Palestine Journal of Mathematics

Vol. 8(2)(2019) , 207–212

Bounds on Randić Color Energy of a Graph

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Communicated by Ayman Badawi

MSC 2010 Classifications: $05C50$.

Keywords and phrases: Colored graph, Randić matrix and Randić color energy.

The author would like to thank the Prof. R. Rangarajan, DOS in Mathematics, University of Mysore, Mysuru, for many useful suggestions.

Abstract. Let G be a colored graph with n vertices, m edges, chromatic number $\chi(G)$ and d_i is the degree of vertex v_i. In this paper, we show that some basic properties of Randić color energy and an upper bound and a lower bound for Randić color energy of a graph in terms of degree of a vertex v_i, number of edges, and determinant of the Randić color matrix.

1 Introduction

Let G be a colored graph if coloring the vertices of a graph such that no two adjacent vertices have the same color. The minimum number of colors assign to vertex of a graph G is called chromatic number of G and it is denoted by $\chi(G)$. The color adjacency matrix [1] $A_C(G)$ are as follows: If $c(v_i)$ is the color of v_i, then

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j); \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j); \\ 0, & \text{otherwise.} \end{cases}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A_c(G)$ are real number and their sum is equal to zero. Color energy [1] of a graph is as the sum of the absolute values of the eigenvalues of $A_c(G)$.

$$\text{i.e. } E_c(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For more research papers on color energy of a graph and its bounds, we can refer [1, 2, 12, 14, 15, 16].
Randić matrix [5] $R(G) = (r_{ij})$ of G is a square symmetric matrix defined by

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & i = j; \\ 0, & v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

Let $\rho_1, \rho_2, \ldots, \rho_n$ be the eigenvalues of the Randić matrix $R(G)$, these eigenvalues are real numbers, and their sum is zero, the Randić energy [5] of a graph G as the sum of the absolute values of the eigenvalues of $R(G)$. Literatures on Randić energy and its bounds and Randić indices can be found in [3, 4, 5, 7, 8, 9, 10, 11, 12].

1.1 Randić color matrix $A_{RC}(G)$ and Randić color energy $E_{RC}(G)$

Let G be a simple colored graph with n vertices. The Randić color matrix [13] $A_{RC}(G) = (r_{ij})$ is a square $n \times n$ matrix defined by

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j); \\ \frac{-1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j); \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_{RC}(G)$ is $|\rho I - A_{RC}(G)|$. Let $\rho_1, \rho_2, \ldots, \rho_n$ be eigenvalues of Randić color matrix $A_{RC}(G)$. Since $A_{RC}(G)$ is real and symmetric matrix, so its eigenvalues are real numbers and that their sum is zero. If the eigenvalues of $A_{RC}(G)$ are $\rho_1, \rho_2, \ldots, \rho_n$ with their multiplicities are $m_1, m_2, \ldots, m_r$ then spectrum of $A_{RC}(G)$ is denoted by $Spec_{RC}(G) = \begin{pmatrix} \rho_1 & \rho_2 & \ldots & \rho_{n-1} & \rho_n \\ m_1 & m_2 & \ldots & m_{r-1} & m_r \end{pmatrix}$. The Randić color energy [13] $E_{RC}(G)$ of a colored graph G is defined as

$$E_{RC}(G) = \sum_{i=1}^{n} |\rho_i|.$$

2 Bounds for Randić Color Energy of a Graph

Lemma 2.1. *Let G be a colored graph and let $\rho_1, \rho_2, \ldots, \rho_n$ be the eigenvalues of Randić color matrix $A_{RC}(G)$. Then*

$$\sum_{i=1}^{n} \rho_i = 0$$

and

$$\sum_{i=1}^{n} \rho_i^2 = 2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right],$$

where m is number of edges in G, m' is the number of pairs of non-adjacent vertices having the same color in G, d_i, d_j are degree of adjacent vertices of different color and d_i', d_j' are degree of non-adjacent vertices of same color in G.

Proof. The sum of the eigenvalues of $A_{RC}(G)$ is the diagonal elements of $A_{RC}(G)$ is

$$\sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} r_{ii} = 0$$

Consider, the sum of squares of the eigenvalues of $A_{RC}(G)$ is trace of $[A_{RC}(G)]^2$,

$$\begin{aligned} \sum_{i=1}^{n} \rho_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\, r_{ji} \\ &= \sum_{i=1}^{n}(r_{ii})^2 + \sum_{i\neq j} r_{ij}\, r_{ji} \\ &= \sum_{i=1}^{n}(r_{ii})^2 + 2\sum_{i<j}(r_{ij})^2 \\ \sum_{i=1}^{n} \rho_i^2 &= 2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]. \end{aligned}$$

Theorem 2.2. *Let G_1 and G_2 be two colored graphs with n vertices and m_1, m_2 are number of edges in G_1 and G_2 respectively. Let $\rho_1, \rho_2, \ldots, \rho_n$ are eigenvalues of $A_{RC}(G_1)$ and $\rho_1', \rho_2', \ldots, \rho_n'$ are eigenvalues of $A_{RC}(G_2)$. Then*

$$\sum_{i=1}^{n} \rho_i\, \rho_i' \leq 2\sqrt{\left[\frac{m_1}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_1)_c'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]\left[\frac{m_2}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_2)_c'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}.$$

Proof. By the Cauchy-Schwartz inequality [17], we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right), \text{ for any real numbers } a_i, b_i.$$

If $a_i = \rho_i$, $b_i = \rho_i'$ we get

$$\left(\sum_{i=1}^{n} \rho_i \rho_i'\right)^2 \leq \left(\sum_{i=1}^{n} \rho_i^2\right)\left(\sum_{i=1}^{n} (\rho_i')^2\right)$$

$$\left(\sum_{i=1}^{n} \rho_i \rho_i'\right)^2 \leq 4\left[\frac{m_1}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_1)_c'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]\left[\frac{m_2}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_2)_c'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] \quad [\textit{by Lemma } 2.1]$$

$$\Rightarrow \sum_{i=1}^{n} \rho_i \rho_i' \leq 2\sqrt{\left[\frac{m_1}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_1)_c'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]\left[\frac{m_2}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_2)_c'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}.$$

3 Bounds for Randić color energy

McClelland [11] gave upper and lower bounds for ordinary energy of a graph. Similar bounds for Randić color energy $E_{RC}(G)$ are given in the following theorem.

Theorem 3.1. ***(Upper Bound)*** *Let G be a graph with n vertices and m edges. Then*

$$E_{RC}(G) \leq \sqrt{2n\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}.$$

Proof. Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

If $a_i = 1$ and $b_i = |\rho_i|$, then

$$\left(\sum_{i=1}^{n} |\rho_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1^2\right)\left(\sum_{i=1}^{n} |\rho_i|^2\right)$$

$$[E_{RC}(G)]^2 \leq n\, 2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]$$

$$E_{RC}(G) \leq \sqrt{2n\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}.$$

Theorem 3.2. ***(Lower Bound)*** *Let G be a graph with n vertices and m edges. Then*

$$E_{RC}(G) \geq \sqrt{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] + n(n-1)D^{\frac{2}{n}}}, \textit{ where } D = \left|\prod_{i=1}^{n} \rho_i\right|.$$

Proof. Consider

$$[E_{RC}(G)]^2 = \left[\sum_{i=1}^{n} |\rho_i|\right]^2 = \sum_{i=1}^{n} |\rho_i|^2 + \sum_{i \neq j} |\rho_i|\,|\rho_j| \tag{3.1}$$

By arithmetic mean and geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\rho_i|\,|\rho_j| \geq \left(\prod_{i \neq j} |\rho_i|\,|\rho_j|\right)^{\frac{1}{n(n-1)}}$$

$$\sum_{i \neq j} |\rho_i|\,|\rho_j| \geq n(n-1)\left(\prod_{i=1}^{n} |\rho_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$\sum_{i\neq j} |\rho_i|\,|\rho_j| \quad \geq \quad n(n-1)\left(\prod_{i=1}^{n} |\rho_i|\right)^{2/n}. \tag{3.2}$$

Using (3.2) in (3.1), we have

$$[E_{RC}(G)]^2 \geq \sum_{i=1}^{n} |\rho_i|^2 + n(n-1)\left|\prod_{i=1}^{n} |\rho_i|\right|^{2/n}$$

$$[E_{RC}(G)]^2 \geq 2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] + n(n-1)D^{\frac{2}{n}}$$

$$E_{RC}(G)] \geq \sqrt{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] + n(n-1)D^{\frac{2}{n}}}.$$

4 Bounds for Randić color spectral radius and Randić color energy

The graph G eigenvalues are labeled in a non-increasing manner, i.e., $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. If G connected, $\rho_1 \geq |\rho_i|, i = 2, 3, \ldots, n$, then eigenvalue ρ_1 is called **spectral radius** [17] of G.

Proposition 4.1. *Let G be a (n, m) colored graph and $\rho_1(G) = max_{1\leq i\leq n}\{|\rho_i|\}$ be the Randić color spectral radius of G. Then*

$$\sqrt{\frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]} \leq \rho_1(G) \leq \sqrt{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}.$$

Proof. Consider,

$$\rho_1^2(G) \quad = \quad max_{1\leq i\leq n}\ \{|\rho_i|\} \leq \sum_{i=1}^{n} |\rho_i|^2 = 2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]$$

$$\rho_1(G) \quad \leq \quad \sqrt{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}$$

Next,

$$n\,\rho_1^2(G) \quad \geq \quad \sum_{i=1}^{n} \rho_i = 2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]$$

$$\rho_1^2(G) \quad \geq \quad \frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]$$

$$\rho_1(G) \quad \geq \quad \sqrt{\frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}$$

$$\therefore \sqrt{\frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]} \leq \rho_1(G) \leq \sqrt{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}.$$

Proposition 4.2. *Let G be a (n, m)-colored graph and $\rho_1, \rho_2, \ldots, \rho_n$ be the Randić color eigenvalues of G. If $n \leq 2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]$ and $\rho_1 \geq \frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]$, then*

$$E_{RC}(G) \leq \frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] +$$
$$\sqrt{(n-1)\left\{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] - \left(\frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]\right)^2\right\}}.$$

Proof. We know that,

$$\sum_{i=1}^{n} \rho_i^2 = 2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]$$

$$\sum_{i=2}^{n} \rho_i^2 = 2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] - \rho_1^2 \tag{4.1}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left(\sum_{i=1}^{n} |\rho_i|\right)^2 &\leq n\sum_{i=1}^{n} |\rho_i|^2 \\ \left(\sum_{i=2}^{n} |\rho_i|\right)^2 &\leq (n-1)\sum_{i=2}^{n} |\rho_i|^2 \end{aligned}$$

and hence

$$\sum_{i=2}^{n} |\rho_i| \leq \sqrt{(n-1)\sum_{i=2}^{n} |\rho_i|^2} \tag{4.2}$$

using (4.1) in (4.2), we get

$$\begin{aligned} E_{RC}(G) - \rho_1 &\leq \sqrt{(n-1)\left\{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] - \rho_1^2\right\}} \\ E_{RC}(G) &\leq \rho_1 + \sqrt{(n-1)\left\{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] - \rho_1^2\right\}} \end{aligned}$$

Consider the function,

$$F(x) = x + \sqrt{(n-1)\left\{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] - x^2\right\}}$$

Then,

$$F'(x) = 1 - \frac{x\sqrt{(n-1)}}{\sqrt{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] - x^2}}$$

$F(x)$ is decreasing in

$$\left(\sqrt{\frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}, \sqrt{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2} + \frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}\right)$$

we have,

$$\sqrt{\frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2}+\frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]} < \frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2}+\frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] \leq \rho_1 \leq \sqrt{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2}+\frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]}$$

by Proposition 4.1, we obtain

$$E_{RC}(G) \leq \frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2}+\frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] + \sqrt{(n-1)\left\{2\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2}+\frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right] - \left(\frac{2}{n}\left[\frac{m}{\left(\sqrt{d_i d_j}\right)^2}+\frac{(m_c)'}{\left(\sqrt{d_i' d_j'}\right)^2}\right]\right)^2\right\}}.$$

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Received: June 12, 2017.
Accepted: December 20, 2017.