Bounds on Randić Color Energy of a Graph

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Abstract. Let $G$ be a colored graph with $n$ vertices, $m$ edges, chromatic number $\chi(G)$ and $d_i$ is the degree of vertex $v_i$. In this paper, we show that some basic properties of Randić color energy and an upper bound and a lower bound for Randić color energy of a graph in terms of degree of a vertex $v_i$, number of edges, and determinant of the Randić color matrix.

1 Introduction

Let $G$ be a colored graph if coloring the vertices of a graph such that no two adjacent vertices have the same color. The minimum number of colors assign to vertex of a graph $G$ is called chromatic number of $G$ and it is denoted by $\chi(G)$. The color adjacency matrix $A_C(G)$ are as follows: If $c(v_i)$ is the color of $v_i$, then

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j); \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j); \\ 0, & \text{otherwise}. \end{cases}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A_c(G)$ are real number and their sum is equal to zero. Color energy [1] of a graph is as the sum of the absolute values of the eigenvalues of $A_c(G)$.

$$E_c(G) = \sum_{i=1}^{n} |\lambda_i|.$$ 

For more research papers on color energy of a graph and its bounds, we can refer [1, 2, 12, 14, 15, 16].

Randić matrix [5] $R(G) = (r_{ij})$ of $G$ is a square symmetric matrix defined by

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & v_i \text{ and } v_j \text{ are adjacent}; \\ 0, & i = j; \\ 0, & v_i \text{ and } v_j \text{ are not adjacent}. \end{cases}$$

Let $\rho_1, \rho_2, \ldots, \rho_n$ be the eigenvalues of the Randić matrix $R(G)$, these eigenvalues are real numbers, and their sum is zero, the Randić energy [5] of a graph $G$ as the sum of the absolute values of the eigenvalues of $R(G)$. Literatures on Randić energy and its bounds and Randić indices can be found in [3, 4, 5, 7, 8, 9, 10, 11, 12].

1.1 Randić color matrix $A_{RC}(G)$ and Randić color energy $E_{RC}(G)$

Let $G$ be a simple colored graph with $n$ vertices. The Randić color matrix [13] $A_{RC}(G) = (r_{ij})$ is a square $n \times n$ matrix defined by
\[ r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j); \\ \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j); \\ 0, & \text{otherwise.} \]

The characteristic polynomial of \( A_{RC}(G) \) is \( |\rho I - A_{RC}(G)| \). Let \( \rho_1, \rho_2, \ldots, \rho_n \) be eigenvalues of Randić color matrix \( A_{RC}(G) \). Since \( A_{RC}(G) \) is real and symmetric matrix, so its eigenvalues are real numbers and that their sum is zero. If the eigenvalues of \( A_{RC}(G) \) are \( \rho_1, \rho_2, \ldots, \rho_n \) with their multiplicities are \( m_1, m_2, \ldots, m_r \), then spectrum of \( A_{RC}(G) \) is denoted by \( \text{Spec}_{RC}(G) = \left( \rho_1 \rho_2 \ldots \rho_{n-1} \rho_n \\ m_1 m_2 \ldots m_{r-1} m_r \right) \). The Randić color energy [13] \( E_{RC}(G) \) of a colored graph \( G \) is defined as

\[ E_{RC}(G) = \sum_{i=1}^{n} |\rho_i|. \]

### 2 Bounds for Randić Color Energy of a Graph

**Lemma 2.1.** Let \( G \) be a colored graph and let \( \rho_1, \rho_2, \ldots, \rho_n \) be the eigenvalues of Randić color matrix \( A_{RC}(G) \). Then

\[ \sum_{i=1}^{n} \rho_i = 0 \]

and

\[ \sum_{i=1}^{n} \rho_i^2 = 2 \left[ \frac{m}{\left( \sqrt{d_i d_j} \right)^2} + \frac{(m'_c)}{\left( \sqrt{d'_i d'_j} \right)^2} \right], \]

where \( m \) is the number of edges in \( G \), \( m'_c \) is the number of pairs of non-adjacent vertices having the same color in \( G \), \( d_i, d_j \) are degree of adjacent vertices of different color and \( d'_i, d'_j \) are degree of non-adjacent vertices of same color in \( G \).

**Proof.** The sum of the eigenvalues of \( A_{RC}(G) \) is the diagonal elements of \( A_{RC}(G) \) is

\[ \sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} r_{ii} = 0. \]

Consider, the sum of squares of the eigenvalues of \( A_{RC}(G) \) is trace of \( [A_{RC}(G)]^2 \),

\[ \sum_{i=1}^{n} \rho_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} r_{ji} = \sum_{i=1}^{n} (r_{ii})^2 + \sum_{i \neq j} r_{ij} r_{ji} = \sum_{i=1}^{n} (r_{ii})^2 + 2 \sum_{i<j} (r_{ij})^2 = 2 \left[ \frac{m}{\left( \sqrt{d_i d_j} \right)^2} + \frac{(m'_c)}{\left( \sqrt{d'_i d'_j} \right)^2} \right]. \]

**Theorem 2.2.** Let \( G_1 \) and \( G_2 \) be two colored graphs with \( n \) vertices and \( m_1 \), \( m_2 \) are number of edges in \( G_1 \) and \( G_2 \) respectively. Let \( \rho_1, \rho_2, \ldots, \rho_n \) are eigenvalues of \( A_{RC}(G_1) \) and \( \rho'_1, \rho'_2, \ldots, \rho'_n \) are eigenvalues of \( A_{RC}(G_2) \). Then

\[ \sum_{i=1}^{n} \rho_i \rho'_i \leq 2 \left[ \frac{m_1}{\left( \sqrt{d_i d_j} \right)^2} + \frac{(m'_c)}{\left( \sqrt{d'_i d'_j} \right)^2} \right] \left[ \frac{m_2}{\left( \sqrt{d_i d_j} \right)^2} + \frac{(m'_c)}{\left( \sqrt{d'_i d'_j} \right)^2} \right]. \]

**Proof.** By the Cauchy-Schwartz inequality [17], we have

\[ (\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2) (\sum_{i=1}^{n} b_i^2), \]

for any real numbers \( a_i, b_i \).
If \( a_i = \rho_i \), \( b_i = \rho'_i \) we get

\[
\left( \sum_{i=1}^{n} \rho_i \rho'_i \right)^2 \leq \left( \sum_{i=1}^{n} \rho_i^2 \right) \left( \sum_{i=1}^{n} (\rho'_i)^2 \right) \]

\[
\left( \sum_{i=1}^{n} \rho_i \rho'_i \right)^2 \leq 4 \left[ \frac{m_1}{(\sqrt{d_i d_j})^2} + \frac{(\rho_1')^2}{(\sqrt{d'_i d'_j})^2} \right] \left[ \frac{m_2}{(\sqrt{d_i d_j})^2} + \frac{(\rho_2')^2}{(\sqrt{d'_i d'_j})^2} \right] \quad \text{[by Lemma 2.1]}
\]

\[
\Rightarrow \sum_{i=1}^{n} \rho_i \rho'_i \leq 2 \sqrt{\left[ \frac{m_1}{(\sqrt{d_i d_j})^2} + \frac{(\rho_1')^2}{(\sqrt{d'_i d'_j})^2} \right] \left[ \frac{m_2}{(\sqrt{d_i d_j})^2} + \frac{(\rho_2')^2}{(\sqrt{d'_i d'_j})^2} \right]}
\]

### 3 Bounds for Randić color energy

McClelland [11] gave upper and lower bounds for ordinary energy of a graph. Similar bounds for Randić color energy \( E_{RC}(G) \) are given in the following theorem.

**Theorem 3.1. (Upper Bound)** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then

\[
E_{RC}(G) \leq \sqrt{2n \left[ \frac{m}{(\sqrt{d_i d_j})^2} + \frac{(\rho_1')^2}{(\sqrt{d'_i d'_j})^2} \right]}.
\]

**Proof.** Cauchy-Schwartz inequality, we have

\[
(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2) (\sum_{i=1}^{n} b_i^2).
\]

If \( a_i = 1 \) and \( b_i = |\rho_i| \), then

\[
(\sum_{i=1}^{n} |\rho_i|)^2 \leq (\sum_{i=1}^{n} 1^2) (\sum_{i=1}^{n} |\rho_i|^2).
\]

\[
[E_{RC}(G)]^2 \leq n 2 \left[ \frac{m}{(\sqrt{d_i d_j})^2} + \frac{(\rho_1')^2}{(\sqrt{d'_i d'_j})^2} \right]
\]

\[
E_{RC}(G) \leq \sqrt{2n \left[ \frac{m}{(\sqrt{d_i d_j})^2} + \frac{(\rho_1')^2}{(\sqrt{d'_i d'_j})^2} \right]}.
\]

**Theorem 3.2. (Lower Bound)** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then

\[
E_{RC}(G) \geq \sqrt{2 \left[ \frac{m}{(\sqrt{d_i d_j})^2} + \frac{(\rho_1')^2}{(\sqrt{d'_i d'_j})^2} \right] + n(n-1)D^2 \frac{1}{2}}, \text{ where } D = |\prod_{i=1}^{n} \rho_i|.
\]

**Proof.** Consider

\[
[E_{RC}(G)]^2 = \left[ \sum_{i=1}^{n} |\rho_i| \right]^2 = \sum_{i=1}^{n} |\rho_i|^2 + \sum_{i \neq j} |\rho_i||\rho_j| \tag{3.1}
\]

By arithmetic mean and geometric mean inequality, we have

\[
\frac{1}{n(n-1)} \sum_{i \neq j} |\rho_i||\rho_j| \geq \left( \prod_{i \neq j} |\rho_i||\rho_j| \right)^{\frac{1}{n(n-1)}}
\]

\[
\sum_{i \neq j} |\rho_i||\rho_j| \geq n(n-1) \left( \prod_{i=1}^{n} |\rho_i|^{2(n-1)} \right)^{\frac{1}{n(n-1)}}
\]
\[
\sum_{i \neq j} |\rho_i| |\rho_j| \geq n(n-1) \left( \prod_{i=1}^{n} |\rho_i| \right)^{2/n}.
\] (3.2)

Using (3.2) in (3.1), we have
\[
[E_{RC}(G)]^2 \geq \sum_{i=1}^{n} |\rho_i|^2 + n(n-1) \left| \prod_{i=1}^{n} |\rho_i| \right|^{2/n}
\]
\[
[E_{RC}(G)]^2 \geq 2 \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right] + n(n-1) D^{\frac{2}{n}}
\]
\[
E_{RC}(G) \geq \sqrt{2 \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right] + n(n-1) D^{\frac{2}{n}}}.
\]

4 Bounds for Randić color spectral radius and Randić color energy

The graph \( G \) eigenvalues are labeled in a non-increasing manner, i.e., \( \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n \). If \( G \) connected, \( \rho_1 \geq |\rho_i|, i = 2, 3, \ldots, n \), then eigenvalue \( \rho_1 \) is called spectral radius [17] of \( G \).

**Proposition 4.1.** Let \( G \) be a \((n,m)\) colored graph and \( \rho_1(G) = \max_{1 \leq i \leq n} \{|\rho_i|\} \) be the Randić color spectral radius of \( G \). Then
\[
\sqrt{\frac{2}{n} \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right]} \leq \rho_1(G) \leq \sqrt{2 \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right]}
\]

**Proof.** Consider,
\[
\rho_1^2(G) = \max_{1 \leq i \leq n} \{|\rho_i|\} \leq \sum_{i=1}^{n} |\rho_i|^2 = 2 \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right]
\]
\[
\rho_1(G) \leq \sqrt{2 \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right]}
\]

Next,
\[
n \rho_1^2(G) \geq \sum_{i=1}^{n} \rho_i = 2 \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right]
\]
\[
\rho_1^2(G) \geq \frac{2}{n} \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right]
\]
\[
\rho_1(G) \geq \sqrt{\frac{2}{n} \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right]}
\]
\[
\therefore \sqrt{\frac{2}{n} \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right]} \leq \rho_1(G) \leq \sqrt{2 \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right]}
\]

**Proposition 4.2.** Let \( G \) be a \((n,m)\)-colored graph and \( \rho_1, \rho_2, \ldots, \rho_n \) be the Randić color eigenvalues of \( G \). If \( n \leq 2 \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right] \) and \( \rho_1 \geq \frac{3}{4} \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)^{\prime}}{\sqrt{d_i d_j}} \right] \), then
\[
E_{RC}(G) \leq \frac{2}{n} \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right] + \\
\sqrt{(n-1) \left\{ 2 \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right] - \left( \frac{2}{n} \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right] \right)^2 \right\}.}
\]

**Proof.** We know that,

\[
\sum_{i=1}^{n} \rho_i^2 = 2 \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right]
\]

\[
\sum_{i=2}^{n} \rho_i^2 = 2 \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right] - \rho_1^2 \tag{4.1}
\]

By Cauchy-Schwarz inequality, we have

\[
\left( \sum_{i=1}^{n} |\rho_i| \right)^2 \leq n \sum_{i=1}^{n} |\rho_i|^2
\]

\[
\left( \sum_{i=2}^{n} |\rho_i| \right)^2 \leq (n-1) \sum_{i=2}^{n} |\rho_i|^2
\]

and hence

\[
\sum_{i=2}^{n} |\rho_i| \leq \sqrt{(n-1) \sum_{i=2}^{n} |\rho_i|^2} \tag{4.2}
\]

using (4.1) in (4.2), we get

\[
E_{RC}(G) - \rho_1 \leq \sqrt{(n-1) \left\{ 2 \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right] - \rho_1^2 \right\}}
\]

\[
E_{RC}(G) \leq \rho_1 + \sqrt{(n-1) \left\{ 2 \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right] - \rho_1^2 \right\}}
\]

Consider the function,

\[
F(x) = x + \sqrt{(n-1) \left\{ 2 \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right] - x^2 \right\}}
\]

Then,

\[
F'(x) = 1 - \frac{x \sqrt{(n-1)}}{\sqrt{2 \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right] - x^2}}
\]

\[
F(x) \text{ is decreasing in}
\]

\[
\left( \sqrt{\frac{2}{n} \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right]}, \sqrt{2 \left[ \frac{m}{(\sqrt{d_i}d_j)} + \frac{(m_e)^\prime}{(\sqrt{d'_i}d'_j)} \right]} \right)
\]
we have,
\[ \sqrt{\frac{2}{n}} \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)'}{\sqrt{d_i d_j}'} \right] \leq \sqrt{2} \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)'}{\sqrt{d_i d_j}'} \right] \]
by Proposition 4.1, we obtain
\[ E_{RC}(G) \leq \frac{2}{n} \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)'}{\sqrt{d_i d_j}'} \right] + \sqrt{(n-1) \left\{ 2 \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)'}{\sqrt{d_i d_j}'} \right] - \left( \frac{2}{n} \left[ \frac{m}{\sqrt{d_i d_j}} + \frac{(m_c)'}{\sqrt{d_i d_j}'} \right] \right)^2 \right\} } \]

References


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