} $$

where $H_p(z)$ is Struve function of order p [11, p.2].

$$ {}_qW^{1,1}_{2\nu+2\lambda-1,1,1,2}(z) = \frac{(\frac{z}{2})^{\nu}\Gamma(\lambda+n+1)}{\Gamma(1+n)} J^q_{\nu,\lambda}(z), \tag{1.4} $$

where $J^q_{\nu,\lambda}(z)$ is Bessel maitland function [17].

$$ {}_1W^{1,1}_{\nu-1,2,1,1}(z) = J_\nu(z), \tag{1.5} $$

where $J_\nu(z)$ is Bessel function of first kind(see [5]-[7]).

$$ {}_qW^{1,1}_{p-1,1,-1,1}(z) = {}_qI_p(z), \tag{1.6} $$

where ${}_qI_p(z)$ is the Galue type generalization of modified Bessel function [1].

The jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ is defined by [7, p.340]

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{array}{cc} -n, \quad 1+\alpha+\beta+n; & \frac{1-z}{2} \\ 1+\alpha; & \end{array}\right] \tag{1.7}$$

or

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \frac{(1+\alpha)_n\ (1+\alpha+\beta)_{n+k}}{k!\ (n-k)!(1+\alpha)_k\ (1+\alpha+\beta)_n}(\frac{z-1}{2})^k. \tag{1.8}$$

We list below a number of important polynomials which can be expressed in terms of Jacobi polynomial for different values of the parameters α and β

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}$$

$$P_n^{(0,0)}(z) = P_n(z) \qquad ;\alpha=\beta=0 \tag{1.9}$$

$$P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(z) = \frac{(\nu+\frac{1}{2})_n}{(2\nu)_n}C_n^{\nu} \qquad ;\alpha=\beta=\nu-\frac{1}{2}, \tag{1.10}$$

where $P_n(z)$ and $C_n^{\nu}(z)$ are Legendre and Gegenbauer polynomial respectively [7]

In 1921, the four Appell functions were unified and generalized by Kampé de Fériet, who defined a general hyper geometric function of two variables. The notation introduce by Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy. We recall here the definition of a more general double hypergeometric function in a slightly modified notation(see [7]):

$$F^{p:q:k}_{l:m:n}\left[\begin{array}{ccc} (a_p): & (b_q): & (c_k); \\ (\alpha_l): & (\beta_m): & (\gamma_n); \end{array} x,y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\ \prod_{j=1}^{q}(b_j)_r\ \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\ \prod_{j=1}^{m}(\beta_j)_r\ \prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!} \tag{1.11}$$

where for convergence
(i)$p+q<l+m+1,\ p+k<l+n+1, |x|<\infty, |y|<\infty$ or
(ii)$p+q=l+m+1,\ p+k=l+n+1$ and

$$\left\{\begin{array}{ll} |x|^{1/(p-l)}+|y|^{1/(p-l)}<1, & ifp>l \\ max\{|x|,|y|\}<1, & ifp\leq l \end{array}\right\}$$

Srivastava and Daoust defined extremely multi-variable hypergeometric function is as follows:

$$F^{p:q_1;....q_r}_{l:m_1;....m_r}\left[\begin{array}{cc} (a_j:\alpha_j^1,..\alpha_j^r)_{1,p}\ :(c_j^1,r_j^1)_{1,q_1};....;(c_j^{(r)},r_j^{(r)})_{1,q_r}; & \\ & x_1,x_2,....x_r \\ (b_j:\beta_j^1,..\beta_j^r)_{1,l}:(d_j^1,\delta_j^1)_{1,m_1};....;(d_j^{(r)},\delta_j^{(r)})_{1,m_r}; & \end{array}\right]$$

$$= \sum_{n_1,n_2,....n_r=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{n_1\alpha_j^1+....+n_r\alpha_j^{(r)}}\ \prod_{j=1}^{q_1}(c_j^1)_{n_1r_j^1}....\prod_{j=1}^{q_r}(c_j^{(r)})_{n_rr_j^{(r)}}}{\prod_{j=1}^{l}(b_j)_{n_1\beta_j^1+....+n_r\beta_j^{(r)}}\ \prod_{j=1}^{m_1}(d_j^1)_{n_1\delta_j^1}....\prod_{j=1}^{m_r}(d_j^{(r)})_{n_r\delta_j^{(r)}}}\frac{x_1}{n_1!},.....,\frac{x_r}{n_r!} \tag{1.12}$$

where the multiple hypergeometric series converges absolutely and $(\lambda)_\nu$ denotes the well known pochhammer symbol [7].
In our study, we also need to recall the following Oberhittinger integral formula [6]

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}dx = 2\lambda\, a^{-\lambda}(\frac{a}{2})^\mu \frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(\lambda+\mu+1)}, \tag{1.13}$$

provided $0 < R(\mu) < R(\lambda)$.

2 Main Results

In this section, we established three generalized integral formulas in terms of Theorems 2.1, Theorem 2.2 and 2.3, which are expressed in terms of Srivastava and Daoust function [7], while theorem 2.3 is expressed in terms of Kampé de Fériet function [7].

Theorem 2.1. *The following integral formula holds true for $R(q) > 0$, $R(\delta) > 0$, $0 < R(\mu) < R(\lambda+p)$ and $x > 0$,*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\, {}_qW^{\gamma,\eta}_{p,b,c,\delta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{ty}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(\eta)\,\Gamma(\frac{p}{\delta}+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times F^{4:\,0:\,0}_{5:\,0:\,1}\left[\begin{array}{c} (\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,3),\ (1+\alpha+\beta:1,2), \\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,3),\ (\eta:\gamma,\gamma),\ (\frac{p}{\delta}+\frac{b+2}{2}:q,q), \end{array}\right.$$

$$\left.\begin{array}{c} (1+\alpha:1,1):\ \_\,;\ \_ \\ (1+\alpha+\beta:1,1):\ \_\,;\ (1+\alpha,1) \end{array}\ \middle|\ -\frac{cy^2}{4a^2},\ \frac{cty^3}{8a^3}\right]. \tag{2.1}$$

Proof. In order to derive (2.1), we denote the left-hand side of (2.1) by I, expanding ${}_qW^{\gamma,\eta}_{p,b,c,\delta}(z)$ and $P_n^{(\alpha,\beta)}(z)$ in their series form and then using the following Lemma(see [7])

$$\sum_{n=0}^\infty\sum_{k=0}^n B(k,n) = \sum_{n=0}^\infty\sum_{k=0}^\infty B(k,n+k) \tag{2.2}$$

$$I = (y/2)^{p+1}\sum_{n=0}^\infty\sum_{k=0}^\infty \frac{(-c)^{n+k}(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}(-t)^k}{(1+\alpha)_k(1+\alpha+\beta)_{n+k}\Gamma(\gamma n+\eta)\Gamma(qn+\frac{p}{\delta}+\frac{b+2}{2})}\frac{(\frac{y^2}{4})^n}{n!}\frac{(\frac{y^3}{8})^k}{k!}$$

$$\times\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-(\lambda+p+1+2n+k)}dx \tag{2.3}$$

Now using (1.13) in (2.3) and after simplification we give

$$I = (y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(\eta)\,\Gamma(\frac{p}{\delta}+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times\sum_{n=0}^\infty\sum_{k=0}^\infty \frac{(\lambda+p+2)_{2n+3k}(\lambda+p-\mu+1)_{2n+3k}(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}}{(\lambda+p+1)_{2n+3k}(\lambda+p+\mu+2)_{2n+3k}(\eta)_{\gamma n+\gamma k}(\frac{p}{\delta}+\frac{b+2}{2})_{qn+qk}}$$

$$\times\frac{1}{(1+\alpha)_k(1+\alpha+\beta)_{n+k}}\frac{(-\frac{cy^2}{4a^2})^n}{n!}\frac{(\frac{cty^3}{8a^3})^k}{k!} \tag{2.4}$$

finally summing up the above series(2.4) with the help of (1.12), we arrive at the right hand side of (2.1). □

Remark 2.2. On setting $p=\nu-1$, q=c=1, b=2, $\gamma=1, \eta=1, \delta=1$ in (2.1), we get the known result of Khan et al[13, p.343].

Theorem 2.3. *The following integral formula holds true for* $R(q)>0,\ R(\delta)>0,\ 0<R(\mu)<R(\lambda+p)$ *and* $x>0$*,*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\,{}_qW^{\gamma,\eta}_{p,b,c,\delta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{txy}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=(y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(\eta)\,\Gamma(\frac{p}{\delta}+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$
$$\times F^{4:\,0:\,1}_{5:\,0:\,1}\left[\begin{array}{c}(\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,2),\ (1+\alpha+\beta:1,2),\\ \\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,4),\ (\eta:\gamma,\gamma),\ (\frac{p}{\delta}+\frac{b+2}{2}:q,q),\end{array}\right.$$
$$\left.\begin{array}{c}(1+\alpha:1,1):\quad \_\,;\quad (2\mu:2)\\ \\ (1+\alpha+\beta:1,1):\quad \_\,;\quad (1+\alpha,1)\end{array}\right|-\frac{cy^2}{4a^2},\ \frac{cty^3}{16a^2}\Bigg]. \tag{2.5}$$

Proof. The proof of the theorem 2.2 is same as the proof of theorem 2.1 under same convergent conditions. Expanding ${}_qW^{\gamma,\eta}_{p,b,c,\delta}(z)$ and $P_n^{(\alpha,\beta)}(z)$ in their series form and then using the Lemma(2.2), we get

$$I'=(y/2)^{p+1}\sum_{n=0}^\infty\sum_{k=0}^\infty\frac{(-c)^{n+k}(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}(-t)^k}{(1+\alpha)_k(1+\alpha+\beta)_{n+k}\Gamma(\gamma n+\eta)\Gamma(qn+\frac{p}{\delta}+\frac{b+2}{2})}\frac{(\frac{y^2}{4})^n}{n!}\frac{(\frac{y^3}{8})^k}{k!}$$
$$\times\int_0^\infty x^{\mu+k}(x+a+\sqrt{x^2+2ax})^{-(\lambda+p+1+2n+k)}dx \tag{2.6}$$

now using (1.13) in (2.6) and after simplification we get

$$I'=(y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(\eta)\,\Gamma(\frac{p}{\delta}+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$
$$\times\sum_{n=0}^\infty\sum_{k=0}^\infty\frac{(\lambda+p+2)_{2n+3k}(\lambda+p-\mu+1)_{2n+2k}(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}(2\mu)_{2k}}{(\lambda+p+1)_{2n+3k}(\lambda+p+\mu+2)_{2n+4k}(\eta)_{\gamma n+\gamma k}(\frac{p}{\delta}+\frac{b+2}{2})_{qn+qk}}$$
$$\times\frac{1}{(1+\alpha)_k(1+\alpha+\beta)_{n+k}}\frac{(-\frac{cy^2}{4a^2})^n}{n!}\frac{(\frac{cty^3}{16a^2})^k}{k!}$$

finally summing up the above series with the help of (1.12), we arrive at the right hand side of (2.3). □

Theorem 2.4. *The following integral formula holds true for* $R(q)>0,\ R(\delta)>0,\ 0<R(\mu)<R(\lambda+p)$ *and* $x>0$*,*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\,{}_qW^{\gamma,\eta}_{p,b,c,\delta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{ty}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=(y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{(1+\alpha)_n\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{n!\,\Gamma(\eta)\,\Gamma(\frac{p}{\delta}+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$
$$\times F^{4:\ \ 1:\ \ 4}_{4:\ \gamma+q:\ 3}\left[\begin{array}{lcr}\Delta(2:\lambda+p+2),\quad \Delta(2:\lambda+p-\mu+1): & & 1;\\ \\ \Delta(2:\lambda+p+1),\quad \Delta(2:\lambda+p+\mu+2): & & \Delta(\gamma:\eta),\Delta(q:\frac{p}{\delta}+\frac{b+2}{2});\end{array}\right.$$

$$\left.\begin{matrix}\Delta(2:1+\alpha+\beta+n),\Delta(2:-n)\\ \Delta(2:1+\alpha)\ ,\frac{1}{2}\end{matrix}\right| -\frac{cy^2}{4a^2},\ \frac{ty^2}{16a^2}\Bigg]$$

$$+t(y)^{p+2}(2)^{-\mu-p-1}(a)^{\mu-\lambda-p-2}\frac{(1+\alpha)_n n(1+\alpha+\beta+n)\Gamma(2\mu)\ \Gamma(\lambda+p+3)\ \Gamma(\lambda+p-\mu+2)}{(n)!(1+\alpha)\ \Gamma(\eta)\ \Gamma(\frac{p}{\delta}+\frac{b+2}{2})\ \Gamma(\lambda+p+2)\Gamma(\lambda+p+\mu+3)}$$

$$\times F^{4:\ 1:\ \ 4}_{4:\ \gamma+q:\ 3}\left[\begin{matrix}\Delta(2:\lambda+p+3), & \Delta(2:\lambda+p-\mu+2): & 1;\\ \Delta(2:\lambda+p+2), & \Delta(2:\lambda+p+\mu+3): & \Delta(\gamma:\eta),\Delta(q:\frac{p}{\delta}+\frac{b+2}{2});\end{matrix}\right.$$

$$\left.\begin{matrix}\Delta(2:2+\alpha+\beta+n),\Delta(2:-n+1)\\ \Delta(2:2+\alpha)\ ,\frac{3}{2}\end{matrix}\right| -\frac{cy^2}{4a^2},\ \frac{ty^2}{16a^2}\Bigg]. \tag{2.7}$$

Proof. In order to derive (2.7), we denote the left-hand side of (2.7) by $I^{''}$ and expanding ${}_qW^{\gamma,\eta}_{p,b,c,\delta}(z)$ and $P_n^{(\alpha,\beta)}(z)$ in their series form and then interchanging order of integration and summation(which is verified by uniform convergence of the involved series under the given conditions), we get

$$I^{''}=(y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{(1+\alpha)_n\Gamma(2\mu)\ \Gamma(\lambda+p+2)\ \Gamma(\lambda+p-\mu+1)}{n!\ \Gamma(\eta)\ \Gamma(\frac{p}{\delta}+\frac{b+2}{2})\ \Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times\sum_{m=0}^{\infty}\sum_{k=0}^{n}\frac{(\lambda+p+2)_{2m+k}(\lambda+p-\mu+1)_{2m+k}(1+\alpha+\beta)_{n+k}}{(\lambda+p+1)_{2m+k}(\lambda+p+\mu+2)_{2m+k}(\eta)_{\gamma\ m}(\frac{p}{\delta}+\frac{b+2}{2})_q}$$

$$\times\frac{(1)_m(1+\alpha)_n}{(1+\alpha)_k(1+\alpha+\beta)_n(n-k)!}\frac{(-\frac{cty^2}{4a^2})^m}{m!}\frac{(-\frac{ty}{2a})^k}{k!}$$

now separating the $k-$series into its even and odd terms and then using the result $(A)_{m+n}=(A)_m(A+m)_n$. Finally after a simplification and summing the series with help of (1.11), we arrive at the right hand side of (2.7). This is the completes proof of Theorem 2.4 □

Remark 2.5. On setting $p=\nu-1$, q=c=1, b=2, $\gamma=1$, $\eta=1$, $\delta=1$ in (2.3), we get the known result of Khan et al [13].

3 Special Cases

In this section, we establish some integral formulas involving the product of Jacobi polynomial with generalized Struve function $H_{p,b,c}(z)$, Struve function $H_p(z)$, Bessel's function $J_\nu(z)$ and Galue type generalization of modified Bessel function ${}_qI_p(z)$ and then we derive some other integrals involving the product of generalized galue type Struve function(GTSF) with Legendre polynomial $P_n(z)$, Gagenbauer polynomial $C_n^\nu(z)$ as special cases of our main results.

Corollary 3.1. *The following integral formula holds true under the same condition of Theorem 1 and (1.2), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\ H_{p,b,c}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{ty}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=(y)^{p+1}(2)^{-\mu-p+1}(a)^{\mu-\lambda-p-1}\frac{1}{\sqrt{\pi}}\frac{\Gamma(2\mu)\ \Gamma(\lambda+p+2)\ \Gamma(\lambda+p-\mu+1)}{\Gamma(p+\frac{b+2}{2})\ \Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times F^{4:\ 0:\ 0}_{5:\ 0:\ 1}\left[\begin{matrix}(\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,3),\ (1+\alpha+\beta:1,2),\\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,3),\ (\frac{3}{2}:1,1),\ (p+\frac{b+2}{2}:1,1),\end{matrix}\right.$$

$$\left.\begin{matrix}(1+\alpha:1,1):\ \_;\ \_\\ (1+\alpha+\beta:1,1):\ \_;\ (1+\alpha,1)\end{matrix}\right| -\frac{cy^2}{4a^2},\ \frac{cty^3}{8a^3}\Bigg]. \tag{3.1}$$

Corollary 3.2. *The following integral formula holds true under the same condition of Theorem 2 and (1.2), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\, H_{p,b,c}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{txy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{-\mu-p+1}(a)^{\mu-\lambda-p-1}\frac{1}{\sqrt{\pi}}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(p+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times F^{4:\,0:\,1}_{5:\,0:\,1}\left[\begin{array}{c}(\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,2),\ (1+\alpha+\beta:1,2),\\ \\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,4),\ (\eta:\gamma,\gamma),\ (\frac{p}{\delta}+\frac{b+2}{2}:q,q),\end{array}\right.$$

$$\left.\begin{array}{c}(1+\alpha:1,1):\ \_;\ (2\mu:2)\\ \\ (1+\alpha+\beta:1,1):\ \_;\ (1+\alpha,1)\end{array}\quad\Bigg|-\frac{cy^2}{4a^2},\ \frac{cty^3}{16a^2}\right]. \tag{3.2}$$

Corollary 3.3. *The following integral formula holds true under the same condition of Theorem 1 and (1.3), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\, H_{p}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{ty}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{-\mu-p+1}(a)^{\mu-\lambda-p-1}\frac{1}{\sqrt{\pi}}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(p+\frac{3}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times F^{4:\,0:\,0}_{5:\,0:\,1}\left[\begin{array}{c}(\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,3),\ (1+\alpha+\beta:1,2),\\ \\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,3),\ (\frac{3}{2}:1,1),\ (p+\frac{3}{2}:1,1),\end{array}\right.$$

$$\left.\begin{array}{c}(1+\alpha:1,1):\ \_;\ \_\\ \\ (1+\alpha+\beta:1,1):\ \_;\ (1+\alpha,1)\end{array}\quad\Bigg|-\frac{y^2}{4a^2},\ \frac{ty^3}{8a^3}\right]. \tag{3.3}$$

Corollary 3.4. *The following integral formula holds true under the same condition of Theorem 2 and (1.3), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\, H_{p}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{txy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{-\mu-p+1}(a)^{\mu-\lambda-p-1}\frac{1}{\sqrt{\pi}}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(p+\frac{3}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times F^{4:\,0:\,1}_{5:\,0:\,1}\left[\begin{array}{c}(\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,2),\ (1+\alpha+\beta:1,2),\\ \\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,4),\ (\frac{3}{2}:1,1),\ (p+\frac{3}{2}:1,1),\end{array}\right.$$

$$\left.\begin{array}{c}(1+\alpha:1,1):\ \_;\ (2\mu:2)\\ \\ (1+\alpha+\beta:1,1):\ \_;\ (1+\alpha,1)\end{array}\quad\Bigg|-\frac{y^2}{4a^2},\ \frac{ty^3}{16a^2}\right]. \tag{3.4}$$

Corollary 3.5. *The following integral formula holds true under the same condition of Theorem 2 and (1.5), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\, J_{\nu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{txy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{\nu}(2)^{1-\mu-p}(a)^{\mu-\lambda-\nu}\frac{\Gamma(2\mu)\,\Gamma(\lambda+\nu+1)\,\Gamma(\lambda+\nu-\mu)}{\Gamma(\nu+1)\,\Gamma(\lambda+\mu)\Gamma(\lambda+\nu+\mu+1)}$$

$$\times F^{4:\,0:\,1}_{5:\,0:\,1}\left[\begin{array}{l}(\lambda+\nu+1:2,3),\ (\lambda+\nu-\mu:2,2),\ (1+\alpha+\beta:1,2),\\ (\lambda+\nu:2,3),\ (\lambda+\nu+\mu+1:2,4),\ (1:1,1),\ (\nu+1:1,1),\end{array}\right.$$

$$\left.\begin{array}{l}(1+\alpha:1,1):\ \_;\ (2\mu:2)\\ (1+\alpha+\beta:1,1):\ \_;\ (1+\alpha,1)\end{array}\ \middle|\ -\frac{y^2}{4a^2},\ \frac{ty^3}{16a^2}\right]. \tag{3.5}$$

Corollary 3.6. *The following integral formula holds true under the same condition of Theorem 1 and (1.6), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\ {}_qI_p\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{ty}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{1-\mu-p}(a)^{\mu-\lambda-p}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(p+1)\,\Gamma(\lambda+p)\Gamma(\lambda+p+\mu+1)}$$

$$\times F^{4:\,0:\,0}_{5:\,0:\,1}\left[\begin{array}{l}(\lambda+p+1:2,3),\ (\lambda+p-\mu+1:2,3),\ (1+\alpha+\beta:1,2),\\ (\lambda+p:2,3),\ (\lambda+p+\mu+1:2,3),\ (1:1,1),\qquad (p+1:q,q),\end{array}\right.$$

$$\left.\begin{array}{l}(1+\alpha:1,1):\ \_;\ \_\\ (1+\alpha+\beta:1,1):\ \_;\ (1+\alpha,1)\end{array}\ \middle|\ \frac{y^2}{4a^2},\ -\frac{ty^3}{8a^3}\right]. \tag{3.6}$$

Corollary 3.7. *The following integral formula holds true under the same condition of Theorem 2 and (1.6), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\ {}_qI_p\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n^{(\alpha,\beta)}\left(1-\frac{txy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{1-\mu-p}(a)^{\mu-\lambda-p}\frac{1}{\sqrt{\pi}}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+1)\,\Gamma(\lambda+p-\mu)}{\Gamma(p+1)\,\Gamma(\lambda+p)\Gamma(\lambda+p+\mu+1)}$$

$$\times F^{4:\,0:\,1}_{5:\,0:\,1}\left[\begin{array}{l}(\lambda+p+1:2,3),\ (\lambda+p-\mu:2,2),\ (1+\alpha+\beta:1,2),\\ (\lambda+p:2,3),\ (\lambda+p+\mu+1:2,4),\ (1:1,1),\ (p+1:q,q),\end{array}\right.$$

$$\left.\begin{array}{l}(1+\alpha:1,1):\ \_;\ (2\mu:2)\\ (1+\alpha+\beta:1,1):\ \_;\ (1+\alpha,1)\end{array}\ \middle|\ \frac{y^2}{4a^2},\ -\frac{ty^3}{16a^2}\right]. \tag{3.7}$$

Corollary 3.8. *The following integral formula holds true under the same condition of Theorem 1 and (1.9), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\ {}_qW^{\gamma,\eta}_{p,b,c,\delta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n\left(1-\frac{ty}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(\eta)\,\Gamma(\frac{p}{\delta}+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times F^{4:\,0:\,0}_{5:\,0:\,1}\left[\begin{array}{l}(\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,3),\ (1:1,2),\\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,3),\ (\eta:\gamma,\gamma),\ (\frac{p}{\delta}+\frac{b+2}{2}:q,q),\end{array}\right.$$

$$\left.\begin{array}{l}(1:1,1):\ \_;\ \_\\ (1:1,1):\ \_;\ (1,1)\end{array}\ \middle|\ -\frac{cy^2}{4a^2},\ \frac{cty^3}{8a^3}\right]. \tag{3.8}$$

Corollary 3.9. *The following integral formula holds true under the same condition of Theorem 2 and (1.9), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\,{}_qW^{\gamma,\eta}_{p,b,c,\delta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)P_n\left(1-\frac{txy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(\eta)\,\Gamma(\frac{p}{\delta}+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times F^{4:\,0:\,1}_{5:\,0:\,1}\left[\begin{array}{c}(\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,2),\ (1:1,2),\\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,4),\ (\eta:\gamma,\gamma),\ (\frac{p}{\delta}+\frac{b+2}{2}:q,q),\end{array}\right.$$

$$\left.\begin{array}{c}(1:1,1):\ \_;\ (2\mu:2)\\ (1:1,1):\ \_;\ (1,1);\end{array}\ \middle|\ -\frac{cy^2}{4a^2},\ \frac{cty^3}{16a^2}\right]. \tag{3.9}$$

Corollary 3.10. *The following integral formula holds true under the same condition of Theorem 1 and (1.10), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\,{}_qW^{\gamma,\eta}_{p,b,c,\delta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)C^{\nu}_n\left(1-\frac{ty}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(\eta)\,\Gamma(\frac{p}{\delta}+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times F^{4:\,1:\,0}_{5:\,1:\,1}\left[\begin{array}{c}(\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,3),\ (2\nu:1,2),\\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,3),\ (\eta:\gamma,\gamma),\ (\frac{p}{\delta}+\frac{b+2}{2}:q,q),\end{array}\right.$$

$$\left.\begin{array}{c}(\nu+\frac{1}{2}:1,1):\ (2\nu:1);\ \_\\ (2\nu:1,1):\ (\nu+\frac{1}{2}:1);\ (\nu+\frac{1}{2}:1)\end{array}\ \middle|\ -\frac{cy^2}{4a^2},\ \frac{cty^3}{8a^3}\right]. \tag{3.10}$$

Corollary 3.11. *The following integral formula holds true under the same condition of Theorem 2 and (1.10), we get*

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\,{}_qW^{\gamma,\eta}_{p,b,c,\delta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)C^{\nu}_n\left(1-\frac{txy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= (y)^{p+1}(2)^{-\mu-p}(a)^{\mu-\lambda-p-1}\frac{\Gamma(2\mu)\,\Gamma(\lambda+p+2)\,\Gamma(\lambda+p-\mu+1)}{\Gamma(\eta)\,\Gamma(\frac{p}{\delta}+\frac{b+2}{2})\,\Gamma(\lambda+p+1)\Gamma(\lambda+p+\mu+2)}$$

$$\times F^{4:\,1:\,1}_{5:\,1:\,1}\left[\begin{array}{c}(\lambda+p+2:2,3),\ (\lambda+p-\mu+1:2,2),\ (2\nu:1,2),\\ (\lambda+p+1:2,3),\ (\lambda+p+\mu+2:2,4),\ (\eta:\gamma,\gamma),\ (\frac{p}{\delta}+\frac{b+2}{2}:q,q),\end{array}\right.$$

$$\left.\begin{array}{c}(\nu+\frac{1}{2}:1,1):\ (2\nu:1),\ (2\mu:2)\\ (2\nu:1,1):\ (\nu+\frac{1}{2}:1),\ (\nu+\frac{1}{2},1)\end{array}\ \middle|\ -\frac{cy^2}{4a^2},\ \frac{cty^3}{16a^2}\right]. \tag{3.11}$$

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