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On sum and ratio formulas for Lucas-balancing numbers

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Abstract. In this paper we give explicit sum formulas for consecutive Lucas-balancing numbers, consecutive even/odd Lucas-balancing numbers, squares of consecutive Lucas-balancing numbers, squares of consecutive even/odd Lucas-balancing numbers and pronic product of Lucas-balancing numbers. Sums of these numbers with alternative signs are also considered. When indices of Lucas-balancing sequence are in arthritic progression, ratios of sum/differences follow certain interesting patterns.

1 Introduction

As defined by Behera and second author of this paper in [1], balancing numbers and balancers are solutions of the diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r).$$

It is customary to denote the n^{th} balancing number by B_n and the corresponding balancer by R_n. Further, $C_n=\sqrt{8B_n^2+1}$ is called the n^{th} Lucas-balancing number [9]. The Binet forms of B_n, C_n and R_n are respectively

$$B_n=\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}},\ C_n=\frac{\alpha^{2n}+\beta^{2n}}{2} \quad \text{and} \quad R_n=\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}-\frac{1}{2}$$

where $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$. For basic details of balancing, Lucas-balancing numbers and balancers the readers are advised to refer to [10].

The objective of this paper is to develop certain interesting sum formulas involving Lucas-balancing numbers in terms of balancing and Lucas-balancing numbers. Certain similar sum formulas involving Fibonacci, Lucas and Pell-Lucas numbers were studied by cerin [2, 3, 4, 5, 6], while sum formulas involving balancing numbers were developed by the authors in [8] and Ray [12]. Davala [7], Ray and Sahu in [12] independently developed some binomial and convolution sums involving balancing and Lucas-balancing numbers using the Binet forms and generating functions.

The following known results will be helpful in the subsequent sections. We will be frequently using these results with/without further reference. For details of the proofs from (a) to (d) the readers are advised to refer to [10].

For any non-negative integer m, n and x

(a) $C_{m+n}=B_mC_{n+1}-B_{m-1}C_n$

(b) $C_{m\pm n}=C_mC_n\pm 8B_mB_n$

(c) $B_{m\pm1}=3B_m\pm C_m$

(d) $B_m=3B_{m+1}-C_{m+1}$

(e) $2R_n=C_n-2B_n-1$

(f) $C_m=B_{n+1}C_{m+n}-B_nC_{m+n+1}$

(g) $C_{m+(n+1)x} = 2C_{m+nx}C_x - C_{m+(n-1)x}$

The proofs of (e) and (f) are as follows:

$$\begin{aligned} B_{n+1}C_{m+n} - B_nC_{m+n+1} =& (3B_n + C_n)C_{m+n} - B_n(3C_{m+n} - 8B_{m+n}) \\ =& C_{m+n}C_n - 8B_{m+n}B_n = C_m, \end{aligned}$$

from which (e) follows. Further (f) follows from

$$\begin{aligned} 2C_{m+nx}C_x - C_{m+(n-1)x} =& 2C_{m+nx}C_x - (B_{x+1}C_{m+nx} - B_xC_{m+nx+1}) \\ =& C_{m+nx}(2C_x - B_{x+1}) + B_xC_{m+nx+1} \\ =& -C_{m+nx}B_{x-1} + B_xC_{m+nx+1} = C_{m+(n+1)x}. \end{aligned}$$

2 Sum formulas involving Lucas-balancing numbers

In this section, we obtain certain sum formulas involving linear and nonlinear combinations of Lucas-balancing numbers.

Theorem 2.1. *For natural numbers k and m*

(a) $\sum\limits_{i=0}^{m} C_i = \frac{1}{2}[B_{m+1} + B_m + 1]$,

(b) $\sum\limits_{i=0}^{m} (-1)^i C_i = \begin{cases} R_{m+1} + 1 \; if \; m \; is \; even, \\ -R_{m+1} \; if \; m \; is \; odd. \end{cases}$

(c) $\sum\limits_{i=0}^{m} C_{k+2i} = C_{k+m}B_{m+1}$

(d) $\sum\limits_{i=0}^{m} (-1)^i C_{k+2i} = \frac{1}{6}[(-1)^m C_{k+2m+1} + C_{k-1}]$

Proof. The proof of (a) follows from induction. We use the Binet form of Lucas-balancing numbers to prove (b), (c) and (d).

Proof of (b) :

$$\begin{aligned} \sum_{i=0}^{m} (-1)^i C_i &= \frac{1}{2}\sum_{i=0}^{m} (-1)^i[\alpha^{2i} + \beta^{2i}] = \frac{1}{2}\left[\sum_{i=0}^{m} (-\alpha^2)^i + \sum_{i=0}^{m} (-\beta^2)^i\right] \\ &= \frac{1}{2}\left[\frac{(-1)^m\alpha^{2m+2} + 1}{\alpha^2 + 1} + \frac{(-1)^m\beta^{2m+2} + 1}{\beta^2 + 1}\right] \\ &= \frac{1}{2}\left[\frac{(-1)^m\alpha^{2m+2} + 1}{2\sqrt{2}\alpha} - \frac{(-1)^m\beta^{2m+2} + 1}{2\sqrt{2}\beta}\right] \\ &= \frac{1}{4\sqrt{2}}\left[(-1)^m\alpha^{2m+1} - \beta - (-1)^m\beta^{2m+1} + \alpha\right] \\ &= \frac{1}{4\sqrt{2}}[(-1)^m\left(\alpha^{2m+1} - \beta^{2m+1}\right) + 2\sqrt{2}] \\ &= \begin{cases} R_{m+1} + 1 \; if \; m \; is \; even, \\ -R_{m+1} \; if \; m \; is \; odd. \end{cases} \end{aligned}$$

Proof of (c) :

$$\begin{aligned}
\sum_{i=0}^{m} C_{k+2i} &= \frac{1}{2}\sum_{i=0}^{m}[\alpha^{2k+4i}+\beta^{2k+4i}] \\
&= \frac{1}{2}\left[\alpha^{2k}\sum_{i=0}^{m}(\alpha^4)^i+\beta^{2k}\sum_{i=0}^{m}(\beta^4)^i\right] \\
&= \frac{1}{2}\left[\left(\frac{\alpha^{2k+4m+4}-\alpha^{2k}}{\alpha^4-1}\right)+\left(\frac{\beta^{2k+4m+4}-\beta^{2k}}{\beta^4-1}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{\alpha^{2k+4m+4}-\alpha^{2k}}{4\sqrt{2}\alpha^2}\right)-\left(\frac{\beta^{2k+4m+4}-\beta^{2k}}{4\sqrt{2}\beta^2}\right)\right] \\
&= \frac{1}{8\sqrt{2}}[\alpha^{2k+4m+2}-\alpha^{2k-2}-\beta^{2k+4m+2}+\beta^{2k-2}] \\
&= \frac{1}{2}[B_{k+2m+1}-B_{k-1}] \\
&= C_{k+m}B_{m+1}.
\end{aligned}$$

Proof of (d) :

$$\begin{aligned}
\sum_{i=0}^{m}(-1)^i C_{k+2i} &= \frac{1}{2}\sum_{i=0}^{m}(-1)^i[\alpha^{2k+4i}+\beta^{2k+4i}] \\
&= \frac{1}{2}\left[\alpha^{2k}\sum_{i=0}^{m}(-\alpha^4)^i+\beta^{2k}\sum_{i=0}^{m}(-\beta^4)^i\right] \\
&= \frac{1}{2}\left[\left(\frac{(-1)^m\alpha^{2k+4m+4}+\alpha^{2k}}{\alpha^4+1}\right)+\left(\frac{(-1)^m\beta^{2k+4m+4}+\beta^{2k}}{\beta^4+1}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{(-1)^m\alpha^{2k+4m+4}+\alpha^{2k}}{6\alpha^2}\right)+\left(\frac{(-1)^m\beta^{2k+4m+4}+\beta^{2k}}{6\beta^2}\right)\right] \\
&= \frac{1}{12}[(-1)^m\alpha^{2k+4m+2}+\alpha^{2k-2}+(-1)^m\beta^{2k+4m+2}+\beta^{2k-2}] \\
&= \frac{1}{6}[(-1)^m C_{k+2m+1}+C_{k-1}].
\end{aligned}$$

□

The proof of following theorem is similar to that of Theorem 2.1. As a further reference, the readers are advised to go through [7].

Theorem 2.2. *For natural numbers k and m*

(a) $\sum_{i=0}^{m} C_{k+i}^2 = \frac{1}{2}[C_{m+2k}B_{m+1}+m+1]$,

(b) $\sum_{i=0}^{m} (-1)^i C_{k+i}^2 = \frac{1}{12}[(-1)^m C_{2m+2k+1}+C_{2k-1}+3(1+(-1)^m)]$,

(c) $\sum_{i=0}^{m} C_{k+2i}^2 = \frac{1}{12}[C_{2m+2k}B_{2m+2}+6(m+1)]$,

(d) $\sum_{i=0}^{m} (-1)^i C_{k+2i}^2 = \frac{1}{68}[(-1)^m C_{4m+2k+2}+C_{2k-2}+17(1+(-1)^m)]$.

(e) $\sum_{i=0}^{m} C_{k+i}C_{k+i+1} = \frac{1}{2}[C_{m+2k+1}B_{m+1}+3(m+1)]$,

(f) $\sum_{i=0}^{m} (-1)^i C_{k+i}C_{k+i+1} = \frac{1}{12}[(-1)^m C_{2m+2k+2}+C_{2k}+9(1+(-1)^m)]$,

(g) $\sum_{i=0}^{m} C_{k+2i}C_{k+2i+1} = \frac{1}{12}[C_{2m+2k+1}B_{2m+2} + 18(m+1)]$,

(h) $\sum_{i=0}^{m} (-1)^i C_{k+2i}C_{k+2i+1} = \frac{1}{68}[(-1)^m C_{4m+2k+3} + C_{2k-1} + 51(1+(-1)^m)]$.

In the following theorem we present few sum formulas involving balancing and Lucas-balancing numbers. We give the proofs of (a) and (f) only, while the proofs of $(b), (c), (d)$ and (e) can be proved by induction or by the use of Binet formulas for balancing and Lucas-balancing numbers.

Theorem 2.3. *For* $n \in \mathbb{N}$

(a) $\sum_{i=0}^{m} B_{k+ni}C_{k+ni} = \frac{1}{2B_n}B_{2k+nm}B_{(m+1)n}$

(b) $\sum_{i=0}^{m} (-1)^i B_{k+ni}C_{k+ni} = \frac{1}{4C_n}[(-1)^m B_{2k+n(2m+1)} - B_{2k-n}]$

(c) $\sum_{i=0}^{m} B_{k+ni}C_{k+ni+1} = \frac{1}{2B_n}B_{2k+nm+1}B_{mn+n} - \frac{1}{2}(m+1)$

(d) $\sum_{i=0}^{m} (-1)^i B_{k+ni}C_{k+ni+1} = \frac{1}{4C_n}[(-1)^m B_{2k+n(2m+1)+1} + B_{2k-n+1} - C_n(1+(-1)^m)]$

(e) $\sum_{i=0}^{m} B_{k+i}C_{k+i+n} = \frac{1}{2}B_{2k+m+1}B_{m+1} - \frac{1}{2}(m+1)B_n$

(f) $\sum_{i=0}^{m} (-1)^i B_{k+i}C_{k+i+n} = \frac{1}{12}[(-1)^m B_{2k+2m+n+1} + B_{2k+n-1} - B_{2n}(1+(-1)^m)]$

Proof. The proof of (a) is based on induction on m. For $m = 1$,

$$\sum_{i=0}^{1} B_{k+ni}C_{k+ni} = B_kC_k + B_{k+n}C_{k+n} = B_{2k+n}C_n = \frac{1}{2B_n}B_{2k+n}B_{2n}$$

and hence the statement is true for $m = 1$. Let us assume that the statement be true for $m = l$. Consider, $m = l+1$,

$$\begin{aligned}\sum_{i=0}^{l+1} B_{k+ni}C_{k+ni} &= \frac{1}{2B_n}B_{2k+nl}B_{(l+1)n} + B_{k+n(l+1)}C_{k+n(l+1)}\\ &= \frac{1}{2B_n}B_{2k+nl}B_{(l+1)n} + \frac{1}{2}B_{2k+2n(l+1)}\\ &= \frac{1}{2B_n}\left[B_{2k+nl}B_{(l+1)n} - B_{2k+2n(l+1)}B_{-n}\right]\\ &= \frac{1}{2B_n}B_{2k+n(l+1)}B_{(l+2)n},\end{aligned}$$

the statement is true for $m = l+1$, hence the proof of (c) follows, the proof of (f) follows from,

$$\begin{aligned}\sum_{i=0}^{m} (-1)^i B_{k+i}C_{k+i+n} &= \sum_{i=0}^{m} \frac{(-1)^i}{8\sqrt{2}}[\alpha^{4k+4i+2n} - \beta^{4k+4i+2n} - \alpha^{2n} + \beta^{2n}]\\ &= \frac{1}{2}\sum_{i=0}^{m}(-1)^i B_{2k+2i+n} + \frac{1}{2}\sum_{i=0}^{m}(-1)^i B_n\end{aligned}$$

(from Throem 2.1 (d) of [8])

$$= \frac{1}{12}[(-1)^m B_{2k+2m+n+1} + B_{2k+n-1} - B_n(1+(-1)^m)]$$

□

In the following theorem, we obtain a higher order recurrence relation for Lucas-balancing numbers and a sum formula involving weighted sum of consecutive Lucas-balancing numbers.

Theorem 2.4. *For* $n \in \mathbb{N}$

(a) $C_{n+2} = 5C_{n+1} + 4\sum_{k=1}^{n} C_i + 2$

(b) $nC_1 + (n-1)C_2 + \cdots + 2C_{n-1} + C_n = \frac{1}{4}\left[C_{n+1} - (2n+3)\right]$

Proof. The proof of (a) follows from the recurrence relation for Lucas-balancing numbers. We use induction to prove (b). Let us define an integer sequence $\{Z_n\}$ as

$$Z_n = nC_1 + (n-1)C_2 + \cdots + 2C_{n-1} + C_n.$$

We will prove that $Z_n = \frac{1}{4}\left[C_{n+1} - (2n+3)\right]$. It is easy to see that the assertion is true for $n = 1$. Assume that the assertion is true for $n = k$. To complete the proof, we need to show that the assertion is true for $n = k+1$.

For each natural number n, $Z_{n+1} - Z_n = \sum\limits_{i=1}^{n+1} C_i$. Hence,

$$\begin{aligned} Z_{k+1} &= Z_k + \sum_{i=1}^{k+1} C_i \\ &= \frac{1}{4}\left[C_{k+1} - (2k+3)\right] + \frac{1}{4}\left[C_{k+3} - 5C_{k+2} - 2\right] \\ &= \frac{1}{4}\left[C_{k+3} - 5C_{k+2} + C_{k+1} - (2k+5)\right] \\ &= \frac{1}{4}\left[C_{k+2} - (2k+5)\right]. \end{aligned}$$

Thus, the assertion is true for $n = k+1$. □

The following theorem provides a binomial sum involving Lucas-balancing numbers.

Theorem 2.5. *For* $n, r \in \mathbb{N}$, $\sum\limits_{k=0}^{n} \binom{n}{k}(-6)^k C_{k+r} = (-1)^n C_{2n+r}$.

Proof. To prove this assertion we use the Binet formula of Lucas-balancing numbers.

$$\begin{aligned} \sum_{k=0}^{n}\binom{n}{k}(-6)^k C_{k+r} &= \sum_{k=0}^{n}\binom{n}{k}(-6)^k\left[\frac{\alpha^{2k+2r}+\beta^{2k+2r}}{2}\right] \\ &= \frac{\alpha^{2r}}{2}\sum_{k=0}^{n}\binom{n}{k}(-6\alpha^2)^k + \frac{\beta^{2r}}{2}\sum_{k=0}^{n}\binom{n}{k}(-6\beta^2)^k \\ &= \frac{\alpha^{2r}}{2}[1-6\alpha^2]^n + \frac{\beta^{2r}}{2}[1-6\beta^2]^n \\ &= \frac{\alpha^{2r}}{2}[-\alpha^4]^n + \frac{\beta^{2r}}{2}[-\beta^4]^n \\ &= (-1)^n\left[\frac{\alpha^{4n+2r}+\beta^{4n+2r}}{2}\right] \\ &= (-1)^n C_{2n+r}. \end{aligned}$$

□

3 Formulas involving ratio of linear combinations

In this section, we present some quotients involving sum or differences of Lucas-balancing numbers that simplify to linear expressions of balancing numbers or Lucas-balancing numbers. In some cases, the subscripts of Lucas-balancing numbers involved in the ratio are in arithmetic progression.

The following index reduction formulas for sequences B_n and C_n will play a crucial role in the balancing and Lucas-balancing numbers, were developed by Ray [11].

Theorem 3.1. *(Index reduction formulas): If x, y, z, w and r are integers and $x+y=z+w$ then*

(a) $B_{x+r}C_{y+r} - B_{z+r}C_{w+r} = B_xC_y - B_zC_w$

(b) $C_{x+r}C_{y+r} - C_{z+r}C_{w+r} = C_xC_y - C_zC_w.$

(c) $B_{x+r}B_{y+r} - B_{z+r}B_{w+r} = B_xB_y - B_zB_w.$

Theorem 3.2. *If m and n are natural numbers then each of* $\frac{C_{m+2n+1} \pm C_m}{C_{m+n+1} \pm C_{m+n}}$ *and* $\frac{C_{m+3n} \pm C_m}{C_{m+2n} \pm C_{m+n}}$ *are independent of m . Also the following identities hold.*

(i) $\frac{C_{m+2n+1} - C_m}{C_{m+n+1} - C_{m+n}} = B_{n+1} + B_n,$

(ii) $\frac{C_{m+2n+1} + C_m}{C_{m+n+1} + H_{m+n}} = B_{n+1} - B_n,$

(iii) $\frac{C_{m+3n} - C_m}{C_{m+2n} - C_{m+n}} = 2C_n + 1,$

(iv) $\frac{C_{m+3n} + C_m}{C_{m+2n} + C_{m+n}} = 2C_n - 1.$

Proof. Using Theorem 3.1 (a), we have

$$\begin{aligned}&(C_{m+n+1} - C_{m+n})(B_{n+1} + B_n)\\&= (C_{m+n+1}B_{n+1} - C_{m+n}B_n) + (B_nC_{m+n+1} - B_{n+1}C_{m+n})\\&= C_{m+2n+1} - C_m\end{aligned}$$

from which (i) follows. Further (iv) follows from

$$\begin{aligned}&(C_{m+2n} + C_{m+n})(2C_n - 1)\\&= 2C_{m+2n}C_n + 2C_{m+n}C_n - C_{m+2n} - C_{m+n}\\&= 2C_{m+2n}C_n + C_{m+n}C_n - 8B_nB_{m+n} - C_{m+n}\\&= 2C_{m+2n}C_n + C_m - C_{m+n}\\&= C_{m+3n} + C_m.\end{aligned}$$

□

Theorem 3.3. *For natural numbers m, n and k,*

$$\frac{C_{m+2n+2k} - C_m}{C_{m+n+2k} - C_{m+n}} = \frac{B_{n+k}}{B_k} = \frac{B_{m+2n+2k} - B_m}{B_{m+n+2k} - B_{m+n}}.$$

Proof. From Theorem 3.1 (a), we have

$$B_{n+k}C_{n+m} - B_kC_m = B_nC_{n+m+k} = B_{n+k}C_{n+m+2k} - B_kC_{n+2m+2k},$$

rearrangement gives

$$\frac{C_{m+2n+2k} - C_m}{C_{m+n+2k} - C_{m+n}} = \frac{B_{n+k}}{B_k}.$$

The proof of

$$\frac{B_{n+k}}{B_k} = \frac{B_{m+2n+2k} - B_m}{B_{m+n+2k} - B_{m+n}}$$

is from Theorem 3.2 [7]. □

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